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Soliton formations for magnetohydrodynamic viscous flow over a nonlinear stretching sheet

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Abstract. In the present paper, the main focus is to study soliton formations of a two-dimensional magnetohydrodynamic flow over a nonlinear stretching sheet with the help of transformed rational function method. The fluid is electrically conductive, normal to the stretching sheet and there is no induced magnetic field. The flow problem is described by the continuity and momentum equation with suitable boundary conditions. For solving the model, the nonlinearity poses a great challenge. Nonlinear partial differential equation has been converted into a nonlinear ordinary differential equation by using similarity transformations, and then a trial solution is assumed. The results indicate complete consistency and effectiveness of the suggested scheme compared with the existing literature.

Keywords. Magnetic field; viscous flow; transformed rational function method; stretching sheet; Maple 18.

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1. Introduction

The study of magnetohydrodynamic (MHD) flow has become an important research area for researchers, engineers and scientists because of its wide industrial applications. The MHD flow has many manufacturing and industrial applications such as in petroleum industries, polymer industries, plastic sheets production, purification of raw oil, production of copper wires and in many other industries. In fluid mechanics, most of the scientific problems are nonlinear. For instance, the nature of the MHD viscous flow over the stretching sheet is nonlinear. Most of the physical problems are highly complex in nature. The fast improvement in nonlinear sciences has led to a wide range of well-organised and trustworthy techniques, which help to deal with such physical problems. Nonlinear problems are still difficult to solve either numerically or analytically.

Solitary wave phenomena was first observed by John Scott Russell. The concept of a soliton has now become ubiquitous in modem nonlinear science and indeed can be found in various branches of physics and mathematics. A soliton can be defined as a stationary localised nonlinear wave, whose profile is determined by a balance of dispersion and nonlinearity. John Scott Russell in 1834 was riding a horse along a narrow canal in Scotland when he observed a ‘rounded smooth well-defined heap of water’ propagating ‘without change of form or speed’. These waves were later named as solitons (from the Latin word solitarius – solitary). In 1895, Diederik Kortweg and Gustav de Varies derived an equation (now known as KdV equation), which describes the surface waves, including solitons on shallow water surfaces. Because of their stability and robustness, solitons have found plenty of applications in physics, hydrodynamics, nonlinear optics, biology, chemistry, etc. Various forms of solitary wave solutions are there in nature: the wave phenomena in elastic media, kink-shaped tanh solutions, plasma waves, bell-shaped sech solutions and applications in biogenetics, optical fibres, chemical kinematics, solid-state physics, condensed matter physics, etc. The Korteweg–de Vries (KdV) equation and Boussinesq equation describe the water wave phenomena and yield travelling wave solutions.

There are various numerical and analytical techniques to find exact solution of mathematically modelled problems. Besides this, if exact solutions of nonlinear evaluation equations are available, it assists in the stability analysis and facilitates numerical solvers in comparison. Many techniques such as the Backlund transformation method [1], the variational iteration method [2], the tanh-function method [3–5], the first integration method [6], the exp-function method [7–10], the truncated Painleve expansion method [11], the Weierstrass elliptic function method [12], the Jacobi elliptic function expansion method [13–15], the rational expansion method [16], the F-expansion method [17], Hirota bilinear method [18], the $\exp(-\varphi)$ expansion method [19] etc. are available to find out exact solution of nonlinear evaluation equations. Some results on solitary wave solution are, the solution of nonlinear cubic–quintic reaction–diffusion equation [20], some new solutions of MHD flow [21] and exp-function method for fractional differential equations [22]. For some topical results about the integral transform methods and on exact solutions, see [23–25,27].

