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LIMIT AUTOMORPHISMS OF THE C^*-ALGEBRAS GENERATED BY ISOMETRIC REPRESENTATIONS FOR SEMIGROUPS OF RATIONALS

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Abstract: We consider inductive sequences of Toeplitz algebras whose connecting homomorphisms are defined by collections of primes. The inductive limits of these sequences are C^*-algebras generated by representations for semigroups of rationals. We study the limit endomorphisms of these C^*-algebras induced by morphisms between copies of the same inductive sequences of Toeplitz algebras. We establish necessary and sufficient conditions for these endomorphisms to be automorphisms of the algebras.

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Introduction

It is well known that the properties of objects and morphisms in the categories of Banach algebras have corresponding analogs in algebraic and topological categories and vice versa (see, for example, [1, 2]).

This article was motivated by several sources. On the one hand, these sources include the articles [3–11] on Toeplitz algebras which are also called semigroup C^*-algebras. On the other hand, they also comprise the papers [12–18] which are devoted to mappings of topological groups. The article [19] contains an application of the properties of mappings of P-adic solenoids to crossed products of C^*-algebras.

In [3, 4], Coburn studied the Toeplitz algebra for the additive semigroup of nonnegative integers. In [5], Douglas considered the case of subsemigroups of the additive group of the reals. In [6, 8], Murphy studied the general case of ordered groups and, in particular, proved that the correspondence between ordered groups and Toeplitz algebras is a continuous functor. These authors showed that isometric representations of semigroups have the universality property (see § 1). This property is involved in our constructions of $*$-homomorphisms between the C^*-algebras generated by nonunitary isometries.

This article is devoted to the limit endomorphisms of Toeplitz algebras that are generated by isometric representations of additive semigroups of nonnegative rationals. We consider these semigroup C^*-algebras as the inductive limits of inductive sequences of Toeplitz algebras with connecting homomorphisms defined by sequences of primes. By a limit endomorphism we mean a $*$-homomorphism induced by a morphism between two copies of the same inductive sequence of algebras. We prove the properties of limit endomorphisms that can be regarded as the corresponding operator-algebraic analogs of the properties of mappings of P-adic solenoids (see, for example, [12, 13, 15, 16, 20–22]) and group homomorphisms of rationals. Here we do not use functorial constructions such as the above-mentioned continuous functor and the Pontryagin duality (see Remark 4 in § 3 concerning the use of functors for the proofs).

The article consists of an Introduction and three sections. In § 1, we introduce the notations and recall the definitions and results of use in the sequel. Then, in § 2, we consider the inductive sequences of Toeplitz algebras defined by sequences of primes. § 3 is devoted to limit endomorphisms and contains the main results of the article. We establish the necessary and sufficient conditions for these endomorphisms to be $*$-automorphisms. The conditions are formulated in number-theoretic, algebraic, and functional terms.

§ 1. Notations and Prerequisites

Throughout the article, $P = (p_1, p_2, \dots)$ stands for an arbitrary sequence of primes (1 is not listed

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in the set of all primes). Alongside P, we consider the additive group of rationals

$$\mathbb{Q}_P := \left\{ \frac{m}{p_1 p_2 \dots p_n} \mid m \in \mathbb{Z},\ n \in \mathbb{N} \right\},$$

where, as usual, $\mathbb{Z}$ and $\mathbb{N}$ designate the additive group of integers and the set of naturals respectively.

Let Γ stand either for $\mathbb{Z}$ or $\mathbb{Q}_P$. Let $\Gamma^+ := \Gamma \cap [0; +\infty)$ be the positive cone in the ordered group Γ. Denote by $l^2(\Gamma^+)$ the Hilbert space of all square-integrable complex-valued functions on the additive semigroup Γ^+. Let $\{e_a \mid a \in \Gamma^+\}$ be the canonical orthonormal basis in $l^2(\Gamma^+)$, i.e., $e_a(b) = \delta_{a,b}$, where $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ if $a \neq b$.

Consider the C^*-algebra $B(l^2(\Gamma^+))$ of all bounded linear operators on the Hilbert space $l^2(\Gamma^+)$. For every $a \in \Gamma^+$, define an isometry $V_a \in B(l^2(\Gamma^+))$ by setting

$$V_a e_b = e_{a+b}, \tag{1}$$

where $b \in \Gamma^+$. The C^*-subalgebra in the algebra $B(l^2(\Gamma^+))$ generated by $\{V_a \mid a \in \Gamma^+\}$ is denoted by $C_r^*(\Gamma^+)$ and called the *reduced semigroup C^*-algebra of* Γ^+ or the *Toeplitz algebra generated by* Γ^+. For $\Gamma = \mathbb{Z}$ we denote this algebra also by $\mathscr{T}$ and use the symbols T and T^n instead of V_1 and V_n respectively, where $n \in \mathbb{Z}^+$.

