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ABSENCE OF NONTRIVIAL SYMMETRIES TO THE HEAT EQUATION IN GOURSAT GROUPS OF DIMENSION AT LEAST 4

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Abstract: Using the extension method, we study the one-parameter symmetry groups of the heat equation $\partial_t p = \Delta p$, where $\Delta = X_1^2 + X_2^2$ is the sub-Laplacian constructed by a Goursat distribution $\mathrm{span}(\{X_1, X_2\})$ in $\mathbb{R}^n$, where the vector fields X_1 and X_2 satisfy the commutation relations $[X_1, X_j] = X_{j+1}$ (where $X_{n+1} = 0$) and $[X_j, X_k] = 0$ for $j \geq 1$ and $k \geq 1$. We show that there are no such groups for $n \geq 4$ (with exception of the linear transformations of solutions which are admitted by every linear equation).

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1. Introduction

It is known in Riemannian geometry that the spectrum of the Laplace–Beltrami operator contains some but not all information on the geometric invariants of the domain manifold (for example, the dimension and the curvature tensor are uniquely reconstructible. Moreover, if the manifold is compact then so is also the volume and if the manifold has boundary then the area of the boundary is determined uniquely). For obtaining this information, we apply the asymptotics of the fundamental solution to the heat equation for small values of the parameter. One of the most important works in this direction is the article [1] whose ideas were used in many investigations into related topics.

Part of these results was transferred with substantial constraints (for example, in [2, 3]) to sub-Riemannian manifolds, where the *sub-Laplacian* acts as an analog of the Laplace–Beltrami operator: $\Delta_\mu = \sum_{j=1}^m \left(X_j^2 + \mathrm{div}_\mu(X_j)X_j\right)$. Here m is the dimension of the horizontal distribution $\mathrm{span}(\{X_j : 1 \leq j \leq m\})$ (we assume for simplicity that the vector fields $\{X_j : 1 \leq j \leq m\}$ are linearly independent, satisfy the Hörmander condition, and generate the same growth vector at all points), while μ is a nowhere vanishing smooth differential form of degree the dimension of the manifold (the volume form), the operator div_μ (called *divergence*) associates with a vector field X the smooth function defined by the formula $\mathrm{div}_\mu(X) \cdot \mu = d(i_X(\mu))$, where i_X is the operator of the interior product of differential forms (acting by the insertion of a vector field X in the first argument of the form), and d is the exterior derivative. The difficulties that arise in an attempt to describe the geometric characteristics of sub-Riemannian manifolds include (1) the absence of ellipticity for the sub-Laplacian, (2) the possible presence of abnormal geodesics, (3) the noncoincidence of the algebraic, Hausdorff, and geodesic dimensions, (4) problems with defining the curvature, and (5) the requirement of the Hörmander condition and a significant complication of the structure of the manifold in case of the failure of the condition or its fulfillment with different growth vectors, etc.

Knowing some formula (even not in elementary functions) for the fundamental solution that are obtained usually by reducing the order of the heat equation (or, more exactly, the Fourier transform of the latter which looks much easier than the initial equation) would substantially help in finding the necessary asymptotics but such a formula is not always easy to obtain.

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In this article, we study the problem of finding such formulas for one important class of Lie groups—the so-called Goursat groups.

Definition 1. The *Goursat group of dimension* n is the nilpotent Lie group in $\mathbb{R}^n$ with a two-dimensional left-invariant distribution H_1 (called the *Goursat distribution*) that has growth vector $(2, 3, \dots, n-1, n)$ (i.e., $\dim H_k = k+1$ for $1 \le k \le n-1$, where $H_{k+1} = H_k + [H_k, H_k]$).

The multiplication in the Goursat group can be written as follows:

$$\sum_{j=1}^{n} x_j \mathbf{e}_j \star \sum_{k=1}^{n} y_k \mathbf{e}_k = (x_1 + y_1)\mathbf{e}_1 + \sum_{j=2}^{n} \left(x_j + \sum_{k=2}^{j} \left(\frac{{x_1}^{j-k}}{(j-k)!} y_k \right) \right) \mathbf{e}_j;$$

moreover, $\mathbf{0} = \sum_{j=1}^{n} 0\mathbf{e}_j$ is the neutral element.

