

# A COMPLETE TOPOLOGICAL CLASSIFICATION OF THE SPACE OF BAIRE FUNCTIONS ON ORDINALS

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**Abstract:** Considering the spaces  $B_p[1, \alpha]$  of all Baire functions  $x : [1, \alpha] \rightarrow \mathbb{R}$  on the ordinal segments  $[1, \alpha]$  that are endowed with the topology of pointwise convergence, we give a complete topological classification of these spaces.

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## § 1. Introduction

The articles [1–4] gave a complete topological classification of the Banach spaces  $C[1, \alpha]$  of all continuous functions on compact ordinal segments. In [5, 6] there is a linear topological classification of these spaces in the topology of pointwise convergence (i.e., of the spaces  $C_p[1, \alpha]$ ). A similar linear topological classification was given for the space of Baire functions  $B_p[1, \alpha]$  with the topology of pointwise convergence in [7]. In this article, we give a complete topological classification of  $B_p[1, \alpha]$  (see Theorem 2.2 below). It turned out that this linear topological classification coincides with the topological classification.

Our terminology basically follows [8]. The ordinal segments  $[1, \alpha]$  are endowed with the order topology  $\mathfrak{S}$ . If  $\alpha$  is an arbitrary ordinal and  $\lambda$  is an initial ordinal (i.e., a cardinal) at most  $\alpha$ , then we put

$$A_{\lambda, \alpha} = \{t \in [1, \alpha] : \chi(t) = |\lambda|\},$$

where  $\chi(t)$  is the character of  $t \in [1, \alpha]$ . In particular,  $A_{\omega, \alpha}$  is the set of all  $t \in [1, \alpha]$  satisfying  $\chi(t) = \aleph_0$ .

Denote by  $\mathfrak{S}_\omega$  the  $\aleph_0$ -modification of the topology  $\mathfrak{S}$ , i.e., the topology in which all  $G_\delta$ -sets are declared open (i.e., the sets representable as the intersection of countably many elements in  $\mathfrak{S}$ ). The ordinal segment  $[1, \alpha]$  endowed with  $\mathfrak{S}_\omega$  will be denoted by  $[1, \alpha]_\omega$ .

Observe the properties of  $[1, \alpha]_\omega$ :

( $a_\omega$ ) If  $A$  is a countable subset in  $[1, \alpha]_\omega$  then  $A$  is closed and discrete.

( $b_\omega$ )  $[1, \alpha]_\omega$  is Lindelöf and hence normal for every ordinal  $\alpha$ .

Indeed, this is obvious if  $\alpha \leq \omega_1$ , where  $\omega_1$  is the first uncountable ordinal. For  $\alpha > \omega_1$ , the Lindelöfness of  $[1, \alpha]_\omega$  is easy to prove by transfinite induction.

Let  $\alpha$  be a limit ordinal. The least order type of the sets  $A \subset [1, \alpha]$  cofinal in  $[1, \alpha)$  will be called the *cofinality* of  $\alpha$  and denoted by  $\text{cf}(\alpha)$ .

It is easy that  $|\text{cf}(\alpha)| = \chi(\alpha)$  for a limit ordinal  $\alpha$ . An initial ordinal  $\alpha$  is called *regular* if  $\text{cf}(\alpha) = \alpha$ . Otherwise, the initial ordinal is called *singular*.

**DEFINITION 1.1.** Refer to  $x : [1, \alpha] \rightarrow \mathbb{R}$  as a *Baire 1-function* if there is a sequence of continuous functions  $x_n : [1, \alpha] \rightarrow \mathbb{R}$  converging pointwise to  $x$ . The set of all Baire 1-functions will be denoted by  $B^1[1, \alpha]$ . If  $\gamma > 1$  is a countable ordinal, then denote by  $B^\gamma[1, \alpha]$  the family of all functions representable as the pointwise limit of a sequence of functions  $f_n : [1, \alpha] \rightarrow \mathbb{R}$ , where  $f_n \in B^{\beta_n}[1, \alpha]$  and  $\beta_n < \gamma$  are countable ordinals. The same set with the topology of pointwise convergence will be denoted by  $B_p^\gamma[1, \alpha]$ .

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**Proposition 1.2.**  $x : [1, \alpha] \rightarrow \mathbb{R}$  is a Baire 1-function if and only if  $x$  is continuous at all  $t \in [1, \alpha] \setminus A_{\omega, \alpha}$ .

PROOF. Take  $x \in B_p[1, \alpha]$ . Then there is a sequence of  $x_n \in C[1, \alpha]$  converging to  $x$  at every point. Fix  $\beta \in [1, \alpha]$  with the property  $\text{cf}(\beta) > \omega$ . Then for each natural  $n$  there is an ordinal  $\beta_n < \beta$  such that  $x_n(\gamma) = x_n(\beta)$  for all  $\gamma \in (\beta_n, \beta]$  [8] (in other words,  $x_n$  becomes constant in some neighborhood of  $\beta$ ). Put  $\beta_0 = \sup\{\beta_n : n \in \mathbb{N}\}$ . Since  $\text{cf}(\beta) > \omega$ , we have  $\beta_0 < \beta$  and

$$x(\gamma) = \lim_{n \rightarrow \infty} x_n(\gamma) = \lim_{n \rightarrow \infty} x_n(\beta) = x(\beta)$$

for every  $\gamma \in (\beta_0, \beta]$ . Consequently,  $x$  is continuous at  $\beta$ .

Conversely, suppose that  $x : [1, \alpha] \rightarrow \mathbb{R}$  is continuous at all points of uncountable cofinality. Prove that there exists a sequence of continuous mappings  $\{x_n : n \in \mathbb{N}\}$  converging to  $x$  pointwise. The proof will be carried out by transfinite induction. Clearly, if  $\alpha$  is a finite ordinal, then the theorem holds true.

