J. Astrophys. Astr. (2019) 40:19
https://doi.org/10.1007/s12036-019-9588-z

First law of black hole thermodynamics for the Kerr black hole using foliation

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MS received 1 December 2018; accepted 25 April 2019; published online 4 June 2019

Abstract. This article follows a simpler approach (Siddiqui *et al.* 2011, Chin. Phys. Lett. 28, 050401) using foliation's concept with virtual displacement of hyper-surfaces near the event horizon to derive the first law of black hole thermodynamics for the Kerr black hole spacetime.

Keywords. Foliation—first law of black hole thermodynamics—Kerr black hole.

1. Introduction

A connection to thermodynamics and black hole begin from an argument given by Penrose, revealing that we can reverse the entropy increase by using black holes. Penrose and Floyd (1971) constructed a mechanism for extracting energy from the rotating black holes. At that time when Penrose presented the argument, it seemed unbelievable to extract energy from a black hole. However, sooner it is realized that the Kerr spacetime stored energy in its ergo-sphere. Thus no energy in real goes out from the black hole, and it appeared to lose the mass on account of some supposed mass being stored as an energy outside a black hole. Christodoulou and Ruffini (1971) worked out that due to an irreducible mass in the black hole, it had an extra energy of electromagnetic nature stored in it. Thus 'black hole thermodynamics' is an electrifying developing branch of black hole physics. The laws of energy conservation provide solid footing to the subject. Historically, it makes a clear connection among heat conduction and chemical transformation. It also explains conversion between heat and mechanical energy. Later, Maxwell and others gave a better comprehension related to macroscopic variables as a driving force for microscopic mechanical variables. Classically, thermodynamics requires equilibrium state in a system. Hence, the laws laid an axiomatic basement to thermodynamics and are acknowledged after a vigilant investigation. These laws are a comprehensive set of appropriate axioms, and from there the remaining thermodynamics can be obtained. The laws by Bekenstein (1972) and Smarr (1973) enlightened temperature equality as the zeroth law, energy preservation as the first law, entropy affinities as the second law and state for an absence of temperature as the third law of thermodynamics. The reinforcement of these laws took place after the innovation of Hawking (1975) radiation.

The fruitful relation among Einstein field equations and the first law for black hole thermodynamics was first explored by Jacobson (1995) and thereafter by Padmanabhan (2002a, b) who made a formalism to understand the thermodynamics of horizons in space-times. He showed that we can write Einstein equations in the form of the first law of thermodynamics

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V. \tag{1}$$

After that there was a stream of work in that direction for different black hole geometries (Ashtekar 2002; Padmanabhan 2002a, b; Paranjape *et al.* 2006; Akbar & Cai 2006, 2007; Eling *et al.* 2006; Akbar 2007; Akbar & Siddiqui 2007; Cai & Cao 2007; Ali 2014; Ali *et al.* 2015; Gangopadhyay & Dutta 2016; Kim *et al.* 2016; Rudra *et al.* 2016; Heydarzade *et al.* 2017).

1.1 *Foliation and black hole spacetime*

The word *foliation* has been originated from the Latin word *folia* meaning *leaf*. The slicing idea was developed from differential equation theory where the paths of the solution's space were considered to be the leaves of the foliation in the seventeenth century. Poincare, in the late nineteenth century, developed techniques to explore global and qualitative features of dynamic systems in cases where the explicit methods of solutions failed. He discovered this during the research on geometry of solution space having complex phenomena of dynamical model. He strongly emphasized on the topological methods and on the phenomena of qualitative nature that led to foliation. To foliate an n-dimensional manifold M, it is decomposed into sub manifolds, having the same dimension p. The sub manifolds are considered to be the leaves of the foliation. The co-dimension s of foliation is defined as $s = n - p$. A foliation of co-dimension one is called a foliation by *hypersurfaces*. The pioneers of foliation theory were Reeb (1952) and Ehresmann (1961), the former who coined the term 'foliation'. Some physically suitable solutions to Einstein's field equations are singular and represent black hole space-time (Stephani *et al.* 2002). These space-times have special importance due to the horizons in their geometries. For the analysis of the dynamics of these types of geometries, one have to foliate the space-time by null or space-like hypersurfaces (York 1972; Estabrook *et al.* 1973; Smarr & York 1978; Eardley & Smarr 1978; Marsden & Tipler 1980; Iriondo *et al.* 1996; Beig & Nurchadha 1998; Guven & Murchadha 1999; Hussain *et al.* 2002; Qadir & Siddiqui 2002, 2006; Beig & Siddiqui 2007).