We will use the following coordinate representation of H_1:

$$H_1 = \operatorname{span}(\{X_1, X_2\}), \quad X_1 = \partial_1,\ X_2 = \sum_{k=0}^{n-2} \frac{{x_1}^k}{k!} \partial_{k+2}.$$

We can check that $(\mathbb{R}^n, \star, \mathbf{0})$ is indeed a group, the vector fields X_1 and X_2 are invariant under left translations by the operation $\star$, and H_1 has growth vector $(2, 3, \dots, n-1, n)$. Endowing the Goursat group with the n-dimensional Lebesgue measure (which is also a Haar measure for this group) and the corresponding volume form, define the sub-Laplacian: $\Delta = X_1^2 + X_2^2$ (since the Goursat group is unimodular and the left-invariant vector fields X_1 and X_2, the divergent terms are zero, and so we will omit the index μ). The equation $\partial_t p(t,x) = \Delta p(t,x)$ will be called the *heat equation*, and its solution under the (generalized) initial condition $p(0,x) = \delta_{\mathbf{0}}(x)$ will be referred as the *fundamental solution*.

We will validate

Theorem 1. *In Goursat groups, it is impossible to reduce the order of the heat equation for* $n \ge 4$*; otherwise, the heat equation not admit one-parameter symmetry groups that differ from the groups of linear transformations of solutions* (*admissible by every linear equation*).

But, for $n = 3$ (i.e., for the Heisenberg group), this reduction is possible. Moreover, there is an explicit formula for the fundamental solution $\partial_t p = \Delta p$ in the Heisenberg group (see, for example, [4], where it is obtained exactly by reducing the order).

Observe preliminarily that we will not even need an initial solution in the proof: search for nontrivial symmetries of the equation itself leads to an overdetermined system.

2. Preliminary Simplifications

Applying the Fourier transform to the equation $\partial_t p = \Delta p$ with respect to the variables x_k for $k \ge 2$ (the variable dual to x_k is denoted by y_k), we obtain the equation in $\hat{p}$; i.e.,

$$\partial_t \hat{p} = \partial_1^2 \hat{p} - \left(\sum_{k=0}^{n-2} \frac{{x_1}^k}{k!} y_{k+2} \right)^2 \hat{p}.$$

This equation is much easier than the initial since derivative is taken only with respect to t and x_1; the variables $(y_k)_{2 \le k \le n}$ may be assumed to be parameters. The polynomial on the right-hand side has degree $2n-4$.

To simplify notations, we will simply write x instead of x_1, use ∂_x instead of ∂_1, and denote the unknown function simply by p instead of $\hat{p}$ (regarding the transformed equation as if it is distinguished from the initial heat equation) and designate $\left(\sum_{k=0}^{n-2} \frac{{x_1}^k}{k!} y_{k+2} \right)^2$ as $Q(x)$; the equation takes the form $\partial_t p = \partial_x^2 p - Qp$.

As we will find out below, the fact that Q is a polynomial of a special kind does not influence the final result; it is only important that this is a polynomial in x which depends neither on t nor p and has degree $2n-4$.

3. The Extension Method

The method of this article is described in a very general form in [5]. The monograph [5] contains all main definitions that are needed for application of the method and some useful formulas which will be utilized in the calculations below in the proof of Theorem 1.

The key idea of this method is that the unknown function and its derivatives of all necessary orders are declared (along with the initial independent variables) to be variables of a larger space, the so-called *extended* space, or *jet space*. A *jet of order* k at some point is a coset of functions that have the same Taylor expansion at this point up to order k; The jet space J^kM is obtained by the disjoint union of the sets of jets at all points of the base manifold M of independent variables. Associating with the sought function p the surface P consisting of the jets of this function at all points, we can transform the differential equation to some algebraic equation in the jet space which must satisfy all points of P. If the initial equation admitted some symmetry group generated by some vector field S then the equation in the extended space admits some symmetry group whose infinitesimal generator is an *extension* of the latter to the jet space. Symmetries are sought from the fact that the action of the extended field at the algebraic equation obtained from the initial differential equation must turn into identity at the solutions to the differential equation. If some solution was known to the initial differential equation then it could be translated along S—this is what is understood as the order reduction.