Suppose that the assertion is proved for all ordinals less than  $\alpha$ .

CASE 1:  $\alpha$  is an infinite nonlimit ordinal. By the induction hypothesis, there is a sequence of continuous functions  $\{y_n : n \in \mathbb{N}\}$  such that  $\lim_{n \rightarrow \infty} y_n(\gamma) = x(\gamma)$  for all  $\gamma \in [1, \alpha - 1]$ . Extend  $y_n$  to  $[1, \alpha]$  by setting

$$x_n(\gamma) = \begin{cases} y_n(\gamma), & \gamma \in [1, \alpha - 1]; \\ x(\alpha), & \gamma = \alpha. \end{cases}$$

Obviously,  $\{x_n : n \in \mathbb{N}\}$  is a desired sequence of continuous functions.

CASE 2:  $\alpha$  is a limit ordinal and  $\text{cf}(\alpha) = \omega$ . Then there exists an increasing sequence of ordinals  $\{\alpha_n : n \in \mathbb{N}\}$  such that  $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ . Consider the following partition of  $[1, \alpha]$ :

$$[1, \alpha] = [1, \alpha_1] \cup (\alpha_1, \alpha_2] \cup \cdots \cup (\alpha_{k-1}, \alpha_k] \cup \cdots \cup \{\alpha\}.$$

On  $[1, \alpha_1]$  and each of the half-intervals  $(\alpha_{k-1}, \alpha_k]$ , there is a sequence of continuous functions  $\{y_n^k : n \in \mathbb{N}\}$  converging pointwise to  $x$ . Put

$$x_n(\gamma) = \begin{cases} y_n^1(\gamma), & \gamma \in [1, \alpha_1], \\ y_n^2(\gamma), & \gamma \in (\alpha_1, \alpha_2], \\ \vdots \\ y_n^n(\gamma), & \gamma \in (\alpha_{n-1}, \alpha_n], \\ x(\alpha), & \gamma \in (\alpha_n, \alpha]. \end{cases}$$

Clearly, all  $x_n$  are continuous on  $[1, \alpha]$  and converge pointwise on  $[1, \alpha]$  to  $x$ .

CASE 3:  $\alpha$  is a limit ordinal and  $\text{cf}(\alpha) > \omega$ . There is an ordinal  $\gamma_0 < \alpha$  such that  $x(\gamma) = x(\alpha)$  for all  $\gamma \in (\gamma_0, \alpha]$ . By the induction hypothesis, there is a sequence of continuous functions  $\{y_n : n \in \mathbb{N}\}$  on  $[1, \gamma_0]$  converging pointwise to  $x$ . Extend  $y_n$  to  $[1, \alpha]$  by setting

$$x_n(\gamma) = \begin{cases} y_n(\gamma), & \gamma \in [1, \gamma_0], \\ x(\alpha), & \gamma \in (\gamma_0, \alpha]. \end{cases}$$

Clearly,  $\{x_n : n \in \mathbb{N}\}$  converges to  $x$  pointwise on  $[1, \alpha]$ .  $\square$

It is not hard to deduce from the above criterion that the pointwise limit of a sequence of 1-functions is also a 1-function; therefore,  $B_p^\gamma[1, \alpha] = B_p^1[1, \alpha]$  for every countable ordinal  $\gamma$ . This means that the space of all Baire functions on  $[1, \alpha]$  coincides with the space of Baire 1-functions. In what follows, we denote this space by  $B_p[1, \alpha]$ . Under this agreement, Proposition 1.2 implies

**Corollary 1.3.**  $B_p[1, \alpha] = C_p([1, \alpha]_\omega)$ .

**Corollary 1.4.** If  $\alpha < \omega_1$  then each  $x : [1, \alpha] \rightarrow \mathbb{R}$  is a Baire 1-function.

**Proposition 1.5.**  $x : [1, \alpha] \rightarrow \mathbb{R}$  is a Baire function if and only if  $x^{-1}(U)$  has type  $F_\sigma$  for every  $U$  open in  $\mathbb{R}$ .

PROOF. Suppose that  $x : [1, \alpha] \rightarrow \mathbb{R}$  and  $x^{-1}(U)$  has type  $F_\sigma$  in  $[1, \alpha]$  for every open  $U \subset \mathbb{R}$ . This is equivalent to the fact that  $x^{-1}(H)$  has type  $G_\delta$  in  $[1, \alpha]$  for every open set  $H \subset \mathbb{R}$ . By Corollary 1.4, we can assume that  $\alpha \geq \omega_1$ . Consider an ordinal  $\beta \leq \alpha$  such that  $\text{cf}(\beta) > \omega$ . Then  $x^{-1}(x(\beta))$  is a  $G_\delta$ -set containing  $\beta$  and, hence, including some half-interval  $(\gamma, \beta]$ . Thus,  $x$  is constant on  $(\gamma, \beta]$  and so  $x$  is continuous at  $\beta$ . By Proposition 1.2, we conclude that  $x$  is a Baire 1-mapping in the sense of Definition 1.1.

Conversely, let  $U$  be an open subset in  $\mathbb{R}$ . Represent  $U$  as  $U = \bigcup_{k=1}^{\infty} F_k$ , where all  $F_k$ 's are closed in  $\mathbb{R}$  and  $F_k \subset \text{Int } F_{k+1}$ ,  $k = 1, 2, \dots$ . Consider a sequence of continuous functions  $x_n : [1, \alpha] \rightarrow \mathbb{R}$  converging to  $x$  pointwise. It is not hard to verify that

$$x^{-1}(U) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} x_n^{-1}(F_k),$$

which implies that  $x^{-1}(U)$  is an  $F_\sigma$ -set in  $[1, \alpha]$ .  $\square$

DEFINITION 1.6 [9]. Let  $Z$  be a topological space. A function  $x : Z \rightarrow \mathbb{R}$  is called *strictly  $\omega$ -continuous* if for each countable set  $D \subset Z$  there exists a continuous function  $y : Z \rightarrow \mathbb{R}$  with  $y|_D = x|_D$ .