There have been some recent studies on an interesting kind of exact solutions called lumps, lump–kink interaction solutions, lump–soliton interaction solutions and Rossby wave solutions [28–34]. Chinese mathematicians, Wu and He, have introduced a very effective method called exp-function technique, a special case of the transformed rational function method [35], or more generally, the multiple exp-function method [36]. This technique gives solution to many physical nonlinear problems. The study of the literature reveals that the exp-function method is applicable and highly steadfast on differential equations. Importantly, Ebaid [37] proved that by applying exp-function technique to any nonlinear ordinary differential equation $c = p$ and $q = d$ are the unique relations that can be attained by equating linear terms with nonlinear terms involved in the problem.

We use the exp-function method to study solitary wave phenomena of the MHD incompressible viscous flow. It is notable that this method is totally compatible and greatly effective for nonlinear partial differential equations. Also it can be extended to physical models that arise in plasma physics, mathematical engineering, applied sciences and fluid mechanics.

2. Mathematical modelling

Consider an MHD flow over a surface of the nonlinear stretching sheet ($y = 0$), and that fluid is incompressible and electrically conductive under the applied magnetic field $B(x)$ normal to the stretching sheet. We also neglect the induced magnetic field. The continuity and momentum equations governing such type of flow are written as

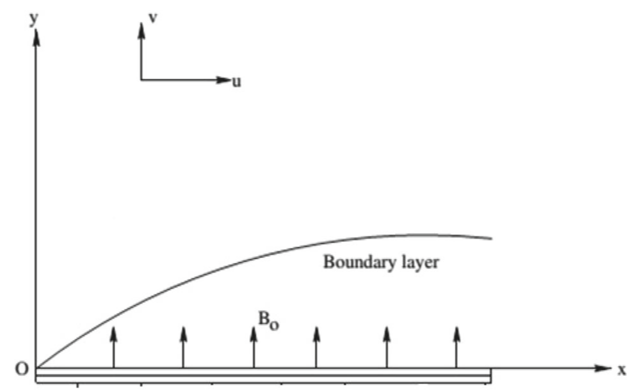

Figure 1. Sketch of the physical problem.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - \sigma\frac{B^2(x)}{\rho}u, \tag{2}$$

where u and v are the components of velocity in the x and y directions, respectively, ρ is the fluid density, $\upsilon = \mu/\rho$ is the kinematics viscosity and σ is the electrical conductivity of the fluid (figure 1).

Here

$$B(x) = B_0 x^{(m-1)/2}. \tag{3}$$

The assumed boundary conditions for the flow are as follows:

$$u(x, 0) = cx^m, \quad v(x, 0) = 0$$

and

$$u(x, \infty) = 0. \tag{4}$$

To solve this problem, we first non-dimensionalised eqs (1)–(4) by introducing the following similarity variables:

$$\eta = y\sqrt{\frac{c(m+1)}{2\upsilon}}x^{(m-1)/2}, \tag{5}$$

$$\psi = \sqrt{\frac{2c\upsilon}{(m+1)}}x^{(m+1)/2} \cdot g(\eta). \tag{6}$$

Using the stream functions

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

$$u = cx^m g'(\eta)$$

and

$$v = -\sqrt{\frac{(m+1)\, cv}{2}}.x^{(m-1)/2}\left(g(\eta) + \frac{m-1}{m+1}\eta g'(\eta)\right), \tag{7}$$

$$\frac{\partial u}{\partial x} = cmx^{m-1}g'(\eta) + c\frac{m-1}{2}x^{m-1}\eta g''(\eta), \tag{8}$$

$$\frac{\partial u}{\partial y} = cx^m\left(x^{(m-1)/2}\sqrt{\frac{c\,(m+1)}{2v}}\right)g''(\eta), \tag{9}$$

$$\frac{\partial^2 u}{\partial y^2} = cx^m x^{(m-1)/2}\sqrt{\frac{c\,(m+1)}{2v}} \times \left(x^{(m-1)/2}\sqrt{\frac{c\,(m+1)}{2v}}\right)g'''(\eta). \tag{10}$$