In application to the problem under consideration, the above yields the following: In our case, $M = \mathbb{R}^2$ (there are two independent variables); and we take $J^2\mathbb{R}^2$ as the extended space because the initial equation has order 2. Let

$$S = \tau(t,x,p)\partial_t + \xi(t,x,p)\partial_x + \psi(t,x,p)\partial_p$$

be an infinitesimal generator of a one-parameter subgroup of the symmetry group of the equation $\partial_t p = \partial_x^2 p - Qp$. Its extension to $J^2\mathbb{R}^2$ is written as

$$\begin{aligned}\underset{2}{S} &= \tau(t,x,p)\partial_t + \xi(t,x,p)\partial_x + \psi(t,x,p)\partial_p \\ &+\psi_{(1,0)}(t,x,p,p_t)\partial_{p_t} + \psi_{(0,1)}(t,x,p,p_x)\partial_{p_x} + \psi_{(2,0)}(t,x,p,p_t,p_{t,t})\partial_{p_{t,t}} \\ &+\psi_{(1,1)}(t,x,p,p_t,p_x,p_{t,x})\partial_{p_{t,x}} + \psi_{(0,2)}(t,x,p,p_x,p_{x,x})\partial_{p_{x,x}}.\end{aligned}$$

Here p with subscripts stands for the variables of the derivatives of the unknown function with respect to the arguments corresponding to these subscripts and the coefficients $\psi_{(\alpha,\beta)}$ of the basic derivative with respect to $p_{\underbrace{t\ldots t}_{\alpha \text{ times}},\underbrace{x\ldots x}_{\beta \text{ times}}}$ (for simplicity of notation, we will briefly write this variable as $p_{t^\alpha x^\beta}$) are computed by the recurrent formulas

$$\begin{aligned}\psi_{(0,0)} &= \psi, \\ \psi_{(\alpha+1,\beta)}+ &= D_t\psi_{(\alpha,\beta)} - (p_{t^{\alpha+1}x^\beta}D_t\tau + p_{t^\alpha x^{\beta+1}}D_t\xi), \\ \psi_{(\alpha,\beta+1)} &= D_x\psi_{(\alpha,\beta)} - (p_{t^{\alpha+1}x^\beta}D_x\tau + p_{t^\alpha x^{\beta+1}}D_x\xi),\end{aligned}$$

in which D_z (here z is one of the independent variables, i.e., t or x) stands for the formal operator of "total" derivative with respect to z; i.e.,

$$D_z = \partial_z + \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} p_{t^\alpha x^\beta,z}\partial_{p_{t^\alpha x^\beta}}.$$

All applications of this operator in fact lead to finite sums since the functions it is applied to in the recurrent formulas depend only on finitely many variables of the form $p_{t^\alpha x^\beta}$.

For deducing these formulas, we can write down the asymptotics of the symmetry group parametrized by a real a as $a \to 0$ as follows:

$$\begin{aligned}\mathfrak{T}_a(t,x,p) &= t + a\tau(t,x,p) + o(a), \\ \mathfrak{X}_a(t,x,p) &= x + a\xi(t,x,p) + o(a), \\ \mathfrak{P}_a(t,x,p) &= p + a\psi(t,x,p) + o(a),\end{aligned}$$

and then calculate the derivatives of the transformed unknown function with respect to the transformed variables up to the first order with respect to a. Then $\psi_{(\alpha,\beta)}$ is found from the fact that, as $a \to 0$,

$$\frac{\partial^{\alpha+\beta}\mathfrak{P}_a}{\partial^\alpha \mathfrak{T}_a \partial^\beta \mathfrak{X}_a} = p_{t^\alpha x^\beta} + a\psi_{(\alpha,\beta)}(t, x, (p_{t^\kappa x^\lambda})_{0\le\kappa\le\alpha,\, 0\le\lambda\le\beta}) + o(a).$$

In the general situation, this is how the extension of a vector field is defined. A recurrent formula for its coefficients written in vector form can be found in [5, Chapter I, § 4, formula (8.4)].

