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PHASELESS INVERSE PROBLEMS THAT USE WAVE INTERFERENCE

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Abstract: We consider the inverse problems for differential equations with complex-valued solutions in which the modulus of a solution to the direct problem on some special sets is a given information in order to determine coefficients of this equation; the phase of this solution is assumed unknown. Earlier, in similar problems the modulus of the part of a solution that corresponds to the field scattered on inhomogeneities in a wide range of frequencies was assumed given. The study of high-frequency asymptotics of this field allows us to extract from this information some geometric characteristics of an unknown coefficient (integrals over straight lines in the problems of recovering the potential and Riemannian distances between the boundary points in the problem of the refraction index recovering). But this is physically much more difficult to measure the modulus of a scattered field than that of the full field. In this connection the question arises how to state inverse problems with the full-field measurements as a useful information. The present article is devoted to the study of this question. We propose to take two plane waves moving in opposite directions as an initiating field and to measure the modulus of a full-field solution relating to interference of the incident waves. We consider also the problems of recovering the potential for the Schrödinger equation and the permittivity coefficient of the Maxwell system of equations corresponding to time-periodic electromagnetic oscillations. For these problems we establish uniqueness theorems for solutions. The problems are reduced to solving some well-known problems.

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§ 1. Statement of the Problem

The study of inverse problems for differential equations with complex-valued solutions whose coefficients are determined with the use of the modulus of a solution to some direct problem on some special sets began comparatively recently. For the first time, this problem was proposed for the Schrödinger equation in Chadan–Sabatier's monograph [1] 40 years ago. The authors explain the necessity of the study of the phaseless inverse problem of recovering the potential in the Schrödinger equation by the fact that physical experiments on accelerators often occur in high energy conditions and thus we cannot really measure the phase of a strongly oscillating field in contrast of the modulus of this field. Newton in his book [2] pointed out that this problem is of a great importance. A possibility of finding the phase by using the modulus of a field in this problem was established in the articles by Klibanov [3–5]. But these articles do not contain any constructive ideas of determining the phase. The first results on the study of the phaseless inverse problem for the Schrödinger equation were obtained in the articles by Klibanov and Romanov [6, 7] and Novikov [8, 9]. In [6, 7] the problem of the potential recovering is reduced to the classical tomography problem of constructing a function from the values of its integrals over straight lines. Some method for the phase construction is proposed [8, 9] that is based on auxiliary measurements in a medium with additional given potentials. Next, similar statements of phaseless inverse problems are studied for the generalized Helmholtz equation in [10–14]. In these articles the problem of determining the refraction coefficient (the quantity inverse to the speed of signals) is considered with the use of

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measurements of the modulus of a solution to the direct problem for high frequencies, where the field is initialized either by the pointwise sources or incident plane waves from infinity. It is demonstrated that the initial problem is reduced to the celebrated inverse kinematic problem of recovering the speed of signals inside a compact domain with the use of traveling time of waves between the boundary points.

More general as compared with the Helmholtz equation is the inverse problem for the system of equations of electrodynamics relating to time-periodic electromagnetic oscillations. The problem is to define the permittivity coefficient modulo the vectors of electric or magnetic intensity measured under high frequencies. It is established in [15, 16] that the phaseless problem of recovering the permittivity coefficient can be also reduced to solving the inverse kinematic problem.

The characteristic peculiarity of the phaseless inverse problems, considered in [6, 7, 10–16], is that the unknown coefficient is assumed a known constant beyond some given compact domain Ω with a smooth boundary S and the modulus of a scattered field initiated by either pointwise sources localized on S or incident plane waves from infinity is given on S. By definition, a scattered field is the difference between the full field in an inhomogeneous medium and the field relating to a homogeneous medium. Namely, specifying a scattered field allows us to reduce the initial problem to the well-known tomography problem and the problem of inverse kinematics. But to measure a scattered field is rather difficult, it is simpler to measure the modulus of a full field. One of such statements with the modulus of a full field given for the Helmholtz equation is considered in [12] (see problem PISP 2) in a linear approximation. In the exact statement the former leads to a strongly nonlinear problem that is not studied at all.

In this article we propose the "correct" statement of the direct problem such that the phaseless inverse problem with the modulus of the full field measured possesses the same properties as that with the scattered field; namely, it admits reduction to the same well-known problems. We restrict exposition to the case of a field initiated by plane incident waves from infinity. In what follows, $\Omega = \{x \in \mathbb{R}^3 \mid |x| < R\}$ is the ball of radius R and $S = \{x \in \mathbb{R}^3 \mid |x| = R\}$ is the boundary of Ω. For definiteness, we consider the Schrödinger equation $(\Delta + k^2 - q(x))u = 0$. In the above-cited articles connected with the Schrödinger equation and the initiating plane wave $u_0(x, k, \nu) = \exp(ikx \cdot \nu)$, where $\nu \in \mathbb{S}^2$ is a unit vector, the modulus of a scattered field $u_{sc} = u - u_0$ is given on the set $S_+(\nu) = \{x \in S \mid x \cdot \nu > 0\}$ as a function of x and the parameters k and ν. Instead of this information, we propose to consider two plane incident waves with directions ν and $-\nu$ and to measure the modulus of a field arising as the result of the wave interference on $S_+(\nu)$; i.e., the modulus of the sum $u(x, k, \nu) + u(x, k, -\nu)$ for all k and $\nu \in \mathbb{S}^2$. It turned out that this information about the field leads to the same results as specification of the modulus of a scattered field.