By substituting eqs (7)–(10) into eq. (2), we obtain

$$g'''(\eta) + g(\eta)g''(\eta) - \gamma\left(g'(\eta)\right)^2 - Mg'(\eta) = 0, \tag{11}$$

where

$$\gamma = \frac{2m}{m+1} \quad \text{and} \quad M = \frac{2\sigma B^2}{\rho c\,(m+1)}.$$

The transformed boundary conditions are

$$g(0) = 0, \quad g'(0) = 1, \quad g'(\infty) = 0. \tag{12}$$

3. Analysis of the method

The general form of the nonlinear partial differential equation is as follows:

$$P\left(u, u_t, u_x, u_y, u_{xy}, u_{xx}, \ldots\right) = 0. \tag{13}$$

The exact solution of the equation is in the form

$$u(x, y, z, t) = u(\eta), \quad \eta = \eta(x, y, z, t).$$

Invoke the wave transformation

$$\eta = kx + sy + lz + \omega t, \tag{14}$$

where k, l, s, ω are constants. We change eq. (13) by using eq. (14) into nonlinear ordinary differential equation (ODE) of the form given below:

$$Q\left(u, u', u'', u''', \ldots\right) = 0, \tag{15}$$

where the prime denotes the derivative with respect to η. If possible, integrate eq. (15) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero.

In the next step, it is important to introduce a new variable $V = V(\eta)$ by a solvable differential equation.

For the first-order differential equation

$$V' = F = F(\eta, V).$$

Two simple solvable cases of function F are as follows:

$$F = F(V) = V, \quad F = F(V) = a + V,$$

where a is a constant. The corresponding first-order equations have the particular solutions $V = \mathrm{e}^{\eta}$ and $V = -1/\eta$ when $a = 0$. This case corresponds to the exp-function method (see [28]).

Consider the rational functions

$$\varepsilon(V) = \frac{p(V)}{q(V)} = \frac{p_m V^m + p_{m-1}V^{m-1} + \cdots + p_0}{q_n V^n + q_{n-1}V^{n-1} + \cdots + q_0},$$

where m and n are natural numbers, p_i and q_i, $0 \le i \le m, n$, are constants but also can be functions of independent variables. We are interested in travelling wave solutions attained by

$$u^{(r)}(\eta) = x(V) = \frac{p(V)}{q(V)},$$

where

$$u^{(r+1)} = F\varepsilon',$$

$$u^{(r+2)} = F\frac{\partial}{\partial V}\left(u^{(r)}\right) = F\varepsilon' + F'\varepsilon', \ldots.$$

Now, we assume that the transformed equation (15) is a rational function equation of V with a given pair of m and n. This can be achieved for all nonlinear equations of the differential polynomial type, where F is a rational function in V. Thus, we need to force the numerator of the resulting rational function in the transformed equation to be zero. This yields a system of algebraic equations in all variables k, s, l, ω, p_i and q_i, $0 \le i \le m, n$, then solve this system to obtain $p(V), q(V)$ and η. After that on integrating $x(V)$ with respect to η, we obtain a class of travelling wave solutions:

$$\begin{aligned} u(x, y, z, t) &= u(\eta) \\ &= \int \cdots \int \frac{p(V(\eta))}{q(V(\eta))}\mathrm{d}\eta \cdots \mathrm{d}\eta \ \ (r \text{ times}) \\ &= \int_0^{\eta}\int_0^{\eta_r} \cdots \int_0^{\eta_2} \frac{p(V(\eta_1))}{q(V(\eta_1))}\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_{r-1}\mathrm{d}\eta_r \\ &\quad + \sum\nolimits_{i=1}^{r} d_i \eta^{r-1}. \end{aligned}$$

In the above, d_i, $1 \le i \le r$, are arbitrary constants. The case of the exp-function method is discussed as follows.