Using the formulas for calculating $\psi_{(\alpha,\beta)}$, we find

$$\underset{2}{S}(p_{x,x} - Qp - p_t) = K_0 + K_1 p_t + K_2 p_x + K_3 p_t^2 + K_4 p_t p_x + K_5 {p_x}^2$$
$$+K_6 p_t p_x^2 + K_7 p_x^3 + K_8 p_{t,x} + K_9 p_{x,x} + K_{10} p_x p_{t,x} + K_{11} p_t p_{x,x} + K_{12} p_x p_{x,x},$$

where the coefficients denoted by K with subscripts are as follows (here Q' stands for the derivative of Q with respect to x):

$$K_0 = -\xi p Q' - \psi Q - \partial_t \psi + \partial_x^2 \psi, \quad K_1 = \partial_t \tau - \partial_p \psi - \partial_x^2 \tau, \quad K_2 = \partial_t \xi - \partial_x^2 \xi + 2\partial_p \partial_x \psi,$$
$$K_3 = \partial_p \tau, \quad K_4 = \partial_p \xi - 2\partial_p \partial_x \tau, \quad K_5 = -2\partial_p \partial_x \xi + \partial_p^2 \psi, \quad K_6 = -\partial_p^2 \tau,$$
$$K_7 = -\partial_p^2 \xi, \quad K_8 = -2\partial_x \tau, \quad K_9 = -2\partial_x \xi + \partial_p \psi,$$
$$K_{10} = -2\partial_p \tau, \quad K_{11} = -\partial_p \tau, \quad K_{12} = -3\partial_p \xi.$$

The expression $\underset{2}{S}(p_{x,x} - Qp - p_t)$ must be identically zero provided that p is a solution (under this condition, p_t is excluded by means the equation $p_t = p_{x,x} - Qp$ whereas $p_{t,x}$ is excluded if the initial differential equation is differentiated with respect to x and denote the derivatives are denoted by the variables of the extended space, i.e., $p_{t,x} = p_{x,x,x} - Q'p - Qp_x$). Expressing these "dependent" variables in terms of the "independent" variables (i.e., p, p_x, $p_{x,x}$, and $p_{x,x,x}$), we find that

$$\underset{2}{S}(p_{x,x} - Qp - p_t)|_{p \text{ is a solution}} = (K_0 - K_1 pQ + K_3 p^2 Q^2 - K_8 pQ')$$
$$+(K_2 - K_4 pQ - K_8 Q - K_{10} pQ')p_x + (K_5 - K_6 pQ - K_{10} Q)p_x^2 + K_7 p_x^3$$
$$+(K_1 - 2K_3 pQ + K_9 - K_{11} pQ)p_{x,x} + (K_4 + K_{12})p_{x,x} p_x + K_6 p_{x,x} p_x^2$$
$$+(K_3 + K_{11})p_{x,x}^2 + K_8 p_{x,x,x} + K_{10} p_{x,x,x} p_x = 0.$$

Equating to zero the coefficients of the monomials consisting of the derivatives (of the first and higher orders) of the unknown functions, we obtain a *system of defining partial differential equations* for τ, ξ, and ψ. Exclude $K_6 = K_7 = K_8 = K_{10} = 0$ immediately from its other equations and observe that $K_3 = -\frac{K_{10}}{2} = 0$ and $K_{11} = \frac{K_{10}}{2} = 0$; moreover, one of the equations ($K_3 + K_{11} = 0$) holds identically. We infer that

$$\begin{cases} K_0 - K_1 pQ = 0, \\ K_2 - K_4 pQ = 0, \\ K_5 = 0, \\ K_1 + K_9 = 0, \\ K_4 + K_{12} = 0. \end{cases}$$

From $K_8 = K_{10} = 0$ we can only obtain that τ depends only on t; $K_6 = 0$ gives no new comparable information. Granted this, from $K_4 + K_{12} = 0$ we have $\partial_p \xi = 0$, i.e., ξ is a function of t and x (and the equation $K_7 = 0$ also adds nothing new). Further, from $K_5 = 0$ it follows that $\partial_p^2 \psi = 0$, whence we can conclude that $\psi = pA_1(t,x) + B_1(t,x)$ with some functions A_1 and B_1, and from $K_1 + K_9 = 0$ it follows that $\partial_t \tau = 2\partial_x \xi$, which turns into $\partial_x^2 \xi = 0$ after differentiation with respect to x with account taken of the fact that τ depends only on t. This means that $\xi = xA_2(t) + B_2(t)$ with some functions A_2 and B_2, and the common solution for τ looks as $\tau(t) = 2\int A_2(t)\,dt$.

