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The Weyl equation under an external electromagnetic field in the cosmic string space–time

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Abstract. In this paper we have considered a massless spinor Dirac particle in the presence of an external electromagnetic field in the cosmic string space–time. To study the Weyl equation in the cosmic string framework using the general definition of Laplacian in the curved space, elements of covariant derivative have been constructed and the Weyl equation has been rewritten in the considered framework. Then we have obtained the equation of energy eigenvalues by using the Nikiforov–Uvarov (NU) method. The wave function has been obtained in terms of Laguerre polynomials. An important result obtained is that the degeneracy of the Minkowski space spectral is broken in the transition from Minkowski to cosmic string space.

Keywords. Weyl equation; cosmic string space–time; electromagnetic field; energy eigenvalues; Nikoforov–Uvarov method.

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1. Introduction

In cosmology, topological defects arise from phase transitions in the early Universe. These include domain walls, cosmic strings and monopoles. Among them, cosmic strings have attracted much attention. In 1976, Kibble [1] proposed the existence of cosmic strings when he was working on the theory of linear vertexes in superconductors. A cosmic string has a conical singularity at the origin and its presence can influence the behaviour of a quantum mechanical system [2]. Cosmic strings appear in a number of physical effects such as γ-ray bursts, the emission of gravitational waves and the generation of high-energy cosmic rays. Also, the string-like defects appear in a number of condensed matter systems, including liquid crystals and graphene-made structures [3]. The presence of a cosmic string changes the solution and shifts the energy levels compared with the flat Minkowski space–time results. It is interesting to observe that these shifts depend on the parameter that defines the angle deficit. These shifts arise from the topological features of the space–time generated by this defect [4–6]. Recently, the Dirac and Klein–Gordon equation has been studied in a curved space and its presence destroys the degeneracy of all the energy levels [4,7]. In [4,8] the effect of topology of the space–time on the energy spectrum of the hydrogen atom is considered. The influence of curvature has been shown in the energy spectrum of electrons [9]. In recent years, the gravitational effect on quantum mechanics has been studied in many-particle systems [10–12]. Santos and Bezerra de Mello [13] investigated the influence of cosmological constant in the geometry of non-Abelian and Abelian cosmic string space–times. Beresnyak [14] derived a hydrodynamic solution for the supersonic flow of the collisional gas before the cosmic string that depends on the angle defect of the string and its speed. Mota and Bakke [15] investigated the influence of non-inertial effects on the ground-state energy of a massive scalar field in the cosmic string space–time. Muniz *et al* [16] studied the Landau levels of a charged particle placed in the space–time of a spinning cosmic string, which is straight and infinitely long. The quantum vacuum interaction energy between two straight parallel cosmic strings has been investigated by Muñoz-Castañeda and Bordag [17] and the topological defects have been studied in other works [18–22]. Kulkarni and Sharma [23] studied the exact solutions of the Dirac equation. The invariant properties of the Dirac equation with external electromagnetic field have been studied in [24]. Also, the

Dirac equation has been investigated in various space–times [25–27]. In 1950, Weyl found powerful geometric reasons which suggested that the Dirac equation must include a nonlinear term representing a spin–spin self-coupling of gravitational origin [28–30]. Weyl was the first person who investigated the Dirac equation in the presence of a massless particle and the result of this survey was the parity conservation. In the quantum field theory, the Weyl equation is used for describing massless spin-1/2 particles [22,31–33].

In this paper, we are going to consider the Weyl equation in the presence of an external electromagnetic field within the cosmic string framework. Therefore, this paper has been organised as follows: in §2 we introduce the Weyl equation and study the influence of cosmic string on it. Next, we find our solutions in terms of Laguerre polynomials. Conclusions are given in §3.