If we take $V = \mathrm{e}^{\eta}$, then the solution function is obtained by

$$u(x, y, z, t) = u(\eta)$$
$$= \int \cdots \int \frac{p_m \mathrm{e}^{m\eta} + p_{m-1}\mathrm{e}^{(m-1)\eta} + \cdots + p_0}{q_n \mathrm{e}^{n\eta} + q_{n-1}\mathrm{e}^{(n-1)\eta} + \cdots + q_0}$$
$$\times \mathrm{d}\eta \cdots \mathrm{d}\eta \ (r \text{ times}).$$

This can yield solutions generated by the exp-function method [8]. It is observed that the transformed rational function method generalised the exp-function method.

According to the exp-function method developed by Chinese mathematicians He and Wu [8], we consider that the wave solutions can be stated as follows:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}. \tag{16}$$

In the above travelling wave solution, c, d, p and q are positive integers, which are calculated, and a_n and b_m are constants.

The equivalent form of the travelling wave solution (16) can be written as

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{17}$$

To determine the values of c and p, we balance the highest order linear term with the highest order nonlinear term in eq. (15). Similarly, for values of d and q, we balance the lowest order linear and nonlinear terms in eq. (15).

By substituting (17) into ODE (15) and equating the coefficient of each power of $\exp(n\eta)$ to zero, we obtain a system of algebraic equations. These algebraic equations are solved for a_n and b_m with the help of symbolic computation software Maple 18. Finally, we attain the travelling wave solution $u(x, y, z, t)$.

4. Solution procedure

Consider eq. (11) which is a nonlinear differential equation:

$$g'''(\eta) + g(\eta)g''(\eta) - \gamma\big(g'(\eta)\big)^2 - Mg'(\eta) = 0.$$

The boundary conditions is given below:

$$g(0) = 0, \quad g'(0) = 1, \quad g'(\infty) = 0,$$

where the prime indicates derivation of g with respect to η.

The solution of eq. (11) can be written in the form of eq. (16). We can frequently select the values of p, q, c and d but we shall understand that the final solution does not reliably weigh on the selection of these values.

Case I. For ease, we set $c = p = 1$ and $q = d = 1$, eq. (16) reduces to the form given as

$$g(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{18}$$

Substituting eq. (18) into eq. (11), we obtain

$$\frac{1}{A}\begin{bmatrix} t_7 \exp(7\eta) + t_6 \exp(6\eta) + t_5 \exp(5\eta) \\ +t_4 \exp(4\eta) + t_3 \exp(3\eta) + t_2 \exp(2\eta) \\ +t_1 \exp(\eta) + t_0 + t_{-1} \exp(-\eta) \\ +t_{-2} \exp(-2\eta) + t_{-3} \exp(-3\eta) \\ +t_{-4} \exp(-4\eta) + t_{-5} \exp(-5\eta) \\ +t_{-6} \exp(-6\eta) + t_{-7} \exp(-7\eta) \end{bmatrix} = 0, \tag{19}$$

where

$$A = (b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^4,$$
$$t_i, \ i = 0, \pm 1, \pm 2, \ldots, \pm 7,$$

are constants obtained by Maple 18. By equating the coefficients of $\exp(n\eta)$ to zero, we obtain

$$\left\{ \begin{array}{l} t_{-7} = 0, t_{-6} = 0, t_{-5} = 0, t_{-4} = 0, t_{-3} = 0, \\ t_{-2} = 0, t_{-1} = 0, t_0 = 0, t_1 = 0, t_2 = 0, \\ t_3 = 0, t_4 = 0, t_5 = 0, t_6 = 0, t_7 = 0 \end{array} \right\}. \tag{20}$$

First solution set:
Consider

$$a_{-1} = 0, a_0 = Mb_0 - b_0, a_1 = Mb_1 - b_1, b_{-1} = 0,$$
$$b_0 = b_0, b_1 = b_1.$$