1.2 *Foliation and first law of black hole thermodynamics*

Siddiqui *et al.* (2011) used the concept of foliation and presented a simple and elegant way of obtaining the first law of black hole thermodynamics for the Schwarzschild and Reissner–Nordstrom space-times. The foliations of these space-times were used in such a way that the horizon corresponds to a particular leaf (hypersurface) of foliation. Then the field equations were worked out for the induced metric of the hypersurfaces and they showed that the field equations for the induced metric at the horizon can be expressed as the first law of black hole thermodynamics. An important aspect of this approach is that one has to essentially deal with $(n-1)$-dimensional induced metric for an n-dimensional spacetime, which significantly simplifies the calculations to obtain such results. Thus manipulation in lower dimensional gravity helps and is used as an arena for investigating various problems that arise in higher dimensions but are not solvable there. Those that have been investigated include quantum gravity in lower dimensions (Ashtekar *et al.* 1989) and black hole evaporation in two dimensions (Mann & Mclenagham 1994). Also, black hole solutions of the Einstein field equations in lower dimension share many important features with higher dimensional black holes. Both have an event horizon. They occur as an endpoint of gravitational collapse. Both show mass inflation and have non-vanishing Hawking temperature and interesting thermodynamic features (Banados *et al.* 1992; Akbar & Siddiqui 2007). The space-time in lower dimensions provide a simple toy model for a number of studies including super-string and supergravity theories. Another important aspect of lower dimension space-time is that it significantly simplifies the calculations in numerical relativity. The reason for the simplicity of lower dimension lies in the fact that in higher dimension, e.g., (3 + 1) spacetime, the curvature tensor decomposes into a curvature scalar, R, a Ricci tensor, $R_{\mu\nu}$ and a remaining trace-free Weyl tensor, $C^{\sigma}{}_{\mu\nu\rho}$, whereas in lower dimension, e.g., (2 + 1) dimension, the Weyl tensor vanishes identically and the full curvature tensor is determined by the Ricci tensor and its trace

$$R_{\mu\nu\rho\delta} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \tag{2}$$

Thus the fundamental difference between lower- and higher-dimensional gravities, i.e., (2 + 1)- and (3 + 1)-dimensional space-times, originates in the fact that the curvature tensor in the (2 + 1) dimension depends linearly on the Ricci tensor. Therefore, the structure of the (2 + 1) dimensional gravity is simple enough to allow a number of exact computations that are impractical in the (3 + 1) dimension (Carlip 1995). In the next section, we follow a formulation presented by Siddiqui *et al.* (2011) for the Schwarzschild and Reissner–Nordstrom black holes to derive the first law of thermodynamics of a Kerr black hole space-time. Section 3 contains conclusion and discussion.

2. First law of black hole thermodynamics for the Kerr space-time

Consider Einstein's field equations for (3 + 1)-dimensional space-time (in gravitational units $G = 1$)

$$G_{ab} = -8\pi T_{ab} \quad (a, b = 0, 1, 2, 3), \tag{3}$$

where G_{ab} and T_{ab} are Einstein's tensor and stress–energy tensor respectively.