2. Solution of the Weyl equation in the presence of an external electromagnetic field

The Weyl equation is used for describing the massless spin-1/2 particles whereas the Dirac equation describes the massive spin-1/2 particles. The Weyl equation in the presence of an external electromagnetic field in a curved space–time can be written as

$$\mathrm{i}\sigma^a e^\mu_a(\nabla_\mu + \mathrm{i}eA_\mu)\psi = 0, \tag{1}$$

in which σ^a are the Pauli matrices and the covariant derivative is $\nabla_\mu = \partial_\mu + \Gamma_\mu$. When $\alpha = 1$, Γ_μ identically vanishes, and we obtain the Weyl equation in flat space–time. The geometry of a cosmic string in cylindrical coordinates is defined by the following line elements:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\phi^2 + \mathrm{d}z^2 \tag{2}$$

with $-\infty < (t, z) < \infty$ and $r \geq 0$, $0 \leq \phi \leq 2\pi$. The parameter $\alpha = 1 - 4\tilde{m}/c^2$ is the angular deficit and runs in the interval (0, 1]. $\tilde{m}$ represents the linear mass density of the cosmic string. The components of the non-coordinate basis $e^{(a)}_\mu$ are called tetrads that satisfy the property [34]

$$g^{\mu\upsilon}(x) = e^\mu_{(a)}(x) e^\upsilon_{(b)}(x) \eta^{ab} \tag{3}$$

and the tensor $\eta^{ab} = \mathrm{diag}(- + ++)$ is the Minkowski space–time metric tensor. The tetrad and its inverse are defined as

$$e^a_\mu(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\alpha r\sin\phi & 0 \\ 0 & \sin\phi & \alpha r\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e^\mu_a(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi/\alpha r & \cos\phi/\alpha r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

By substituting the tetrads in eq. (3) we have

$$g^{\mu\upsilon} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\alpha^2 r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_{\mu\upsilon} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^2 r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

$\omega^a_{\mu b}$ represents the spin connection, which is given by

$$\omega^a_{\mu b} = \eta_{\bar{a}\bar{c}} e^{\bar{c}}_\upsilon e^\sigma_{\bar{b}} \Gamma^\upsilon_{\sigma\mu} - \eta_{\bar{a}\bar{c}} e^\upsilon_{\bar{b}} \partial_\mu e^{\bar{c}}_\upsilon. \tag{6}$$

$\Gamma^\upsilon_{\sigma\mu}$ are the Christoffel symbols of the second kind and can be obtained from [35]

$$\Gamma^\mu_{ij} = \frac{1}{2} g^{\mu k}\left[\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right]. \tag{7}$$

The non-zero components of spin connections are

$$\omega^1_{\phi 2} = -\omega^2_{\phi 1} = 1 - \alpha. \tag{8}$$

Γ_μ is the spinor affine connection given by

$$\Gamma_\mu = -\frac{1}{8}\omega^a_{\mu b}[\sigma^a, \sigma^b]. \tag{9}$$

It is valid for massless spin-1/2 particles. By using Pauli matrices, the non-vanishing component of the spinorial affine connection is found to be

$$\Gamma_\phi = -\frac{1}{8}(\omega^1_{\phi 2}[\sigma^1, \sigma^2] + \omega^2_{\phi 1}[\sigma^2, \sigma^1])$$
$$= -\frac{1}{2}\omega^1_{\phi 2}\sigma^1\sigma^2, \tag{10}$$

$$\Gamma_\phi = -\mathrm{i}\frac{(1-\alpha)}{2}\sigma^z. \tag{11}$$

In eq. (1), the vector potential associated with a uniform magnetic field parallel to the string $\overrightarrow{\mathbf{B}} = B_0\hat{k}$ is considered as

$$A_\mu = (0, 0, A_\phi, 0), \quad A_\phi = \frac{rB_0}{2}, \tag{12}$$

where it has no dependence on α. So, the Weyl equation takes the form

$$\mathrm{i}\sigma^0\frac{\partial\psi}{\partial t}+\mathrm{i}\sigma^1\Bigg(\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{\alpha r}\frac{\partial}{\partial\phi}+\frac{\mathrm{i}\sin\phi}{\alpha r}\frac{\omega^1_{\phi 2}}{2}\sigma^3 -\frac{\mathrm{i}\sin\phi}{\alpha r}eA_\phi\Bigg)\psi +\mathrm{i}\sigma^2\Bigg(\sin\phi\frac{\partial}{\partial r}+\frac{\cos\phi}{\alpha r}\frac{\partial}{\partial\phi}-\frac{\mathrm{i}\cos\phi}{\alpha r}\frac{\omega^1_{\phi 2}}{2}\sigma^3 +\frac{\mathrm{i}\cos\phi}{\alpha r}eA_\phi\Bigg)\psi+\mathrm{i}\sigma^3\frac{\partial\psi}{\partial z}=0. \tag{13}$$