Inserting these values into eq. (18), we attain exact travelling wave solution $g(\eta)$ of eq. (11) (figure 2):

$$g(\eta) = \frac{(Mb_0 - b_0) + (Mb_1 - b_1)\mathrm{e}^{\eta}}{b_0 + b_1 \mathrm{e}^{\eta}}.$$

Second solution set:
Consider

$$a_{-1} = a_{-1}, a_0 = \frac{a_{-1}b_0}{b_{-1}}, a_1 = \frac{a_{-1}b_1}{b_{-1}},$$
$$b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1.$$

Using the above-mentioned values in the trial solution, the exact travelling wave solution $g(\eta)$ of the flow problem is attained (figure 3):

$$g(\eta) = \frac{a_{-1}\mathrm{e}^{-\eta} + (a_{-1}b_0/b_{-1}) + (a_{-1}b_1/b_{-1})\mathrm{e}^{\eta}}{b_{-1}\mathrm{e}^{-\eta} + b_0 + b_1 \mathrm{e}^{\eta}}.$$

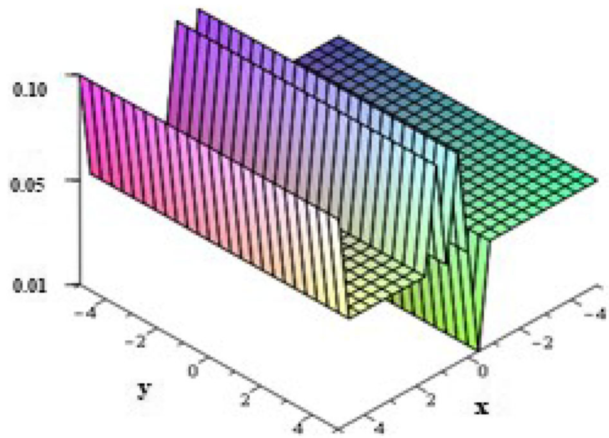

Figure 2. Graphical representation of the first solution set for $b_0 = 2,\ b_1 = 0.1, M = 0.4$.

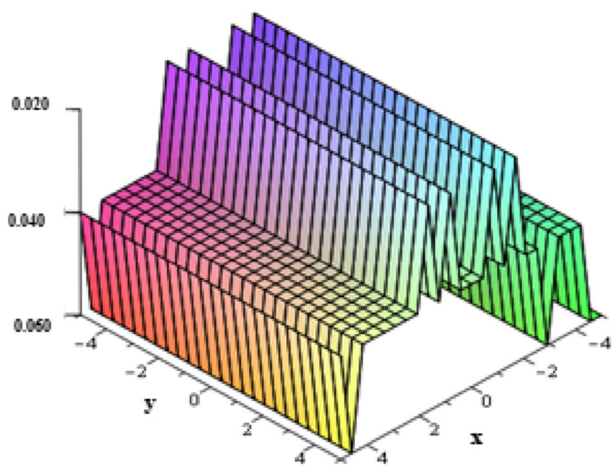

Figure 3. Graphical representation of the second solution set for $b_0 = 2, b_1 = 0.1, a_{-1} = 2, b_{-1} = 1$.

Third solution set:
Now we consider that

$$a_{-1} = \frac{a_1 b_{-1}}{b_1}, a_0 = 0, a_1 = a_1, b_{-1} = b_{-1}, b_0 = 0,$$
$$b_1 = b_1.$$

Using the above values we have the following travelling wave solution $g(\eta)$ of the given eq. (11) (figure 4):

$$g(\eta) = \frac{(a_1 b_{-1}/b_1)e^{-\eta} + a_1 e^{\eta}}{b_{-1}e^{-\eta} + b_1 e^{\eta}}.$$