**Proposition 1.7.** Every function  $x : [1, \alpha]_\omega \rightarrow \mathbb{R}$  is strictly  $\omega$ -continuous.

PROOF. Let  $D \subset [1, \alpha]_\omega$  be a countable subset. By property  $(a_\omega)$ , the function  $x|_D$  is continuous on the closed set  $D$ . Moreover, by property  $(b_\omega)$  and the Tietze–Urysohn Theorem, there is a continuous extension  $y : [1, \alpha]_\omega \rightarrow \mathbb{R}$ .  $\square$

Recall (see [8]) that a topological space  $X$  is called *real-complete* if  $X$  is homeomorphic to a closed subspace of a product of real lines. A real-complete space  $\nu X$  is called the *real compactification* of  $X$  if there exists a homeomorphic embedding  $\nu : X \rightarrow \nu X$  for which the closure  $\overline{\nu(X)}$  coincides with  $\nu X$  and for each continuous function  $f : X \rightarrow \mathbb{R}$  there is a continuous embedding  $\tilde{f} : \nu X \rightarrow \mathbb{R}$ .

**Proposition 1.8.** The real compactification  $\nu B_p[1, \alpha]$  is canonically homeomorphic to the Tychonoff product  $\mathbb{R}^{[1, \alpha]}$ .

PROOF. Recall that for each topological space  $X$  the space  $\nu C_p(X)$  is canonically homeomorphic to the set of all strictly  $\omega$ -continuous functions on  $X$  (see [10]). By Corollary 1.3 and Proposition 1.7 we see that  $\nu B_p[1, \alpha] = \nu C_p([1, \alpha]_\omega) = \mathbb{R}^{[1, \alpha]}$ .  $\square$

## § 2. Proof of the Main Theorem

In [7], we proved the following

**Theorem 2.1.** Let  $\alpha$  and  $\beta$  be infinite ordinals and let  $\alpha \leq \beta$ . Then  $B_p[1, \alpha]$  and  $B_p[1, \beta]$  with the topology of pointwise convergence are linearly homeomorphic if and only if one of the following possibilities holds:

- (1)  $\omega \leq \alpha \leq \beta < \omega_1$ ;
- (2)  $\omega_1 \leq \alpha \leq \beta < \omega_2$ ;
- (3)  $\tau \cdot n \leq \alpha \leq \beta < \tau \cdot (n+1)$ , where  $\tau \geq \omega_2$  is an initial regular ordinal and  $n < \omega$ ;
- (4)  $\tau \cdot \sigma \leq \alpha \leq \beta < \tau \cdot \sigma^+$ , where  $\tau \geq \omega_2$  is an initial regular ordinal,  $\sigma$  is an initial ordinal such that  $\omega \leq \sigma < \tau$ , and  $\sigma^+$  is the least initial ordinal greater than  $\sigma$ ;
- (5)  $\tau^2 \leq \alpha \leq \beta < \tau^+$ , where  $\tau \geq \omega_2$  is an initial regular ordinal and  $\tau^+$  is the least initial ordinal greater than  $\tau$ ;
- (6)  $\tau \leq \alpha \leq \beta < \tau^+$ , where  $\tau \geq \omega_2$  is an initial singular ordinal.

In this article, we prove

**Theorem 2.2.** Let  $\alpha$  and  $\beta$  be infinite ordinals. The spaces  $B_p[1, \alpha]$  and  $B_p[1, \beta]$  are homeomorphic if and only if they are linearly homeomorphic.

Clearly, if  $\alpha$  and  $\beta$  get into one of the segments mentioned in Theorem 1 then  $B_p[1, \alpha]$  and  $B_p[1, \beta]$  are homeomorphic. If  $\alpha$  and  $\beta$  get into different segments and  $\alpha < \tau \leq \beta$  for some initial ordinal  $\tau$  then  $B_p[1, \alpha]$  and  $B_p[1, \beta]$  are nonhomeomorphic since they have different weights.

Thus, for proving Theorem 2.2, it suffices to demonstrate that  $B_p[1, \tau \cdot \sigma]$  and  $B_p[1, \tau \cdot \lambda]$  are nonhomeomorphic if  $\tau \geq \omega_2$  is an initial regular ordinal and  $\sigma$  and  $\lambda$  are initial ordinals satisfying the condition  $1 \leq \sigma < \lambda \leq \tau$ .

**Lemma 2.3.** *Suppose that  $\alpha$  is an arbitrary ordinal,  $\lambda$  is an initial regular ordinal,  $\omega_1 < \lambda \leq \alpha$ , a function  $x : [1, \alpha] \rightarrow \mathbb{R}$  is continuous at the points of  $A_{\omega_1, \alpha}$ , and  $t_0 \in A_{\lambda, \alpha}$ . Then there is an ordinal  $\gamma < t_0$  such that  $x|_{(\gamma, t_0)} = \text{const}$ .*

PROOF. Suppose the contrary. Since  $\text{cf}(t_0) = \lambda > \omega_1$ , for some  $\varepsilon_0 > 0$  and every  $\gamma < t_0$ , there are ordinals  $t_\gamma$  and  $q_\gamma$  satisfying the inequality  $\gamma < t_\gamma < q_\gamma < t_0$  and such that  $|x(t_\gamma) - x(q_\gamma)| \geq \varepsilon_0$ .