To solve eq. (13) we take the following form for the wave function:

$$\psi=\begin{pmatrix}\psi_1(r)\\ \psi_2(r)\end{pmatrix}=\mathrm{e}^{-\mathrm{i}Et+\mathrm{i}m\phi+\mathrm{i}kz}\begin{pmatrix}\varphi_1(r)\\ \varphi_2(r)\end{pmatrix}. \tag{14}$$

Substituting eq. (14) into eq. (13) leads to two coupled differential equations

$$(E-k)\varphi_1(r)+\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\left(\partial_r+\frac{m}{\alpha r}+\frac{1-\alpha}{2\alpha r}+\frac{eB_0}{2\alpha}\right)\times\varphi_2(r)=0, \tag{15}$$

$$(E+k)\varphi_2(r)+\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\left(\partial_r-\frac{m}{\alpha r}+\frac{1-\alpha}{2\alpha r}-\frac{eB_0}{2\alpha}\right)\times\varphi_1(r)=0. \tag{16}$$

By using eqs (15) and (16) we can write the wave function $\varphi_2(r)$ in terms of $\varphi_1(r)$ as follows:

$$\varphi_2(r)=\left[\frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}(\partial_r-(m/\alpha r)+(1-\alpha)/(2\alpha r)-(eB_0)/(2\alpha))}{E+k}\right]\varphi_1(r). \tag{17}$$

Then, we can write the system of equations in terms of $\varphi_1(r)$ as

$$\varphi_1''(r)+\left(\frac{a}{r}\right)\varphi_1'(r)+\left(\frac{1-4m^2-4\alpha+4\alpha m+3\alpha^2}{4\alpha^2 r^2}-\frac{meB_0}{\alpha^2 r}+\frac{-e^2B_0^2-4k^2\alpha^2+4E^2\alpha^2}{4\alpha^2}\right)\varphi_1(r)=0, \tag{18}$$

where $a=(1-\alpha)/\alpha$. To solve the above equation, we use the Nikiforov–Uvarov (NU) method, which has been described in detail in Appendix A and write the solution of eq. (18) as [36]

$$\frac{\mathrm{d}^2\varphi_1(r)}{\mathrm{d}r^2}+\left(\frac{a}{r}\right)\frac{\mathrm{d}\varphi_1(r)}{\mathrm{d}r}+\frac{1}{r^2}(-\xi_1 r^2+\xi_2 r-\xi_3)\varphi_1(r)=0, \tag{19}$$

where

$$\xi_1=\frac{e^2B_0^2+4k^2\alpha^2-4\alpha^2E^2}{4\alpha^2}, \tag{20}$$

$$\xi_2=\frac{-meB_0}{\alpha^2}, \tag{21}$$

$$\xi_3=\frac{-1+4m^2+4\alpha-4\alpha m-3\alpha^2}{4\alpha^2} \tag{22}$$

and

$$\alpha_1=\frac{1-\alpha}{\alpha},\quad \alpha_2=\alpha_3=\alpha_5=0,\quad \alpha_4=\frac{2\alpha-1}{2\alpha},$$
$$\alpha_6=\alpha_9=\xi_1,\quad \alpha_7=-\xi_2,$$
$$\alpha_8=\frac{(2\alpha-1)^2}{4\alpha^2}+\xi_3,\quad \alpha_{10}=1+2\sqrt{\frac{(2\alpha-1)^2}{4\alpha^2}+\xi_3},$$
$$\alpha_{11}=2\sqrt{\xi_1},$$
$$\alpha_{12}=\frac{2\alpha-1}{2\alpha}+\sqrt{\frac{(2\alpha-1)^2}{4\alpha^2}+\xi_3},$$
$$\alpha_{13}=-\sqrt{\xi_1}. \tag{23}$$

From the following energy eigenvalue equation

$$(2n+1)\sqrt{\xi_1}-\xi_2+2\sqrt{\left(\frac{(2\alpha-1)^2}{4\alpha^2}+\xi_3\right)\xi_1}=0 \tag{24}$$