Case II: If $c = p = 1$ and $q = d = 2$, then eq. (16) becomes

$$g(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{21}$$

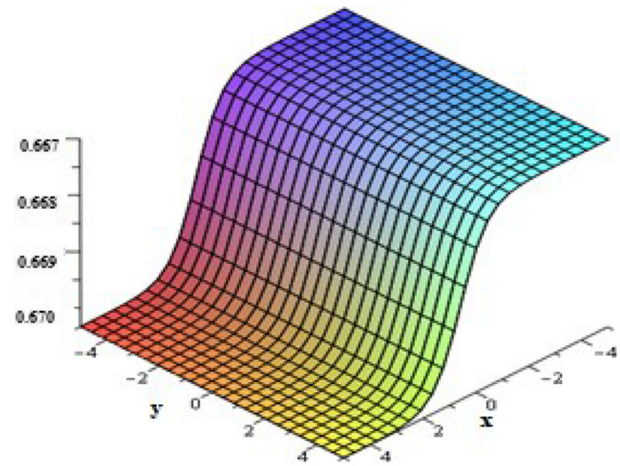

Figure 4. Graphical representation of the third solution set for $b_1 = 0.3, a_1 = 2, b_{-1} = 1$.

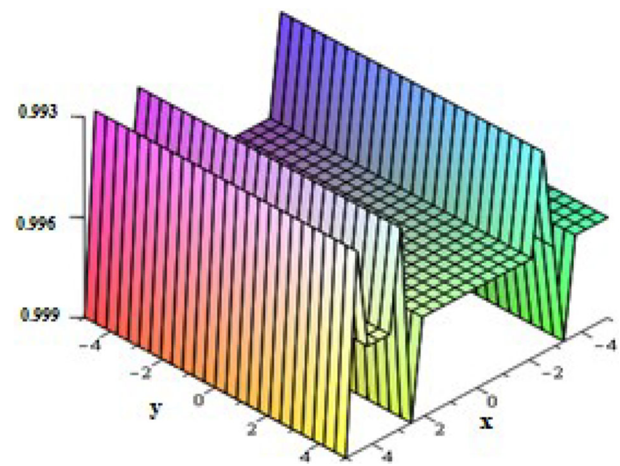

Figure 5. Graphical representation of the fourth solution set for $b_2 = 0.1, b_0 = 0.2, M = 0.3$.

Fourth solution set:
Consider

$$a_{-1} = 0, a_0 = \frac{1}{2}Mb_0 - 2b_0, a_1 = 0,$$
$$a_2 = -2b_2 + \frac{1}{2}Mb_2,$$
$$b_{-1} = 0, b_0 = b_0, b_1 = 0, b_2 = b_2.$$

Inserting the above values in the trial solution (21) we obtain the exact travelling wave solution $g(\eta)$ of eq. (11) (figure 5):

$$g(\eta) = \frac{\left(\frac{1}{2}Mb_0 - 2b_0\right) + \left(-2b_2 + \frac{1}{2}Mb_2\right)e^{2\eta}}{b_0 + b_2 e^{2\eta}}.$$

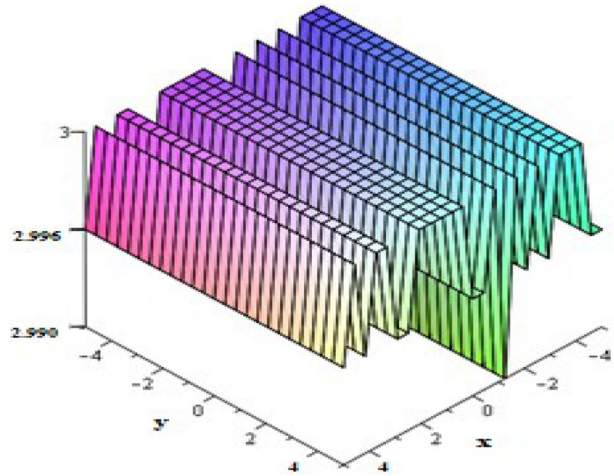

Figure 6. Graphical representation of the fifth solution set for $b_0 = 0.1, b_1 = 2, b_2 = 1, M = 4$.