By transfinite induction, for every  $\xi \in [1, \omega_1)$  we can choose points  $\gamma_\xi$ ,  $t_{\gamma_\xi}$ , and  $q_{\gamma_\xi}$  so that  $\gamma_\xi < t_{\gamma_\xi} < q_{\gamma_\xi} < \gamma_{\xi+1} < t_0$  and  $\gamma_\xi = \sup_{\eta < \xi} \gamma_\eta$  for a limit ordinal  $\xi$ . Then  $\gamma_0 = \sup_{\xi \in [1, \omega_1)} \gamma_\xi = \sup_{\xi \in [1, \omega_1)} q_{\gamma_\xi} = \sup_{\xi \in [1, \omega_1)} t_{\gamma_\xi}$  is an element of  $A_{\omega_1, \alpha}$  and  $x$  is discontinuous at  $\gamma_0$ , which contradicts the hypothesis of the lemma.  $\square$

Introduce some notations. Given  $x \in \mathbb{R}^{[1, \alpha]}$  and an initial ordinal  $\lambda \leq \alpha$ , denote by  $G_\lambda(x)$  the family

$$G_\lambda(x) = \left\{ \bigcap_{s \in S} V_s : V_s \text{ is a standard neighborhood of } x \text{ in } \mathbb{R}^{[1, \alpha]} \text{ and } |S| = \lambda \right\}.$$

Refer to the elements of  $G_\lambda(x)$  as  $\lambda$ -neighborhoods of  $x$ .

Denote the set of all discontinuity points of  $x \in \mathbb{R}^{[1, \alpha]}$  by  $D(x)$ .

Given a regular ordinal  $\tau > \omega_1$  and an initial ordinal  $\sigma$  such that  $\sigma \leq \tau$ , put

$$M_{\tau\sigma} = \{x \in \mathbb{R}^{[1, \tau \cdot \sigma]} : x \text{ is continuous at } t \in [1, \tau \cdot \sigma] \text{ satisfying } \omega_1 \leq \text{cf}(t) < \tau\}.$$

Clearly,  $B_p[1, \tau \cdot \sigma] \subset M_{\tau\sigma}$ .

**Lemma 2.4.** *Suppose that  $\tau > \omega_1$  is an initial regular ordinal,  $\sigma \leq \tau$  is an initial ordinal, and  $x \in M_{\tau\sigma}$ . Then  $D(x)$  is at most countable.*

PROOF. Suppose that  $D(x)$  is uncountable and put

$$t_0 = \min\{t \in [1, \tau \cdot \sigma] : [1, t] \cap D(x) \text{ is uncountable}\}.$$

Clearly,  $\text{cf}(t_0) > \omega$ . Since  $x \in M_{\tau\sigma}$ ; therefore,  $x$  is continuous at  $t$  satisfying  $\text{cf}(t) = \omega_1$ . By Lemma 2.3, there is an ordinal  $\gamma < t_0$  such that  $x|_{(\gamma, t_0)}$  is a constant function (if  $\text{cf}(t_0) = \omega_1$  then such an ordinal  $\gamma$  exists by the continuity of  $x$  at  $t_0$ ). But then, in view of the equality,

$$[1, t_0] \cap D(x) = ([1, \gamma] \cap D(x)) \cup ([\gamma, t_0] \cap D(x)),$$

we conclude that  $[1, \gamma] \cap D(x)$  is uncountable, which contradicts the definition of  $t_0$ .  $\square$

**Lemma 2.5.** *Suppose that  $\tau > \omega_1$  is an initial regular ordinal,  $\sigma$  is an initial ordinal, and  $\sigma \leq \tau$ . Then*

$$M_{\tau\sigma} = \{x \in \mathbb{R}^{[1, \tau \cdot \sigma]} : V \cap B_p[1, \tau \cdot \sigma] \neq \emptyset$$

*for every initial ordinal  $\lambda < \tau$  and every  $\lambda$ -neighborhood  $V$  of  $x$ \}.*

PROOF. Denote the right-hand side of the equality by  $L_{\tau\sigma}$  and prove that this set coincides with  $M_{\tau\sigma}$ . Assume that  $x \notin M_{\tau\sigma}$ , i.e.,  $x$  is discontinuous at some  $t_0$  for which  $\omega_1 \leq \text{cf}(t_0) < \tau$ . Since  $|\text{cf}(t_0)| = \chi(t_0)$ , there exists a base  $\{U_j(t_0)\}_{j \in J}$  of neighborhoods of  $t_0$  such that  $|J| < \tau$ . Since  $x$  is discontinuous at  $t_0$ , there exists a number  $\varepsilon_0 > 0$  such that for each  $j \in J$  there is  $t_j \in U_j(t_0)$  satisfying  $|x(t_j) - x(t_0)| \geq \varepsilon_0$ . Let  $V = \bigcap_{j \in J, n \in \mathbb{N}} V_{j,n}$ , where  $V_{j,n} = V(x, t_j, t_0, 1/n)$  is a standard neighborhood of  $x$  in  $\mathbb{R}^{[1, \tau \cdot \sigma]}$ . If  $y \in V$  then  $y(t_j) = x(t_j)$  and  $y(t_0) = x(t_0)$ . Consequently,  $y$  is discontinuous at  $t_0$ , and so  $y \notin B_p[1, \tau \cdot \sigma]$ . Thus,  $V \cap B_p[1, \tau \cdot \sigma] = \emptyset$ , i.e.,  $x \notin L_{\tau\sigma}$ .

Suppose that  $x \in M_{\tau\sigma}$ , i.e.,  $x$  can be discontinuous only at the points of  $A_{\tau, \tau \cdot \sigma}$  and at the points cofinal to  $\omega$ .