The vaccum solution of the field equations for axis-symmetric gravitational field due to mass M is called the Kerr black hole given (in gravitational units $G = c = 1$) by the metric

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta^2}{\rho^2}(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi)^2 + \frac{\rho^2}{\Delta^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}(a\mathrm{d}t - (r^2 + a^2)\mathrm{d}\phi)^2, \end{aligned} \tag{4}$$

where $\Delta^2 = (r^2 + a^2) - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$.

Consider r = constant = k. We get the induced metric as

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta^2}{\rho^2}(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi)^2 + \rho^2\mathrm{d}\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}(a\mathrm{d}t - (k^2 + a^2)\mathrm{d}\phi)^2, \end{aligned} \tag{5}$$

where $\Delta^2 = (k^2 + a^2) - 2Mk$ and $\rho^2 = k^2 + a^2\cos^2\theta$.

We discuss the thermodynamic properties of the Kerr black hole at the event horizon. The Kerr space-time is associated with temperature T, entropy S and angular velocity Ω as

$$T = \frac{K}{2\pi} = \frac{M(k^2 - a^2)}{2\pi(k^2 + a^2)^2}, \tag{6}$$

$$S = \pi(k^2 + a^2), \tag{7}$$

$$\Omega = \frac{a}{k^2 + a^2}. \tag{8}$$

The mass M is given by

$$M = \frac{(k^2 + a^2)}{2k}$$

and hence

$$\mathrm{d}M = \frac{k^2 - a^2}{2k^2}\mathrm{d}k. \tag{9}$$

Since

$$J = aM,$$

so

$$\mathrm{d}J = a\mathrm{d}M = \frac{a(k^2 - a^2)}{2k^2}\mathrm{d}k \tag{10}$$

and also,

$$\begin{aligned} T\mathrm{d}S &= \frac{M(k^2 - a^2)}{2\pi(k^2 + a^2)^2}2\pi k = \frac{Mk(k^2 - a^2)}{(k^2 + a^2)^2}\mathrm{d}k \\ &= -\frac{(a^2 - k^2)}{2(k^2 + a^2)}\mathrm{d}k \end{aligned} \tag{11}$$

and

$$\Omega \mathrm{d}J = \frac{a}{k^2 + a^2}\mathrm{d}J = \frac{a^2(k^2 - a^2)}{2k^2(k^2 + a^2)}\mathrm{d}k. \tag{12}$$

The (0, 0)-component of the field equation for the induced metric (5) can be written in the form

$$\frac{2Mk - k^2}{(k^2 + a^2)^2} + \frac{4Mka^2}{(k^2 + a^2)^3} - \frac{12M^2k^2a^2}{(k^2 + a^2)^4} = 0.$$

Multiplying by $\frac{(k^2+a^2)^2}{2k^2}$, we get

$$\frac{2Mk - k^2}{2k^2} + \frac{4Mka^2}{2k^2(k^2 + a^2)} - \frac{12M^2k^2a^2}{2k^2(k^2 + a^2)^2} = 0.$$

By subtracting and adding $\frac{Mk(k^2-a^2)}{(k^2+a^2)^2}$, we get

$$\begin{aligned} &\frac{2Mk - k^2}{2k^2} + \frac{4Mka^2}{2k^2(k^2 + a^2)} - \frac{12M^2k^2a^2}{2k^2(k^2 + a^2)^2} \\ &-\frac{Mk(k^2 - a^2)}{(k^2 + a^2)^2} + \frac{Mk(k^2 - a^2)}{(k^2 + a^2)^2} = 0, \end{aligned} \tag{13}$$

by considering a virtual displacement dk of the horizon and then by multiplying it on both sides of the above equation. Using (11), we get

$$\begin{aligned} &\frac{2Mk - k^2}{2k^2}\mathrm{d}k + \frac{4Mka^2}{2k^2(k^2 + a^2)}\mathrm{d}k - \frac{12M^2k^2a^2}{2k^2(k^2 + a^2)^2}\mathrm{d}k \\ &-T\mathrm{d}S + \frac{Mk(k^2 - a^2)}{(k^2 + a^2)^2}\mathrm{d}k = 0. \end{aligned} \tag{14}$$