Fifth solution set:
Consider the given below values, we have

$$a_{-1} = 0, a_0 = Mb_0 - b_0, a_1 = -b_1 + Mb_1,$$
$$a_2 = (M - 1)b_2, b_{-1} = 0, b_0 = b_0, b_1 = b_1, b_2 = b_2.$$

We achieved the following travelling wave solution $g(\eta)$ of eq. (11) (figure 6):

$$g(\eta) = \frac{(Mb_0 - b_0) + (Mb_1 - b_1)\,\mathrm{e}^{\eta} + (M - 1)\,b_2\mathrm{e}^{2\eta}}{b_0 + b_1\mathrm{e}^{\eta} + b_2\mathrm{e}^{2\eta}}.$$

5. Results and discussion

The soliton wave formations for a two-dimensional MHD flow model have been examined via a novel analytical technique. The findings are mentioned and discussed as follows.

A soliton is a non-trivial time-invariant solution of a field equation, which arises due to a delicate balance between the nonlinearity and dispersion of the medium. The self-steepening effects associated with nonlinearities and the spreading out of a disturbance by dispersion give rise to this steady-state pulse (wave of permanent profile). The linear description produces waves that experience dispersion and causes localised disturbance to spread. At amplitudes that are slightly nonlinear, the competition between the two effects of steepening and spreading can intuitively be expected to lead to the aforesaid balance. Depending upon the dispersion, waves of different wave numbers, speeds and amplitudes propagate without changing their shapes. Of course, strong dispersive media can produce waves of high amplitudes but weak dispersive media produce waves of sufficiently small amplitudes. Again, the solitary waves, which can preserve asymptotically its shape and velocity, can also interact and pass through one another in nearly elastic fashion which is a remarkable characteristic. By changing the values of physical and additional free parameters, the velocity and amplitude of solitary waves are controlled. Also, the amplitude is proportional to the velocity of propagation, and taller solitary waves are thinner and move faster.

The solitary wave moves towards the right direction if the velocity is positive or towards the left direction if the velocity is negative and the amplitudes and velocities are controlled by various physical parameters. The solitary waves show more complicated behaviours, which are controlled by various physical and additional free parameters. The figures indicate graphical solutions for different values of physical parameters. The graphical representations in figures 2–6 signify solitary waves for various values of physical and additional free parameters. In all the cases, it is observed that the soliton wave solutions do not strongly depend on values of additional free parameters, and we attain equivalent solitary wave solutions. The graphical outcomes visibly depict soliton waves of various types.

Solitons can be seen as describing real phenomena, such as waves in narrow channels. There are many other examples, such as solitons in fibre optics, condensed matter and so on. They are all very interesting in their own right, but in theoretical physics, solitons are so much more than just a device to describe the phenomena in classical wave theories. The soliton hypothesis in neuroscience is a model that claims to explain how the action potentials are initiated and conducted along the axons based on a thermodynamic theory of nerve pulse propagation. It proposes that the signals travel along the cell's membrane in the form of certain kinds of solitary sound (or density) pulses that can be modelled as solitons.

6. Conclusion

In this paper, we apply a new and modified technique to obtain solitary wave solutions to the MHD viscous flow over a nonlinear stretching sheet. We attain the desired soliton solutions through the exponential functions. With the help of symbolic computation softwares such as Maple, Matlab and Mathematica, finding the exact solution of nonlinear differential equations becomes very easy and convenient. The presented method also works for nonlinear differential equations without linearisation, discretisation and perturbation. The accuracy of the attained results through backward substitution into the original equation with Maple 18 is guaranteed. In short, the obtained results show that this scheme is much more operative, more competent and highly accurate for evaluating exact solution of nonlinear evolution equations. The solitary wave solutions are represented graphically and the numerical results are highly encouraging.

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