If for some reasons it is not convenient for a given plane reference wave initiating oscillations with the direction $\nu \in \mathbb{S}^2$ to employ the wave with the direction $-\nu$ then for a given ν we can use alternative auxiliary initiating plane waves $e^{ikx\cdot\beta^j}$ with the unit directions $\beta^j = \beta^j(\nu) \in \mathbb{S}^2$, $j = 1, 2. \dots, m$, which satisfy the two conditions:

$$-\nu \cdot \beta^j > 0,\ j = 1, 2, \dots, m; \quad S_+(\nu) \subset \bigcup_{j=1}^{m} S_+(-\beta^j). \tag{1.1}$$

In this case for a given j, the modulus of the sum of two waves arising as a result of interference of the incident waves from infinity with the directions ν and β^j is defined on $S_+(-\beta^j)$. The information obtained as a result is sufficient for unique recovery of the coefficient in question.

In Section 2 we consider the phaseless inverse problem of recovering the potential in the Schrödinger equation relying on the idea of the interference of two waves. The statement of the problem is given for the cases of the wave interference initiated by the plane waves $e^{ikx\cdot\nu}$ and $e^{-ikx\cdot\nu}$ or $e^{ikx\cdot\nu}$ and $e^{ikx\cdot\beta^j}$, $j = 1, 2. \dots, m$, incident from infinity. In Section 3 we consider the problem of recovering the permittivity coefficient in the Maxwell system of equations based on the measurements of the sum of two opposite electromagnetic waves incident from infinity. Some auxiliary material of use in the study of the last problem is exposed in Section 4.

§ 2. The Schrödinger Equation

Consider the equation

$$(\Delta + k^2 - q(x))u = 0, \quad x \in \mathbb{R}^3, \tag{2.1}$$

in which k is an oscillation frequency, $k > 0$, and $q(x)$ is a potential. Let Ω be a ball of radius R centered at the origin, $\Omega = \{x \in \mathbb{R}^3 \mid |x| < R\}$. The boundary of Ω is denoted by S. Assume that the potential $q(x)$ satisfies the conditions

$$q(x) \geq 0, \quad q(x) \in C^4(\mathbb{R}^3), \quad \operatorname{supp} q(x) \subset \Omega. \tag{2.2}$$

Put

$$u_0(x, k, \nu) = e^{ikx\cdot\nu}, \quad \nu \in \mathbb{S}^2. \tag{2.3}$$

The function $u_0(x, k, \nu)$ satisfies (2.1) for $q(x) = 0$. We require that the function $u_{sc}(x, k, \nu) = u(x, k, \nu) - u_0(x, k, \nu)$ satisfies the radiation conditions

$$u_{sc}(x, k, \nu) = O(r^{-1}), \quad \frac{\partial u_{sc}}{\partial r} - iku_{sc} = o(r^{-1}) \quad \text{as} \quad r = |x - y| \to \infty. \tag{2.4}$$

In this case $u_{sc}(x, k, \nu)$ describes the wave scattered on the potential $q(x)$ presenting a solution to the equation

$$(\Delta + k^2 - q(x))u_{sc} = q(x)u_0(x, k, \nu), \quad x \in \mathbb{R}^3, \tag{2.5}$$

and the function $u(x, k, \nu) = u_0(x, k, \nu) + u_{sc}(x, k, \nu)$ is a solution to (2.1) initiated by the plane wave $u_0(x, k, \nu)$ incident from infinity. It is established in [7] that under conditions (2.2) the function $u_{sc}(x, k, \nu)$ as $k \to \infty$ has the following asymptotics:

$$u_{sc}(x, k, \nu) = -\frac{ie^{ik(x\cdot\nu)}}{2k} \int_0^\infty q(x - s\nu)\, ds + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty. \tag{2.6}$$

Let k_0 be a positive number. Below we consider two statements of the inverse problem of recovering the potential.

Problem 1. *Let the function*

$$f(x, k, \nu) = |u(x, k, \nu) + u(x, k, -\nu)|^2, \quad x \in S_+(\nu),\ \nu \in \mathbb{S}^2,\ k \geq k_0, \tag{2.7}$$

be defined on $S_+(\nu) = \{x \in S \mid x \cdot \nu > 0\}$. *Given* $f(x, k, \nu)$, *find* $q(x)$.

Problem 2. *Assume that some system of the unit vectors* $\beta^j = \beta^j(\nu)$, $j = 1, 2, \dots, m$, *meets conditions* (1.1), *the functions*

$$g_j(x, k, \nu) = |u(x, k, \nu) + u(x, k, \beta^j)|^2, \tag{2.8}$$
$$x \in S_+(-\beta^j), \quad \nu \in \mathbb{S}^2, \quad k \geq k_0, \quad j = 1, 2, \dots, m,$$

are defined on $S_+(-\beta^j) = \{x \in S \mid -x \cdot \beta^j > 0\}$. *Find* $q(x)$ *using the functions* $g_j(x, k, \nu)$, $j = 1, 2, \dots, m$.