Having performed these simplifications, rewrite the still unused equations in terms of the new functions:

$$-(xA_2 + B_2)pQ' - (pA_1 + B_1)Q - \partial_t(pA_1 + B_1) + \partial_x^2(pA_1 + B_1) - pQ(2A_2 - A_1) = 0,$$

$$\partial_t(xA_2 + B_2) + 2\partial_x A_1 = 0.$$

The second equation in this system can be integrated:

$$-2A_1(t,x) = \frac{x^2}{2}\partial_t A_2(t) + x\partial_t B_2(t) + C(t)$$

with some function C. Insert this expression for A_1 in the first equation which yields $pY + Z = 0$, where Y and Z depend only on t and x, so that $Y = 0$ and $Z = 0$. Since $Z = \partial_x^2 B_1 - \partial_t B_1 - B_1 Q$, the equation $Z = 0$ gives no information apart from the fact that B_1 is a solution to the initial differential equation (as we will see below, this gives a trivial symmetry possessed by every linear equation). Nontrivial symmetries can result only from the equation $Y = 0$, which is written down in extended form as follows:

$$-\frac{1}{2}\left(\frac{x^2}{2}\partial_t^2 A_2 + x\partial_t^2 B_2 + \partial_t C\right) + \frac{1}{2}\partial_t A_2 = -(2QA_2 + Q'(xA_2 + B_2)).$$

This equation is polynomial in x, and the degree of the polynomial on the left-hand side is equal to 2 and the degree of the polynomial on the right-hand side is equal to $2n-4$. For $n \geq 4$, this means that $A_2 = 0$ and $B_2 = 0$ (since the degree of Q' is equal to $2n-5$, which is nevertheless greater than 2 for $n \geq 4$), whence $\xi = 0$ and $\tau = \text{const}$. The equation $Y = 0$ turns into $C = \text{const}$; i.e., $A_1 = \text{const}$, which exactly means that the initial differential equation is linear but this trivial symmetry certainly does not enable us to reduce the order.

Thus, we have proved the following analog of Theorem 1:

Theorem 2. *If Q is a polynomial in x of degree greater than 2 then the equation $\partial_t p(t,x) = \partial_x^2 p(t,x) - Q(x)p(t,x)$ admits no one-parameter symmetry groups different from the groups of linear transformations (admitted by every linear equation).*

Since the initial heat equation in the Goursat group differs from the equation studied above only by the Fourier transform which is written in the form of an integral, Theorem 1 is a corollary to Theorem 2.

Note also that, as was shown in [6], the application of the noncommutative Fourier transform (in contrast to the usual Fourier transform which we used in Section 2) does not simplify the equation obtained in the final result (at least, for $n = 4$): it still contains a polynomial of degree 4.

4. Conclusion

The result of this article can be interpreted in various ways: On the one hand, the reason for the nonintegrability of the equation studied can be regarded as purely algebraic (the degree of the polynomial Q is greater than 2); on the other hand, this phenomenon can be given a geometric meaning. The fact is that for $n = 4$ the Goursat group is nothing but the Engel group in which, as is known, there are abnormal geodesics; they are also present in groups of large dimensions. Because of these geodesics, the following features arise: (1) remaining a hypoelliptic operator, the sub-Laplacian loses directional analytic hypoellipticity; (2) spheres in the Carnot–Carathéodory metric (in a neighborhood of the ends of the abnormal geodesics) are not defined by inequalities with analytic functions. It seems intuitively clear that if the order of the heat equation decreased then it would be solved in integrals of "rather simple" elementary functions, which would be inconsistent with the absence of analytic hypoellipticity for the sub-Laplacian; here the nonintegrability of the heat equation was proved by rather rigorous methods.

All these difficulties are absent in the Heisenberg group. If we write the analogous defining relations and reduce them to an equation for A_2, B_2, and C with the polynomial Q of degree 2 then the system obtained by equating the coefficients at x^2, x^1, and x^0 to zero is no longer overdetermined, and the fundamental solution is found by translating the stationary solution along the found vector field defining the symmetry.

Basing on the above, we may assume that the study of the asymptotics of the fundamental solution will require (in the case of the presence of abnormal geodesics) the search not for the solution itself but for some its approximations with subsequently proving their "uniformity" (in the sense that the passage to the limit from approximations to the solution can be interchanged as $t \to 0$). Perhaps, this problem might be solved in the future.

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