It is easy that  $A_{\tau, \tau \cdot \sigma}$  has the form

$$\begin{aligned} A_{\tau, \tau \cdot \sigma} &= \{\tau \cdot (\xi + 1) : \xi < \sigma\} \quad \text{if } \sigma < \tau; \\ A_{\tau, \tau \cdot \sigma} &= \{\tau \cdot (\xi + 1) : \xi < \tau\} \cup \{\tau \cdot \tau\} \quad \text{if } \sigma = \tau. \end{aligned}$$

By Lemma 2.4,  $D(x)$  is at most countable, and so

$$A_{\tau, \tau \cdot \sigma} \cap D(x) = \{\tau \cdot (\xi_n + 1) : n \in \mathbb{N}\} \quad \text{if } \sigma < \tau$$

for some sequence  $\{\xi_n\}_{n=1}^\infty \subset [1, \sigma)$ . Similarly,

$$A_{\tau, \tau \cdot \sigma} \cap D(x) = \{\tau \cdot (\xi_n + 1) : n \in \mathbb{N}\} \cup \{\tau \cdot \tau\} \quad \text{if } \sigma = \tau.$$

Let  $\lambda < \tau$  and let  $V(x) = \bigcap \{U(x, \eta, 1/n) : \eta \in S, n \in \mathbb{N}\}$  be the  $\lambda$ -neighborhood of  $x$ . Then  $|S| < \tau$ . Since  $S$  is not cofinal to a regular ordinal  $\tau$ , for each  $n \in \mathbb{N}$  there is an ordinal  $\gamma_n$  such that

$$\tau \xi_n < \gamma_n < \tau(\xi_n + 1) \quad \text{and} \quad (\gamma_n, \tau(\xi_n + 1)) \cap S = \emptyset.$$

For  $\sigma = \tau$ , there also is an ordinal  $\gamma_0 < \tau^2$  such that  $(\gamma_0, \tau^2) \cap S = \emptyset$  and  $(\gamma_0, \tau^2) \cap \{\tau(\xi_n + 1)\}_{n=1}^\infty = \emptyset$ .

Consider the function

$$\tilde{x}(t) = \begin{cases} x(\tau(\xi_n + 1)) & \text{if } t \in (\gamma_n, \tau(\xi_n + 1)); \\ x(\tau^2) & \text{if } t \in (\gamma_0, \tau^2); \\ x(t) & \text{otherwise.} \end{cases}$$

Since  $\tilde{x}|_S = x|_S$ , we have  $\tilde{x} \in V(x)$ . On the other hand, assume that  $t \in [1, \tau \cdot \sigma]$  and  $\text{cf}(t) \geq \omega_1$ . Obviously,  $\tilde{x}$  is constant on each of the intervals  $(\gamma_n, \tau(\xi_n + 1)]$  and  $(\gamma_0, \tau^2]$  and so  $\tilde{x}$  is continuous at all  $t \in \bigcup_{n=1}^\infty (\gamma_n, \tau(\xi_n + 1)] \cup (\gamma_0, \tau^2]$ . If  $t \notin \bigcup_{n=1}^\infty (\gamma_n, \tau(\xi_n + 1)] \cup (\gamma_0, \tau^2]$  then  $\tilde{x}$  coincides with  $x$  in some neighborhood of  $t$  and so  $\tilde{x}$  is continuous at  $t$ . If  $t \in \bigcup_{n=1}^\infty (\gamma_n, \tau(\xi_n + 1)] \setminus \bigcup_{n=1}^\infty (\gamma_n, \tau(\xi_n + 1))$  then  $\text{cf}(t) = \omega$ . Thus,  $\tilde{x} \in B_p[1, \tau \cdot \sigma]$ , i.e.,  $V(x) \cap B_p[1, \tau \cdot \sigma] \neq \emptyset$ , and hence  $x \in L_{\tau \cdot \sigma}$ .  $\square$

The proof of the following lemma can be found in [8].

**Lemma 2.6.** *Let  $X$  and  $Y$  be topological spaces and let  $\varphi : X \rightarrow Y$  be a homeomorphism. Then there is a homeomorphism  $\tilde{\varphi} : \nu X \rightarrow \nu Y$  such that  $\tilde{\varphi}(x) = \varphi(x)$  for all  $x \in X$ .  $\square$*

**Proposition 2.7.** *Let  $\tau > \omega_1$  be a regular ordinal, while let  $\sigma$  and  $\eta$  be initial ordinals such that  $\omega \leq \eta < \sigma \leq \tau$ . Then  $B_p[1, \tau \cdot \sigma]$  and  $B_p[1, \tau \cdot \eta]$  are nonhomeomorphic.*

PROOF. Suppose that there exists a homeomorphism  $\varphi : B_p[1, \tau \cdot \sigma] \rightarrow B_p[1, \tau \cdot \eta]$ . Without loss of generality, we may assume that  $\varphi(0) = 0$ . By Lemma 2.6, there exists a homeomorphism

$$\tilde{\varphi} : \nu(B_p[1, \tau \cdot \sigma]) \rightarrow \nu(B_p[1, \tau \cdot \eta])$$

such that  $\tilde{\varphi}|_{B_p[1, \tau \cdot \sigma]} = \varphi$ . By Proposition 1.8 we can assume that  $\tilde{\varphi}$  is a homeomorphism from  $\mathbb{R}^{[1, \tau \cdot \sigma]}$  onto  $\mathbb{R}^{[1, \tau \cdot \eta]}$  that extends  $\varphi$ . Consider the subspaces  $M_{\tau \cdot \sigma} \subset \mathbb{R}^{[1, \tau \cdot \sigma]}$  and  $M_{\tau \cdot \eta} \subset \mathbb{R}^{[1, \tau \cdot \eta]}$ . Lemma 2.5 implies that  $\tilde{\varphi}(M_{\tau \cdot \sigma}) = M_{\tau \cdot \eta}$ .