Theorem 1. *Each of Problems* 1 *and* 2 *has at most one solution. Moreover, each of these problems is reduced to the tomography problem: Find a function* $q(x)$ *inside* Ω *given the integrals*

$$\int_0^\infty q(x - s\nu)\, ds = h(x, \nu), \quad x \in S_+(\nu),\ \nu \in \mathbb{S}^2. \tag{2.9}$$

PROOF. First, we consider Problem 1. In accord with (2.6) and the inclusion of the support of $q(x)$ in Ω, we infer

$$u(x,k,-\nu) = e^{-ikx\cdot\nu} + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty,\ x \in S_+(\nu),\ \nu \in \mathbb{S}^2.$$

Hence,

$$f(x,k,\nu) = |u(x,k,\nu) + u(x,k,-\nu)|^2$$
$$= \left| e^{ikx\cdot\nu} - \frac{ie^{ikx\cdot\nu)}}{2k} \int\limits_0^\infty q(x-s\nu)\,ds + e^{-ikx\cdot\nu} \right|^2 + O\left(\frac{1}{k^2}\right),$$
$$x \in S_+(\nu),\ \nu \in \mathbb{S}^2,\ k \geq k_0.$$

Elementary calculations yield

$$f(x,k,\nu) = 2 + 2\cos(2kx\cdot\nu) + \frac{\sin(2kx\cdot\nu)}{k} \int\limits_0^\infty q(x-s\nu)\,ds + O\left(\frac{1}{k^2}\right),$$
$$x \in S_+(\nu), \quad \nu \in \mathbb{S}^2, \quad k \geq k_0.$$

Fix $x \in S_+(\nu)$ and $\nu \in \mathbb{S}^2$ in this equality and put

$$k = k_l(x.\nu) = [\pi/2 + 2l\pi]/(2x\cdot\nu),$$

where l is a positive integer, $l \in \mathbb{N}$. In this case $\cos(2kx\cdot\nu) = 0$ and $\sin(2kx\cdot\nu) = 1$. Putting

$$h(x,\nu) = \lim_{l\to\infty} k_l[f(x,k_l,\nu) - 2], \quad x \in S_+(\nu),\ \nu \in \mathbb{S}^2,$$

we arrive at (2.9).

In Problem 2, for all $\nu \in \mathbb{S}^2$ and unit vectors $\beta^j = \beta^j(\nu)$, $j = 1, 2, \ldots, m$, satisfying (1.1), we have

$$u(x,k,\beta^j) = e^{ikx\cdot\beta^j} + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty,\ x \in S_+(-\beta^j),\ \beta^j \in \mathbb{S}^2,\ j = 1, 2, \ldots, m.$$

Hence,

$$g_j(x,k,\nu) = |u(x,k,\nu) + u(x,k,\beta^j)|^2$$
$$= \left| e^{ikx\cdot\nu} - \frac{ie^{ikx\cdot\nu}}{2k} \int\limits_0^\infty q(x-s\nu)\,ds + e^{ikx\cdot\beta^j} \right|^2 + O\left(\frac{1}{k^2}\right),$$
$$x \in S_+(-\beta^j), \quad \beta^j \in \mathbb{S}^2, \quad k \geq k_0, \quad j = 1, 2, \ldots, m.$$

In this case

$$g_j(x,k,\nu) = 2 + 2\cos x[k(x\cdot\nu - x\cdot\beta^j)]$$
$$+ \frac{\sin[k(x\cdot\nu - x\cdot\beta^j)]}{k} \int\limits_0^\infty q(x-s\nu)\,ds + O\left(\frac{1}{k^2}\right),$$
$$x \in S_+(-\beta^j), \quad \beta^j \in \mathbb{S}^2, \quad k \geq k_0, \quad j = 1, 2, \ldots, m.$$

Fix $\nu \in \mathbb{S}^2$, $\beta^j \in \mathbb{S}^2$, and $x \in S_+(\nu) \cap S_+(-\beta^j)$ in this equality. In this case $x\cdot\nu > 0$, $-x\cdot\beta^j > 0$ and thus $x\cdot\nu - x\cdot\beta^j > 0$. Put

$$k = k_{lj}(x.\nu, \beta^j) = [\pi/2 + 2l\pi]/(x\cdot\nu - x\cdot\beta^j),$$

where l is a positive integer, $l \in \mathbb{N}$. Define the function

$$h(x,\nu) = \lim_{l\to\infty} k_{lj}[g_j(x,k_{lj},\nu) - 2], \quad x \in S_+(\nu) \cap S_+(-\beta^j),\ \nu \in \mathbb{S}^2.$$

Since the union of the sets $S_+(-\beta^j)$ includes $S_+(\nu)$, the function $h(x,\nu)$ is defined for all $x \in S_+(\nu)$ and $\nu \in \mathbb{S}^2$. Hence, we obtain (2.9) as in Problem 2.

Since $q(x)$ is compactly supported, (2.9) leads to the following tomography problem: Find $q(x)$ from the integrals of $q(x)$ over all straight lines. As is known, the solution to this problem is unique. There exist numerous and effective algorithms of its numerical solution. □

The statement of the problem of recovering the refraction index $n(x)$ in the Helmholtz equation $(\Delta + k^2n^2(x))u(x,\nu) = 0$ is rather similar to those for Problems 1 and 2 and this problem can be studied by means of the same scheme as that for the Schrödinger equation with the use of high frequency asymptotics (see [13]).

§ 3. Equations of Electrodynamics

Consider Maxwell's system of equations which corresponds to a nonmagnetic nonconducting medium and time-periodic electromagnetic oscillations; i.e., we have

$$\operatorname{rot} H = -i\omega\varepsilon(x)E, \quad \operatorname{rot} E = i\omega\mu_0 H. \tag{3.1}$$

In these equations $\mu_0 > 0$ is a constant permeability coefficient, $\varepsilon(x)$ stands for the permittivity coefficient, and $\omega > 0$ is the frequency of oscillations. The vectors $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$ characterize the intensities of electric and magnetic fields, respectively.