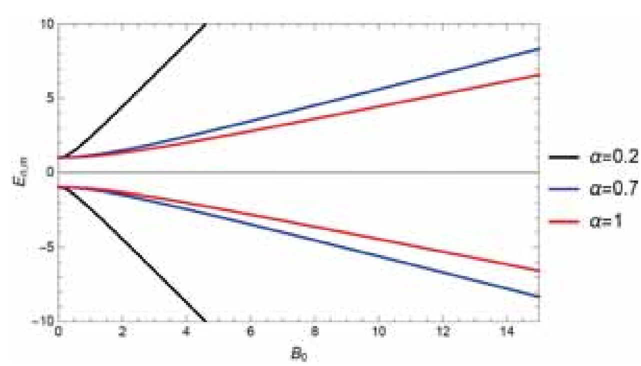

Figure 1. The energy spectrum vs. B_0 for $n=2$ with $e=k=1$, $m=1$ for different α.

the energy spectrum can be obtained as

$$E_{\pm,n,m} = \pm\sqrt{k^2 + \frac{B_0^2 e^2(1-[4m^2/(2+2n-(2m/\alpha))^2])}{4\alpha^2}} \tag{25}$$

and for $\alpha = 1$, we have

$$E_{\pm,n,m} = \pm\sqrt{k^2 + \frac{B_0^2 e^2(1-[4m^2/(2+2n-2m)^2])}{4}}. \tag{26}$$

The wave function is obtained as

$$\varphi_1(r) = r^{([(2\alpha-1)/2\alpha]+\sqrt{[(2\alpha-1)^2/4\alpha^2]+\xi_3})} \times \mathrm{e}^{-\sqrt{\xi_1}r} L_n^{2\sqrt{[(2\alpha-1)^2/4\alpha^2]+\xi_3}}(2\sqrt{\xi_1}r) \tag{27}$$

and by using eq. (17) we obtain

$$\varphi_2(r) = \left[\frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}(\partial_r-(m/\alpha r)+[(1-\alpha)/2\alpha r]-(eB_0/2\alpha))}{E+k}\right] \times\left(r^{([(2\alpha-1)/2\alpha]+\sqrt{[(2\alpha-1)^2/4\alpha^2]+\xi_3})} \times\mathrm{e}^{-\sqrt{\xi_1}r} L_n^{2\sqrt{[(2\alpha-1)^2/4\alpha^2]+\xi_3}}(2\sqrt{\xi_1}r)\right), \tag{28}$$

where the wave function is obtained in terms of Laguerre polynomials L_n.

In figure 1, we observe energy decreases by increasing α and decreasing B_0. The energy spectrum as a functionof B_0 for flat space–time ($\alpha = 1$) is shown in figure 2 for different n that breaks the degeneracy of energy levels. The energy spectrum as a function of α for different B_0 is shown in figure 3. The density of probability $|\Psi(r)|^2$ vs. r for different values of n is shown in figure 4.

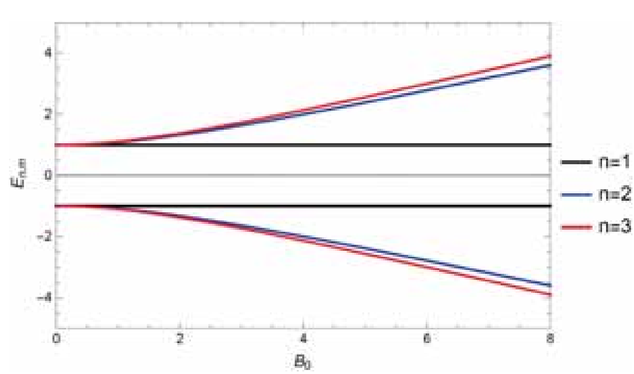

Figure 2. The energy spectrum vs. B_0 for $\alpha = 1$ with $e = k = 1$, $m = 1$ for different n.