Given  $t \in A_{\tau, \tau \cdot \sigma}$ , let  $\chi_t$  be the characteristic function of the singleton  $\{t\}$ . Obviously,  $\{\chi_t\}_{t \in A_{\tau, \tau \cdot \sigma}} \subset M_{\tau \cdot \sigma} \setminus B_p[1, \tau \cdot \sigma]$  and, for every sequence of pairwise distinct points  $t_n \in A_{\tau, \tau \cdot \sigma}$ , the sequence  $\{\chi_{t_n}\}$  converges pointwise to the zero of  $\mathbb{R}^{[1, \tau \cdot \sigma]}$ . Consider the set of functions  $\{\tilde{\varphi}(\chi_t)\}_{t \in A_{\tau, \tau \cdot \sigma}} \subset M_{\tau \cdot \eta} \setminus B_p[1, \tau \cdot \eta]$ . Each of the functions  $\tilde{\varphi}(\chi_t)$  is discontinuous at some point of  $A_{\tau, \tau \cdot \eta} \subset [1, \tau \cdot \eta]$ , i.e. at some point of the form  $\tau \cdot (\delta + 1)$  where  $\delta < \eta$ . Put

$$B_\delta = \{\tilde{\varphi}(\chi_t) \mid \tilde{\varphi}(\chi_t) \text{ is discontinuous at } \tau(\delta + 1)\}.$$

Since  $|A_{\tau, \tau \cdot \eta}| < |\sigma|$  and  $\bigcup_{\delta < \eta} B_\delta = \tilde{\varphi}(\{\chi_t\}_{t \in A_{\tau, \tau \cdot \sigma}})$ , there is a point  $\tau(\delta_0 + 1)$ ,  $\delta_0 < \eta$ , such that  $|B_{\delta_0}| = |\sigma| > |\eta| > |\delta_0|$ . Since  $\tilde{\varphi}(\chi_{t_n}) \rightarrow 0$  for every sequence of pairwise distinct points  $t_n \in A_{\tau, \tau \cdot \sigma}$ ; therefore,  $\{\tilde{\varphi}(\chi_t) \in B_{\delta_0} : \tilde{\varphi}(\chi_t)(\tau(\delta_0 + 1)) \neq 0\}$  is at most countable, and so  $B_{\delta_0}^0 = \{\tilde{\varphi}(\chi_t) \in B_{\delta_0} : \tilde{\varphi}(\chi_t)(\tau(\delta_0 + 1)) = 0\}$  is uncountable. By Lemma 2.3, for every function  $\tilde{\varphi}(\chi_t) \in B_{\delta_0}^0$ , there is an ordinal  $\gamma_t$  such that  $\tilde{\varphi}(\chi_t)|_{(\gamma_t, \tau(\delta_0 + 1))} = \text{const} = C_t$ , where  $C_t \neq 0$  because all functions in  $B_{\delta_0}^0$  are discontinuous at  $\tau(\delta_0 + 1)$ . It is easy that there are uncountably many functions in  $B_{\delta_0}^0$  for which  $|C_t| \geq \varepsilon_0$  for some  $\varepsilon_0 > 0$ . Consider a sequence of such pairwise distinct functions  $\tilde{\varphi}(\chi_{t_n})$  satisfying  $|\tilde{\varphi}(\chi_{t_n})| \equiv C_{t_n} \geq \varepsilon_0$  on  $(\gamma_{t_n}, \tau(\delta_0 + 1))$ . Since  $\text{cf}(\tau(\delta_0 + 1)) > \omega$ , we have  $\gamma_0 = \sup_{n < \omega} \gamma_{t_n} < \tau(\delta_0 + 1)$ , and hence  $|\tilde{\varphi}(\chi_{t_n})(t)| \geq \varepsilon_0$  for every  $t \in (\gamma_0, \tau(\delta_0 + 1))$ . This contradicts the fact that the function sequence  $\{\tilde{\varphi}(\chi_{t_n})\}_{n < \omega}$  vanishes pointwise.  $\square$

**Proposition 2.8.** *Suppose that  $\tau$  is an initial regular ordinal,  $\tau \geq \omega_2$ , while  $m$  and  $n$  are distinct naturals. Then  $B_p[1, \tau \cdot m]$  and  $B_p[1, \tau \cdot n]$  are nonhomeomorphic.*

For proving this theorem, we will need some notations and auxiliary assertions.

We will identify  $B_p[1, \tau \cdot n]$  with  $B_p([1, \tau] \times [n])$ , where  $[n] = \{1, 2, \dots, n\}$ . If  $\tau$  is an uncountable regular ordinal then for each  $x \in B_p[1, \tau]$  there is an ordinal  $\gamma < \tau$  such that  $x|_{[\gamma, \tau]}$  is a constant function. It follows that  $B_p[1, \tau]$  is linearly homeomorphic to the subspace of  $B_p[1, \tau]$  consisting of all functions that vanish from some  $\alpha < \tau$  on. The desired linear homeomorphic embedding  $\varphi : B_p[1, \tau] \rightarrow B_p[1, \tau]$  can be defined by the formula

$$\begin{aligned}\varphi(x)(1) &= x(\tau); \\ \varphi(x)(n) &= x(n-1) - x(\tau) \quad \text{if } 2 \leq n < \omega; \\ \varphi(x)(\alpha) &= x(\alpha) - x(\tau) \quad \text{if } \omega \leq \alpha < \tau.\end{aligned}$$

Denote this subspace by  $B_p^0[1, \tau]$ . Obviously, the subspace  $B_p([1, \tau] \times [n])$  is linearly homeomorphic to the space

$$B_p^0([1, \tau] \times [n]) = \{x \in B_p([1, \tau] \times [n]) : x|_{[\gamma, \tau] \times [n]} \equiv 0 \text{ for some } \gamma < \tau\}$$

for all  $n \in \mathbb{N}$ . Given  $\alpha < \tau$  put

$$B_p^\alpha([1, \tau] \times [n]) = \{x \in B_p^0([1, \tau] \times [n]) : x(t, i) = 0 \text{ for all } t \in [\alpha, \tau] \text{ and all } i \in [n]\}.$$

If  $\alpha$  is a nonlimit ordinal or  $\text{cf}(\alpha) = \omega$  then  $B_p^\alpha([1, \tau] \times [n])$  can be identified with  $B_p([1, \alpha] \times [n])$ .