Let $\Omega \subset \mathbb{R}^3$ and let S be the ball of radius R centered at the origin and its boundary. We assume that the permittivity coefficient is an infinitely differentiable function in $\mathbb{R}^3$ agreeing with a prescribed positive constant ε_0 beyond the domain Ω and the support of $\varepsilon(x) - \varepsilon_0$ lies strictly inside Ω. Denote by $n(x) = \sqrt{\varepsilon(x)\mu_0}$ the refraction index in an inhomogeneous medium and by $n_0 = \sqrt{\varepsilon_0\mu_0}$ its value in a homogeneous medium.

We examine a special solution to equations (3.1) of the plane wave type incident from infinity on the domain Ω which satisfies the radiation conditions. To describe it, we introduce the vector and scalar potentials A and ϕ, respectively, as follows:

$$\mu_0 H = \operatorname{rot} A, \quad E = i\omega A - \nabla\phi. \tag{3.2}$$

Choosing ϕ so that

$$\phi = \frac{1}{i\omega n^2(x)} \operatorname{div} A, \quad n(x) = \sqrt{\mu_0\varepsilon(x)},$$

we obtain the equation for A in the form

$$-\omega^2 n^2(x)A - \Delta A + (\operatorname{div} A)\nabla \log(\varepsilon(x)) = 0. \tag{3.3}$$

In a homogeneous medium (for $n(x) = n_0$), there exists a solution to (3.3) of the plane wave type of the form $A^0(x,\omega,\nu,j^0) = j^0 e^{i\omega n_0 x\cdot\nu}$, where ν and j^0 are arbitrary unit vectors orthogonal to each other. This solution describes a harmonic wave that propagates in the direction ν and is polarized in the direction j^0.

Denote by $A^{sc}(x,\omega,\nu,j^0) = A(x,\omega,\nu,j^0) - A^0(x,\omega,\nu,j^0)$ the vector potential relating to a scattered field. The function $A^{sc}(x,\omega,\nu,j^0)$ satisfies the Sommerfeld radiation condition

$$A^{sc}(x,\omega,\nu,j^0) = O(r^{-1}), \quad \frac{\partial A^{sc}}{\partial r} - i\omega n_0 A^{sc} = o(r^{-1}) \quad \text{as } r = |x| \to \infty. \tag{3.4}$$

The existence of a solution to (3.3) which satisfies (3.4) is established in [16].

Given A, the vectors E and H are calculated as

$$H = \frac{1}{\mu_0} \operatorname{rot} A, \quad E = i\omega \left[A + \frac{1}{\omega^2 n^2(x)} \nabla \operatorname{div} A \right]. \tag{3.5}$$

An electromagnetic field in a homogeneous medium corresponding to a harmonic wave is defined by the equalities

$$\begin{aligned} H^0(x,\omega,\nu,j^0) &= \frac{1}{\mu_0} \operatorname{rot} A^0(x,\omega,\nu,j^0), \\ E^0(x,\omega,\nu,j^0) &= i\omega \left[A^0(x,\omega,\nu,j^0) + \frac{1}{\omega^2 n_0^2} \nabla \operatorname{div} A^0(x,\omega,\nu,j^0) \right], \end{aligned} \tag{3.6}$$

where $A^0(x,\omega,\nu,j^0)$ is determined from the formula $A^0(x,\omega,\nu,j^0) = j^0 e^{i\omega n_0 x\cdot\nu}$.

We can now formulate the problem of recovering the permittivity coefficient which is considered below. Put $S^+(\nu) = \{x \in S \mid \nu \cdot x > 0\}$.

Problem 3. *Given is the function*

$$f(x,\omega,\nu,j^0) = |E(x,\omega,\nu,j^0) + E(x,\omega,-\nu,j^0)|^2, \tag{3.7}$$

$$x \in S^+(\nu), \quad \nu \in \mathbb{S}^2, \quad \omega \geq \omega_0 > 0,$$

defined for all $\nu \in \mathbb{S}^2$, $x \in S^+(\nu)$, and the frequencies ω beginning with some frequency $\omega_0 > 0$. Find $\varepsilon(x)$ inside Ω.

Remark. The electric field E in the statement of Problem 3 can be replaced with the magnetic field H.

Given a unit vector ν, denote by $y^0 = y^0(\nu) = -R\nu$ a point on S and by $\Sigma(\nu) = \{y \in \mathbb{R}^3 \mid (y - y^0)\cdot\nu = 0\}$ the plane passing through y^0 and orthogonal to ν. Also define the function $\varphi(x,\nu)$ as follows: $\varphi(x,\nu) = n_0 x\cdot\nu$ for $x\cdot\nu \leq y^0\cdot\nu$ and it is a solution to the Cauchy problem for the eikonal equation

$$|\nabla_x \varphi(x,\nu)|^2 = n^2(x), \quad \varphi(x,\nu)|_{\Sigma(\nu)} = n_0 y^0\cdot\nu, \tag{3.8}$$

such that $\varphi(x,\nu) > n_0 y^0\cdot\nu$ for $x\cdot\nu > y^0\cdot\nu$. This solution corresponds to the case that $\nabla_x\varphi(x,\nu)\cdot\nu > 0$ for $x\cdot\nu > y^0\cdot\nu$. The characteristics of the eikonal equation are geodesics of the Riemannian metric $d\tau = n(x)|dx|$, $n(x) = \sqrt{\varepsilon(x)\mu_0}$. These geodesics are orthogonal to the plane $\Sigma(\nu)$, since $\nabla_x\varphi(x,\nu) = n_0\nu$ on $\Sigma(\nu)$.