Then using $2Mk = k^2 + a^2$ in the first and fifth term, we get

$$\begin{aligned} &\frac{(k^2 + a^2) - k^2}{2k^2}\mathrm{d}k + \frac{4Mka^2}{2k^2(k^2 + a^2)}\mathrm{d}k \\ &-\frac{12M^2k^2a^2}{2k^2(k^2 + a^2)^2}\mathrm{d}k - T\mathrm{d}S + \frac{(k^2 - a^2)}{2(k^2 + a^2)}\mathrm{d}k = 0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\frac{k^2}{2k^2}\mathrm{d}k + \frac{a^2 - k^2}{2k^2}\mathrm{d}k + \frac{4Mka^2}{2k^2(k^2 + a^2)}\mathrm{d}k \\ &-\frac{12M^2k^2a^2}{2k^2(k^2 + a^2)^2}\mathrm{d}k - T\mathrm{d}S + \frac{(k^2 - a^2)}{2(k^2 + a^2)}\mathrm{d}k = 0. \end{aligned} \tag{16}$$

Rearranging the terms, we get

$$\frac{(k^2-a^2)}{2(k^2+a^2)}\mathrm{d}k+\frac{a^2-k^2}{2k^2}\mathrm{d}k+\frac{k^2}{2k^2}\mathrm{d}k + \frac{4Mka^2}{2k^2(k^2+a^2)}\mathrm{d}k-\frac{12M^2k^2a^2}{2k^2(k^2+a^2)^2}\mathrm{d}k-T\mathrm{d}S=0. \tag{17}$$

Simplifying the first and second terms, and using $2Mk = k^2 + a^2$ in the fourth and fifth terms, we get

$$\frac{-a^2(k^2-a^2)}{2k^2(k^2+a^2)}\mathrm{d}k+\frac{k^2}{2k^2}\mathrm{d}k + \left(\frac{2a^2}{2k^2}-\frac{3a^2}{2k^2}\right)\mathrm{d}k-T\mathrm{d}S=0.$$

Again simplifying, we get

$$\frac{-a^2(k^2-a^2)}{2k^2(k^2+a^2)}\mathrm{d}k+\frac{k^2}{2k^2}\mathrm{d}k-\frac{a^2}{2k^2}\mathrm{d}k-T\mathrm{d}S=0.$$

Combining the second and third terms, we get

$$\frac{-a^2(k^2-a^2)}{2k^2(k^2+a^2)}\mathrm{d}k+\frac{k^2-a^2}{2k^2}\mathrm{d}k-T\mathrm{d}S=0.$$

Using Equations (9) and (12), we get

$$-\Omega\mathrm{d}J+\mathrm{d}M-T\mathrm{d}S=0.$$

Finally, we get

$$\mathrm{d}M=T\mathrm{d}S+\Omega\mathrm{d}J.$$

Hence the first law of thermodynamics holds at horizons. Therefore, foliation of Kerr black hole behaves like a thermal system satisfying the first law of thermodynamics.

3. Conclusion and discussion

In this article, we follow an elegant and simple way of obtaining the first law of black hole thermodynamics, using the concept of foliation for the Kerr black hole space-time. Instead of obtaining field equations of black hole space-time and analysing thermal interpretation at the horizon, here the main idea is to consider a foliation so that the horizon corresponds to a particular hypersurface. Then we work out the field equations for the induced metric of the hypersurfaces and obtain the earlier thermal analysis with very simple manipulations. It will allow us to work on higher dimensional gravities in future, to study different aspects of spacetime without massive calculations. It will be interesting to extend this approach to the Kerr–Newmann black hole and also to other black hole geometries.

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