PROOF OF PROPOSITION 2.8 proceeds by contradiction. Suppose that there is a homeomorphism  $T$  between  $B_p[1, \tau \cdot m]$  and  $B_p[1, \tau \cdot n]$ . We may assume without loss of generality that  $T0 = 0$ .

**Lemma 2.9.** *Suppose that  $\tau$  is an initial regular ordinal,  $\tau \geq \omega_2$ , while  $m$  and  $n$  are naturals. If  $T : B_p^0([1, \tau] \times [m]) \rightarrow B_p^0([1, \tau] \times [n])$  is a homeomorphism then for every  $\gamma \in (\omega, \tau)$  there is  $\alpha_\gamma \in (\gamma, \tau)$  such that  $T(B_p([1, \alpha_\gamma] \times [m])) = B_p([1, \alpha_\gamma] \times [n])$ .*

PROOF. Consider an arbitrary ordinal  $\alpha_1 \in (\gamma, \tau)$ . Then  $d(B_p[1, \alpha_1]) \leq |[1, \alpha_1]|$ , i.e., there is an everywhere dense subset  $A$  in  $B_p[1, \alpha_1]$  of cardinality at most  $|[1, \alpha_1]|$ .

Since  $\tau$  is an uncountable regular ordinal, for each function  $x \in A$  there exists an ordinal  $\beta(x) < \tau$  such that  $Tx|_{[\beta(x), \tau] \times \{i\}} \equiv 0$  for all  $i = 1, 2, \dots, n$ . Then  $\beta_1 = \sup\{\beta(x) : x \in A\} < \tau$  and  $Tx|_{[\beta_1, \tau] \times \{i\}} \equiv 0$  for all  $x \in B_p([1, \alpha_1] \times [m])$  and all  $i = 1, 2, \dots, n$ . We can assume that  $\beta_1 > \alpha_1$ . Analogously, for the ordinal  $\beta_1 < \tau$  there is an ordinal  $\alpha_2$  such that  $\beta_1 < \alpha_2 < \tau$  and  $T^{-1}(B_p([1, \beta_1] \times [n])) \subset B_p([1, \alpha_2] \times [m])$ . Continuing this process, we obtain the increasing sequence of ordinals

$$\gamma < \alpha_1 < \beta_1 < \alpha_2 < \dots < \alpha_k < \beta_k < \alpha_{k+1} < \dots < \tau$$

such that

$$T(B_p([1, \alpha_k] \times [m])) \subset B_p([1, \beta_k] \times [n]), \quad (*)$$

$$T^{-1}(B_p([1, \beta_k] \times [n])) \subset B_p([1, \alpha_{k+1}] \times [m]). \quad (**)$$

Then the ordinal  $\alpha_\gamma = \sup_{k < \omega} \alpha_k = \sup_{k < \omega} \beta_k$  satisfies the equality

$$T(B_p^{\alpha_\gamma}([1, \tau] \times [m])) = B_p^{\alpha_\gamma}([1, \tau] \times [n]).$$

Indeed, if  $x \in B_p^{\alpha_\gamma}([1, \tau] \times [m])$  then  $x_k = x|_{[1, \alpha_k] \times [m]}$  are elements of  $B_p([1, \alpha_k] \times [m])$  and converge to  $x$  pointwise. Using (\*), we infer that  $Tx_k \in B_p([1, \beta_k] \times [n]) \subset B_p^{\alpha_\gamma}([1, \tau] \times [n])$ , and so  $Tx = \lim_{k \rightarrow \infty} Tx_k \in B_p^{\alpha_\gamma}([1, \tau] \times [n])$ . The reverse inclusion is proved similarly on using (\*\*).  $\square$

Since the ordinal  $\gamma < \tau$  in Lemma 2.9 was chosen arbitrarily, we obtain

**Corollary 2.10.** *If  $T : B_p^0([1, \tau] \times [m]) \rightarrow B_p^0([1, \tau] \times [n])$  is a homeomorphism then  $L = \{\alpha \in [1, \tau] : T(B_p^\alpha([1, \tau] \times [m])) = B_p^\alpha([1, \tau] \times [n])\}$  is closed and cofinal in  $[1, \tau]$ .  $\square$*

**Lemma 2.11.** *Suppose that  $\tau$  is an initial regular ordinal,  $\tau \geq \omega_2$ , while  $m$  and  $n$  are naturals. If  $T : B_p^0([1, \tau] \times [m]) \rightarrow B_p^0([1, \tau] \times [n])$  is a homeomorphism then for every  $\gamma < \tau$  there exists  $\alpha_\gamma \in (\gamma, \tau)$  such that for  $x, y \in B_p^0([1, \tau] \times [m])$  the following are equivalent:*

$$\begin{aligned} x|_{[1, \alpha_\gamma] \times \{i\}} &= y|_{[1, \alpha_\gamma] \times \{i\}} \quad \text{for all } i = 1, 2, \dots, m, \\ Tx|_{[1, \alpha_\gamma] \times \{j\}} &= Ty|_{[1, \alpha_\gamma] \times \{j\}} \quad \text{for all } j = 1, 2, \dots, n. \end{aligned}$$

PROOF. Consider an arbitrary ordinal  $\alpha_1 < \tau$ . Fix  $t \in [1, \alpha_1]$  and a natural number  $k \in [n]$ . Consider the continuous function  $f : B_p^0([1, \tau] \times [m]) \rightarrow \mathbb{R}$  defined as  $f(x) = (Tx)(t, k)$ .