We will study Problem 3 assuming that the following two assumptions are made.

Assumption 1. *The function $n(x)$ belongs to $C^\infty(\mathbb{R}^3)$ and the Riemannian metric $d\tau = n(x)|dx|$ is such that every point $x \in \mathbb{R}^3$ can be joined with the plane $\Sigma(\nu)$ only by one geodesic $\Gamma(x,\nu)$ orthogonal to $\Sigma(\nu)$.*

A geodesic field $\Gamma(x,\nu)$ satisfying this assumption is called *regular*. Note that the regularity of the families of geodesic lines is presented in some form in the articles [12–16] connected with the study of inverse problems for the Helmholtz and Maxwell equations without phase information and also in [17–20], where the inverse kinematic problem is studied.

Let $C_+(\nu) \subset \mathbb{R}^3$ be a cylindrical domain consisting of the rays $x = x^0 + s\nu$, $s \in [0,\infty)$ such that $x^0 \in S_+(\nu)$. Obviously, for $x \in \mathbb{R}^3 \backslash (\Omega \cup C_+(\nu))$, geodesics $\Gamma(x,\nu)$ are segments of straight lines orthogonal to the plane $\Sigma(\nu)$.

The Riemannian length of the geodesic $\Gamma(x,\nu)$ is denoted by $\tau(x,\nu)$. Note that the functions $\varphi(x,\nu)$ and $\tau(x,\nu)$ are connected by the simple relation $\varphi(x,\nu) = y^0\cdot\nu - \tau(x,\nu)$ for $x\cdot\nu \leq y^0\cdot\nu$ and $\varphi(x,\nu) = y^0\cdot\nu + \tau(x,\nu)$ for $x\cdot\nu \geq y^0\cdot\nu$.

Assumption 2 is connected with the main term of the asymptotic expansion of the vector potential and the set $S_+(\nu)$ is the support of the data of Problem 3. It is established in [16] that the functions $A(x,\omega,\nu,j^0)$ for high frequencies have the asymptotics

$$A(x,\omega,\nu,j^0) = \exp(i\omega\varphi(x,\nu))\left[\alpha(x,\nu,j^0) + O\left(\frac{1}{\omega}\right)\right] \quad \text{as } \omega \to \infty, \tag{3.9}$$

where the function $\alpha(x,\nu,j^0)$ is a solution to the Cauchy problem for the transport equation

$$\begin{gathered} 2(\nabla\varphi\cdot\nabla)\alpha(x,\nu,j^0) + \alpha(x,\nu,j^0)\Delta\varphi(x,\nu) - (\alpha(x,\nu,j^0)\cdot\nabla\varphi(x,\nu))\nabla\log\varepsilon(x) = 0, \\ \alpha|_{\Sigma(\nu)} = j^0. \end{gathered} \tag{3.10}$$

Obviously, $\alpha(x,\nu,j^0) = j^0$ for $x \in \mathbb{R}^3 \setminus (\Omega \cup C_+(\nu))$.

Assumption 2. *The inequality*

$$\alpha(x,\nu,j^0)\cdot j^0 > 0, \quad x \in S_+(\nu),\ \nu \in \mathbb{S}^2, \tag{3.11}$$

holds for all $x \in S_+(\nu)$ and $\nu \in \mathbb{S}^2$.

Lemma 2, proven in Section 4, provides sufficient conditions on the refraction index $n(x)$ for Assumption 2 to be valid.

Theorem 2. *Under Assumptions 1 and 2, Problem 3 has at most one solution. Moreover, this problem is reduced to solving the inverse kinematic problem: Find $n(x)$ inside Ω, given*

$$h(x,\nu) = \tau(x,\nu), \quad x \in S_+(\nu),\ \nu \in \mathbb{S}^2. \tag{3.12}$$

Proof. We employ the high frequency asymptotics that is found in [16].

In an inhomogeneous medium the asymptotics of $E(x,\omega,\nu)$ is defined by the formula

$$E(x,\omega,\nu,j^0) = i\omega\exp(i\omega\varphi(x,\nu))\left[\alpha^{\perp}(x,\nu,j^0) + O\left(\frac{1}{\omega}\right)\right] \quad \text{as } \omega \to \infty, \tag{3.13}$$

in which $\alpha^{\perp}(x,\nu,j^0) = \alpha(x,\nu,j^0) - n^{-2}(x)(\alpha(x,\nu,j^0)\cdot\nabla\varphi(x,\nu))\nabla\varphi(x,\nu)$ is the projection of the vector $\alpha(x,\nu,j^0)$ into the plane orthogonal to the vector $\nabla\varphi(x,\nu)$.

In a homogeneous medium ($n(x) = n_0$) the asymptotics of $E^0(x,\omega,\nu,j^0)$ is defined by the formula

$$E^0(x,\omega,\nu,j^0) = i\omega\exp(i\omega n_0 x\cdot\nu)\left[j^0 + O\left(\frac{1}{\omega}\right)\right] \quad \text{as } \omega \to \infty. \tag{3.14}$$

In accord with (3.13) and (3.14), we infer

$$E(x,\omega,-\nu,j^0) = E^0(x,\omega,-\nu,j^0) = i\omega\exp(-i\omega n_0 x\cdot\nu)\left[j^0 + O\left(\frac{1}{\omega}\right)\right]$$
$$\text{as } \omega \to \infty,\ x \in S^+(\nu).$$