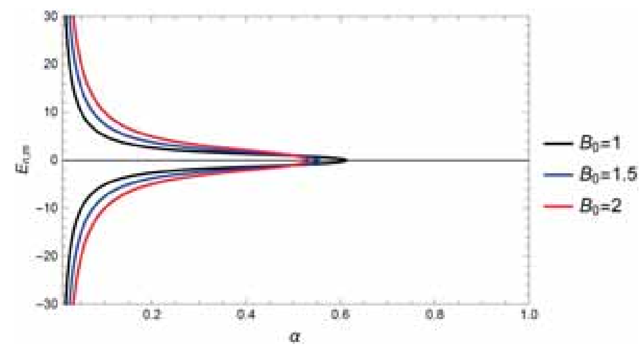

Figure 3. The energy spectrum vs. α for $n = 1$ with $e = k = 1$, $m = 2$ for different B_0.

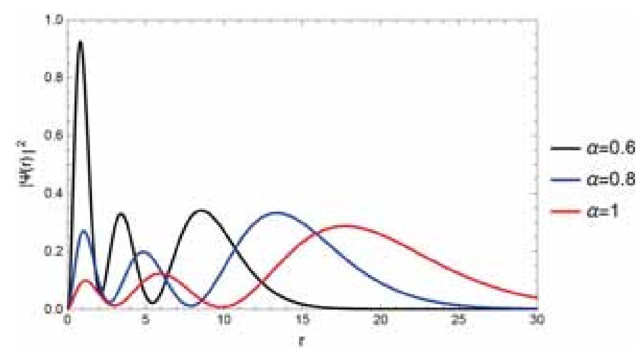

Figure 4. $|\Psi(r)|^2$ vs. r for $n = 2$ with $B_0 = e = k = 1$, $m = 1$ for different α.

3. Conclusion

In this work, we have considered spin-1/2 particles in the presence of gravitational fields of a cosmic string. We obtained the solution of the Weyl equation in a curved space in the cylindrical coordinate. After the determination of elements of covariant derivative and using an ansatz form for the wave function, a system of coupled differential equations were derived. The coupled system of differential equations are decoupled and an explicit differential equation for one of the components of the wave function was found. We have obtained the energy equation and the wave function using the terms of Laguerre polynomials. An important result that we have shown is that the degeneracy of the Minkowski space spectral is broken in the transition from Minkowski to cosmic string space.

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Appendix A. The NU method

We consider the following differential equation [37]:

$$\frac{d^2\Psi(r)}{dr^2}+\frac{\alpha_1-\alpha_2 r}{r(1-\alpha_3 r)}\frac{d\Psi(r)}{dr}+\frac{1}{r^2(1-\alpha_3 r)^2}\times\{-\xi_1 r^2+\xi_2 r-\xi_3\}\Psi(r)=0. \tag{A1}$$

According to the NU method, the equations of energy eigenvalues and eigenfunctions, respectively, are obtained from

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \tag{A2}$$

$$\Psi(r)=r^{\alpha_{12}}(1-\alpha_3 r)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}\times P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 r), \tag{A3}$$

where

$$\alpha_4=\frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),$$
$$\alpha_6=\alpha_5^2+\xi_1,\quad \alpha_7=2\,\alpha_4\,\alpha_5\,-\xi_2,$$
$$\alpha_8=\alpha_4^2+\xi_3,\quad \alpha_9=\alpha_3\,\alpha_7\,+\,\alpha_3^2\,\alpha_8\,+\,\alpha_6,$$
$$\alpha_{10}=\alpha_1\,+\,2\,\alpha_4+2\sqrt{\alpha_8},$$
$$\alpha_{11}=\alpha_2\,-\,2\,\alpha_5+2(\sqrt{\alpha_9}\,+\,\alpha_3\,\sqrt{\alpha_8}),$$
$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\quad \alpha_{13}=\,\alpha_5\,-\,(\sqrt{\alpha_9}\,+\,\alpha_3\,\sqrt{\alpha_8}). \tag{A4}$$

In the special case of $\alpha_3=0$ [38]

$$\lim_{\alpha_3\to 0}P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-\alpha_3 r)=L_n^{\alpha_{10}-1}(\alpha_{11}r), \tag{A5}$$

$$\lim_{\alpha_3\to 0}(1-\alpha_3 r)^{-\alpha_{12}}-\frac{\alpha_{13}}{\alpha_3}=e^{\alpha_{13}r} \tag{A6}$$

and

$$\Psi_n(r)=r^{\alpha_{12}}e^{\alpha_{13}r}L_n^{\alpha_{10}-1}(\alpha_{11}r). \tag{A7}$$

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