It is well known (see [11]) that each continuous function on an everywhere dense subset in a product of real lines depends on countably many coordinates; i.e., there exists a countable set  $A_{(t,k)} \subset [1, \tau]$  such that  $x|_{A_{(t,k)} \times [m]} = y|_{A_{(t,k)} \times [m]}$  implies  $f(x) = f(y)$  for  $x, y \in B_p^0([1, \tau] \times [m])$ . It follows that there is an ordinal  $\beta_1 \in (\alpha_1, \tau)$  for which  $x|_{[1, \beta_1] \times [m]} = y|_{[1, \beta_1] \times [m]}$  implies that  $Tx|_{[1, \alpha_1] \times [n]} = Ty|_{[1, \alpha_1] \times [n]}$ .

Similarly, for the ordinal  $\beta_1$  there is an ordinal  $\alpha_2 \in (\beta_1, \tau)$  such that the condition  $Tx|_{[1, \alpha_2] \times [n]} = Ty|_{[1, \alpha_2] \times [n]}$  implies that  $x|_{[1, \beta_1] \times [m]} = y|_{[1, \beta_1] \times [m]}$ .

Continuing this process, we obtain the sequence

$$\gamma < \alpha_1 < \beta_1 < \alpha_2 < \dots < \alpha_k < \beta_k < \alpha_{k+1} < \dots < \tau$$

for which the condition  $x|_{[1, \beta_k] \times [m]} = y|_{[1, \beta_k] \times [m]}$  implies  $Tx|_{[1, \alpha_k] \times [n]} = Ty|_{[1, \alpha_k] \times [n]}$  and the condition  $Tx|_{[1, \alpha_{k+1}] \times [n]} = Ty|_{[1, \alpha_{k+1}] \times [n]}$  implies  $x|_{[1, \beta_k] \times [m]} = y|_{[1, \beta_k] \times [m]}$ .

Then the claim of the lemma holds for the ordinal  $\alpha_\gamma = \sup_{k < \omega} \alpha_k = \sup_{k < \omega} \beta_k$ .  $\square$

**Corollary 2.12.** *If  $T : B_p^0([1, \tau] \times [m]) \rightarrow B_p^0([1, \tau] \times [n])$  is a homeomorphism then*

$$M = \{\alpha \in [1, \tau] : x|_{[1, \alpha] \times [m]} = y|_{[1, \alpha] \times [m]} \iff Tx|_{[1, \alpha] \times [n]} = Ty|_{[1, \alpha] \times [n]}\}$$

*is closed and cofinal in  $[1, \tau]$ .  $\square$*

PROOF OF PROPOSITION 2.8. Suppose that there is a homeomorphism  $T : B_p^0([1, \tau] \times [m]) \rightarrow B_p^0([1, \tau] \times [n])$ . Since  $\tau \geq \omega_2$ , while  $L$  and  $M$  are cofinal and closed in  $[1, \tau]$ , there is an ordinal  $\gamma_0 \in L \cap M$  such that  $\text{cf}(\gamma_0) = \omega_1$ .

In  $B_p^0([1, \tau] \times [m])$  and  $B_p^0([1, \tau] \times [n])$ , consider the closed subspaces

$$\begin{aligned} E &= \{x \in B_p^0([1, \tau] \times [m]) : x(\gamma_0, i) = 0, i = 1, 2, \dots, m\}, \\ H &= \{x \in B_p^0([1, \tau] \times [n]) : x(\gamma_0, i) = 0, i = 1, 2, \dots, n\}. \end{aligned}$$

Given  $x \in E$ , let

$$x'(t, i) = \begin{cases} x(t, i) & \text{if } t \leq \gamma_0, \\ 0 & \text{if } t > \gamma_0. \end{cases}$$

Since  $\gamma_0 \in L$ , we have  $T(x') \in B_p^0([1, \gamma_0] \times [n])$ , and so  $T(x')(\gamma_0, i) = 0$  for  $i = 1, 2, \dots, n$ . On the other hand, since  $x'|_{[1, \gamma_0] \times [m]} = x|_{[1, \gamma_0] \times [m]}$  and  $\gamma_0 \in M$ , we infer  $T(x')|_{[1, \gamma_0] \times [n]} = T(x)|_{[1, \gamma_0] \times [n]}$ , and, by the continuity of  $T(x)$  and  $T(x')$  at  $\gamma_0$ , we conclude that  $T(x')(\gamma_0, i) = T(x)(\gamma_0, i)$  for all  $i = 1, 2, \dots, n$ . Thus,  $T(x)(\gamma_0, i) = 0$  for all  $i = 1, 2, \dots, n$ , whence  $T(E) \subset H$ . We similarly find that  $T^{-1}(H) \subset E$ , i.e.,  $T(E) = H$ . But this is impossible since the subspaces  $E$  and  $H$  have different codimensions in  $B_p^0([1, \tau] \times [m])$  and  $B_p^0([1, \tau] \times [n])$  respectively (see [12, Lemma 4]).

**Proposition 2.13.** *Let  $\tau \geq \omega_2$  be a regular ordinal and  $n < \omega$ . Then  $B_p[1, \tau \cdot \omega]$  and  $B_p[1, \tau \cdot n]$  are nonhomeomorphic spaces.*

PROOF. The claim becomes obvious if we notice that  $B_p[1, \tau \cdot \omega]$  is homeomorphic to its square and  $B_p[1, \tau \cdot n]$  is not homeomorphic to its square by Proposition 2.8.  $\square$

Propositions 2.8 and 2.13 finish the proof of Theorem 2.2.

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