Putting $x \in S^+(\nu)$, we infer that

$$\begin{aligned} \frac{1}{\omega^2}f(x,\omega,\nu,j^0) = \frac{1}{\omega^2}|E(x,\omega,\nu,j^0) + E(x,\omega,-\nu,j^0)|^2 = |\alpha^{\perp}(x,\nu,j^0)|^2 + 1 \\ +2\alpha^{\perp}(x,\nu,j^0)\cdot j^0\cos\left(\omega(\varphi(x,\nu) + n_0 x\cdot\nu)\right) + O\left(\frac{1}{\omega}\right) \quad \text{as } \omega \to \infty. \end{aligned} \tag{3.15}$$

Prove that

$$\alpha^{\perp}(x,\nu,j^0)\cdot j^0 = \alpha(x,\nu,j^0)\cdot j^0, \quad x \in S_+(\nu). \tag{3.16}$$

This follows from the fact that $\nabla\varphi(x,\nu)=\nu$ for $x\in S_+(\nu)$. Indeed, if we assume that $\nabla\varphi(x,\nu)\neq\nu$ at some point $x\in S_+(\nu)$ then the geodesic line $\Gamma(x,\nu)$ after its propagation beyond Ω (beyond the domain it is extended as a straight line) intersects some geodesics orthogonal to the plane $\Sigma(\nu)$ and disjoint with Ω (hence, they are straight lines). This contradicts Assumption 1 about the regularity of the field of geodesic lines. So, (3.16) takes place. In accord with Assumption 2, $\alpha^{\perp}(x,\nu,j^0)\cdot j^0>0$ for $x\in S_+(\nu)$.

Fix $x\in S^+(\nu)$ and $\nu\in\mathbb{S}^2$. In this case $\varphi(x,\nu)+n_0x\cdot\nu=\eta(x,\nu)>0$. The function $f(x,\omega,\nu,j^0)/\omega^2$ is a function of only one variable ω almost periodic in this variable. Since ω is not bounded above, the period $\eta(x,\nu)\neq 0$ of this function is uniquely defined from (3.16) with any accuracy. Hence, the function $\varphi(x,\nu)=\eta(x,\nu)-n_0x\cdot\nu$ for all $x\in S^+(\nu)$, $\nu\in\mathbb{S}^2$ is uniquely defined as well. Note that $\tau(x,\nu)=h(x,\nu)=x\cdot\nu$ for $x\in S\setminus S^+(\nu)$, $\nu\in\mathbb{S}^2$. As a result, we arrive at the following problem: Find $n(x)$ inside Ω, given the traveling times $\tau(x,\nu)$ between $x\in S$ and the plane $\Sigma(\nu)$ for all $\nu\in\mathbb{S}^2$. Only by form this statement differs from the standard inverse kinematic problem on recovering the speed field inside Ω for prescribed traveling times between the boundary points. The known results (see [17–20]) justify the uniqueness theorem for solutions to this problem. $\square$

§ 4. Sufficient Conditions for Validity of Assumption 2

Using the eikonal equation, we can write out the equation for the geodesics $\Gamma(x,\nu)$. Introduce the orthogonal triple of vectors e_k, $k=1,2,3$, as follows: $e_1=\nu$, $e_2=j^0$, $e_3=e_1\times e_2$. Represent $y\in\Sigma(\nu)$ as $y=y^0(\nu)+a_2e_2+a_3e_3$ and consider the pencil of geodesics $\xi=\xi(s,a_2,a_3,\nu)$ intersecting the plane $\Sigma(\nu)=\{y\in\mathbb{R}^3 \mid (y-y^0(\nu))\cdot\nu=0\}$ orthogonally. Recall that $y^0(\nu)$ belongs to S and $\Sigma(\nu)$ is tangent to S at this point. The varying parameter s on a geodesic is counted from the plane $\Sigma(\nu)$, with $s=0$ at a point $\xi\in\Sigma(\nu)$ and $s>0$ for points on geodesics in the half-plane $(x-y^0(\nu))\cdot\nu>0$. For $s<0$, the geodesic is the straight line $\xi=y^0(\nu)+a_2e_2+a_3e_3+s\nu$. Denote $p(x,\nu)=\nabla\varphi(x,\nu)$. For fixed ν and $s>0$, the geodesic lines and the quantities $p(\xi,\nu)$ and $\tau(\xi,\nu)$ depending on the parameters s,a_2,a_3 and solving the Cauchy problem for the system of ordinary differential equations

$$\begin{gathered}\frac{d\xi}{ds}=\frac{p}{n^2(\xi)},\quad \frac{dp}{ds}=\nabla\log n(\xi),\quad \frac{d\tau}{ds}=1,\\ \xi|_{s=0}=y^0(\nu)+a_2e_2+a_3e_3,\quad p|_{s=0}=n_0\nu,\quad \tau|_{s=0}=0.\end{gathered}\tag{4.1}$$

From this system it follows that $s>0$ coincides with the Riemannian length counted from the plane $\Sigma(\nu)$. The system (4.1) defines the functions $\xi=\xi(s,a_2,a_3,\nu)$ and $p=p(s,a_2,a_3,\nu)$. To find them for a fixed ν as functions of the variables x_1, x_2, and x_3, we need to solve the equation $x=\xi(s,a_2,a_3,\nu)$, for s, a_2, and a_3 and express the latter trough x. In view of Assumption 1 we can do so uniquely. Thereby, we have a one-to-one correspondence between the coordinates of a point x and the parameters s,a_2,a_3. In this case $s=s(x,\nu)=\tau(x,\nu)$, $a_2=a_2(x,\nu)$, $a_3=a_3(x,\nu)$, and the equations for geodesics are representable as $\xi=\zeta(x.\nu)\equiv\xi(s(x,\nu),a_2(x,\nu),a_3(x,\nu),\nu)$. The function $\nabla\varphi(x,\nu)=p(x,\nu)$ is determined uniquely as well.

Introduce the Jacobian

$$J=\det\begin{pmatrix}\frac{\partial\xi_1}{\partial s} & \frac{\partial\xi_2}{\partial s} & \frac{\partial\xi_3}{\partial s}\\ \frac{\partial\xi_1}{\partial a_2} & \frac{\partial\xi_2}{\partial a_2} & \frac{\partial\xi_3}{\partial a_2}\\ \frac{\partial\xi_1}{\partial a_3} & \frac{\partial\xi_2}{\partial a_3} & \frac{\partial\xi_3}{\partial a_3}\end{pmatrix}.$$

Lemma 1. *Along the geodesics line* $\Gamma(x,\nu)$ *we have*

$$\frac{d\log J}{ds}=\operatorname{div}(n^{-2}(\xi)\nabla\varphi(\xi,\nu)).\tag{4.2}$$

PROOF. Calculating the derivative of J along the trajectories, we have

$$\frac{dJ}{ds} = \det\begin{pmatrix} \frac{\partial}{\partial s}\left(\frac{\partial \xi_1}{\partial s}\right) & \frac{\partial \xi_2}{\partial s} & \frac{\partial \xi_3}{\partial s} \\ \frac{\partial}{\partial a_2}\left(\frac{\partial \xi_1}{\partial s}\right) & \frac{\partial \xi_2}{\partial a_2} & \frac{\partial \xi_3}{\partial a_2} \\ \frac{\partial}{\partial a_3}\left(\frac{\partial \xi_1}{\partial s}\right) & \frac{\partial \xi_2}{\partial a_3} & \frac{\partial \xi_3}{\partial a_3} \end{pmatrix} + \det\begin{pmatrix} \frac{\partial \xi_1}{\partial s} & \frac{\partial}{\partial s}\left(\frac{\partial \xi_2}{\partial s}\right) & \frac{\partial \xi_3}{\partial s} \\ \frac{\partial \xi_1}{\partial a_2} & \frac{\partial}{\partial a_2}\left(\frac{\partial \xi_2}{\partial s}\right) & \frac{\partial \xi_3}{\partial a_2} \\ \frac{\partial \xi_1}{\partial a_3} & \frac{\partial}{\partial a_3}\left(\frac{\partial \xi_2}{\partial s}\right) & \frac{\partial \xi_3}{\partial a_3} \end{pmatrix}$$

$$+ \det\begin{pmatrix} \frac{\partial \xi_1}{\partial s} & \frac{\partial \xi_2}{\partial s} & \frac{\partial}{\partial s}\left(\frac{\partial \xi_3}{\partial s}\right) \\ \frac{\partial \xi_1}{\partial a_2} & \frac{\partial \xi_2}{\partial a_2} & \frac{\partial}{\partial a_2}\left(\frac{\partial \xi_3}{\partial s}\right) \\ \frac{\partial \xi_1}{\partial a_3} & \frac{\partial \xi_2}{\partial a_3} & \frac{\partial}{\partial a_3}\left(\frac{\partial \xi_3}{\partial s}\right) \end{pmatrix}.$$

To calculate the determinants, we use the formulas

$$\frac{\partial}{\partial s}\left(\frac{\partial \xi_k}{\partial s}\right) = \frac{\partial}{\partial s}(n^{-2}(\xi)p_k) = \sum_{m=1}^{3} \frac{\partial(n^{-2}(x)p_k)}{\partial \xi_m}\frac{\partial \xi_m}{\partial s}, \quad k = 1,2,3,$$

$$\frac{\partial}{\partial a_l}\left(\frac{\partial \xi_k}{\partial s}\right) = \frac{\partial}{\partial a_l}(n^{-2}(\xi)p_k) = \sum_{m=1}^{3} \frac{\partial(n^{-2}(\xi)p_k)}{\partial x_m}\frac{\partial \xi_m}{\partial a_l}, \quad k = 1,2,3,\ l = 2,3,$$

and we can find that

$$\frac{dJ}{ds} = J \operatorname{div}(n^{-2}(\xi)p(\xi,\nu)) = J \operatorname{div}(n^{-2}(\xi)\nabla\varphi(\xi,\nu)).$$

The last formula validates (4.2). $\square$

Since in view of (4.1) for $s = 0$

$$\frac{\partial \xi}{\partial s} = \frac{\nu}{n_0} = \frac{e_1}{n_0}, \quad \frac{\partial \xi}{\partial a_2} = e_2, \quad \frac{\partial \xi}{\partial a_3} = e_3,$$

$J = 1/n_0$ for $s = 0$. Integrating (4.2), we arrive at

$$J = \frac{1}{n_0}\exp \int\limits_{\Gamma(x,\nu)} \operatorname{div}(n^{-2}(\xi)\nabla\varphi(\xi,\nu))\, ds. \tag{4.3}$$

Formula (4.3) implies that J is positive along $\Gamma(x,\nu)$. Actually, it follows from Assumption 1.

Lemma 2. *Under Assumption 1, let the refraction index $n(x)$ satisfy the conditions*

$$0 < n(x) \le n_1,\ x \in \mathbb{R}^3; \quad \operatorname{supp}(n(x) - n_0) \subset \Omega; \quad |\nabla(n(x))^{-1}| \le \delta,\ \delta > 0. \tag{4.4}$$

Then Assumption 2 holds under the condition

$$3\delta n_1 \operatorname{diam}\Omega < 2\log 2. \tag{4.5}$$

PROOF. Divide both parts of (3.10) by $2n^2$ and employ (4.1), (4.2), and the equality $\varepsilon(x) = n^2(x)/\mu_0$. Then we can write the Cauchy problem for defining $\alpha(x,\nu)$ as

$$\frac{d\alpha}{ds} + \frac{1}{2}\alpha\left(\frac{d\log J}{ds} - \nabla n^{-2}(\xi)\cdot p\right) + \frac{1}{4}(\alpha\cdot p)\nabla n^{-2}(\xi) = 0, \quad \alpha|_{\Sigma(\nu)} = j^0. \tag{4.6}$$

Introduce the new function $\hat\alpha = \sqrt{J}\alpha$. In this case (4.6) is transformed to the problem

$$\frac{d\hat\alpha}{ds} - \frac{1}{2}\hat\alpha(\nabla n^{-2}(\xi)\cdot p) + \frac{1}{4}(\hat\alpha\cdot p)\nabla n^{-2}(\xi) = 0, \quad \hat\alpha|_{s=0} = \frac{j^0}{\sqrt{n_0}}. \tag{4.7}$$

Take $x \in S_+(\nu)$ arbitrarily and denote the intersection point of the geodesic $\Gamma(x,\nu)$ with $S_-(\nu) = S \backslash S_+(\nu)$ by ξ^1, and the corresponding value s relating to this point by $s_1 = s_1(x,\nu)$. Since $\nabla n^{-2}(\xi) = 0$ for all $\xi \in (\mathbb{R}^3 \backslash \Omega)$, (4.7) implies that $\hat{\alpha}(\xi,\nu) = j^0/\sqrt{n_0}$ for $\xi \in \Gamma(\xi^1,\nu)$. Integrating (4.7) and involving the initial data we infer that $\hat{\alpha}$ satisfies the integral equation along the geodesic $\Gamma(x,\nu)$, with $\xi \in (\Gamma(x,\nu) \backslash \Gamma(\xi^1,\nu))$, and

$$\hat{\alpha}(\xi,\nu) = \frac{j^0}{\sqrt{n_0}} + \frac{1}{2}\int\limits_{s_1}^{s} \Big[\hat{\alpha}(\xi',\nu)(\nabla n^{-2}(\xi') \cdot p(\xi',\nu)) - \frac{1}{2}(\hat{\alpha}(\xi',\nu) \cdot p(\xi',\nu))\nabla n^{-2}(\xi')\Big] ds', \tag{4.8}$$

where $s = s(\xi,\nu)$, $\xi' = \xi'(s', a_2, a_3, \nu)$, and s' is a variable parameter on the geodesic. Using for $\xi \in (\Gamma(x,\nu) \setminus \Gamma(\xi^1,\nu))$ the inequalities $|p| \le n(\xi)|$ and $n(\xi)|\nabla n^{-2}(\xi)| \le 2\delta$, we find that

$$|\hat{\alpha}(\xi,\nu)| \le \frac{1}{\sqrt{n_0}} + \frac{3\delta}{2}\int\limits_{s_1}^{s} |\hat{\alpha}(\xi',\nu)|\, ds', \quad s \ge s_1. \tag{4.9}$$

Hence,

$$|\hat{\alpha}(\xi,\nu)| \le \frac{1}{\sqrt{n_0}} e^{3\delta(s-s_1)/2}, \quad s_1 \le s \le s(x,\nu). \tag{4.10}$$

Equation (4.8) yields

$$\hat{\alpha}(\xi,\nu) \cdot j^0 = \frac{1}{\sqrt{n_0}} + \frac{1}{2}\int\limits_{s_1}^{s} \Big[(\hat{\alpha}(\xi',\nu) \cdot j^0)(\nabla n^{-2}(\xi') \cdot p(\xi',\nu)) - \frac{1}{2}(\hat{\alpha}(\xi',\nu) \cdot p(\xi',\nu))(\nabla n^{-2}(\xi') \cdot j^0)\Big] ds', \quad s \ge s_1.$$

Thus,

$$\hat{\alpha}(\xi,\nu) \cdot j^0 \ge \frac{1}{\sqrt{n_0}} - \frac{3\delta}{2}\int\limits_{s_1}^{s} |\alpha(\hat{\xi'},\nu)|\, ds'$$

$$\ge \frac{1}{\sqrt{n_0}}\left(1 - \frac{3\delta}{2}\int\limits_{s_1}^{s} e^{3\delta(s'-s_1)/2} ds'\right) = \frac{1}{\sqrt{n_0}}(2 - e^{3\delta(s-s_1)/2}). \tag{4.11}$$

For $s_1 \le s \le s(x,\nu) = \tau(x,\nu)$, we have that $s - s_1 \le n_1|x - \xi^1| \le n_1 \operatorname{diam}\Omega$ and so from (4.11) it follows that

$$\hat{\alpha}(x,\nu) \cdot j^0 \ge \frac{1}{\sqrt{n_0}}(2 - e^{3\delta n_1 \operatorname{diam}\Omega/2}), \quad x \in S_+(\nu). \tag{4.12}$$

If (4.5) is fulfilled then we find that

$$\hat{\alpha}(x,\nu) \cdot j^0 > 0$$

for $x \in S_+(\nu)$. The latter is equivalent to the inequality $\alpha(x,\nu) \cdot j^0 > 0$ for $x \in S_+(\nu)$ which agrees with Assumption 2. $\square$

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