Similarly, the semigroup C^*-algebra can be defined for an arbitrary cancellative semigroup. This algebra is generated by the left regular representation of the given semigroup. For additional literature and a brief history of the study of semigroup C^*-algebras, the reader is referred, for example, to [23, 24].

A mapping $\rho : \Gamma^+ \to B : a \mapsto W_a$ into a unital C^*-algebra B is called an *isometric homomorphism* (or a *representation*) if $W_a^* W_a = 1$ and $W_{a+b} = W_a W_b$ for all $a, b \in \Gamma^+$. Obviously, the mapping

$$\pi : \Gamma^+ \to C_r^*(\Gamma^+) : a \mapsto V_a \tag{2}$$

is an isometric homomorphism. It is known that it possesses the following universality property:

Theorem (the universality property of an isometric homomorphism). *Given an isometric homomorphism $\rho : \Gamma^+ \to B$, there exists a unique $*$-homomorphism $\rho^* : C_r^*(\Gamma^+) \to B$ such that the diagram*

$$\begin{tikzcd} & \Gamma^+ \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ C_r^*(\Gamma^+) \arrow[rr, "\rho^*"'] & & B \end{tikzcd} \tag{3}$$

commutes; i.e., $\rho^* \circ \pi = \rho$. Moreover, if $\rho(a)$ is not unitary for each $a > 0$ then the $*$-homomorphism ρ^* is injective.

A proof of this property can be found in [6, Theorems 1.3 and 2.9]. Observe also that some independent proofs of a few results from [6] are collected in [25, Corollaries 2.5 and 2.6].

Recall the definitions of inductive sequence and inductive limit (for details, see, for example, [2, Chapter 6; 26, Chapter 6]). We deal with the two categories, namely, with the category of C^*-algebras and their $*$-homomorphisms and also with the category of abelian groups and their homomorphisms. In what follows, these categories will be denoted by $\mathscr{C}$.

An *inductive sequence* or a *direct sequence* in the category $\mathscr{C}$ is a collection $\{A_n, \varphi_n\}_{n=1}^{+\infty}$ consisting of objects A_n and morphisms $\varphi_n : A_n \to A_{n+1}$ of $\mathscr{C}$. For representing this inductive sequence, the diagram

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \tag{4}$$

is often used. The *inductive limit* or the *direct limit* of the inductive sequence (4) is a pair $(A, \{\varphi_{n,\infty}\}_{n=1}^{+\infty})$ consisting of an object A and a sequence of morphisms $\{\varphi_{n,\infty} : A_n \to A\}_{n=1}^{+\infty}$ of $\mathscr{C}$ possessing the following *universality property*:

(1) for each $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\varphi_{n,\infty}"'] & & A_{n+1} \arrow[dl, "\varphi_{n+1,\infty}"] \\ & A & \end{tikzcd} \tag{5}$$

commutes; i.e., $\varphi_{n,\infty} = \varphi_{n+1,\infty} \circ \varphi_n$.

(2) for every sequence of morphisms $\{\psi_{n,\infty} : A_n \to B\}_{n=1}^{+\infty}$ in $\mathscr{C}$ satisfying the condition $\psi_{n,\infty} = \psi_{n+1,\infty} \circ \varphi_n$ for each $n \in \mathbb{N}$, there is a unique morphism $\psi : A \to B$ such that the diagram

$$\begin{tikzcd} & A_n \arrow[dl, "\varphi_{n,\infty}"'] \arrow[dr, "\psi_{n,\infty}"] & \\ A \arrow[rr, "\psi"'] & & B \end{tikzcd} \tag{6}$$

commutes; i.e., $\psi_{n,\infty} = \psi \circ \varphi_{n,\infty}$ for each $n \in \mathbb{N}$.

The object A itself is often called the inductive limit. It is well known that inductive limits can always be constructed in the category $\mathscr{C}$. For an inductive sequence $\{A_n, \varphi_n\}_{n=1}^{+\infty}$, its inductive limit is denoted by $\varinjlim\{A_n, \varphi_n\}$. Moreover, each inductive sequence in the category $\mathscr{C}$ has its inductive limit which is unique up to isomorphism in $\mathscr{C}$ [2, Chapter 6; 26, Chapter 6].

For numbers $m < n$, we will also consider the connecting morphisms

$$\varphi_{m,n} : \varphi_{n-1} \circ \dots \circ \varphi_{m+1} \circ \varphi_m : A_m \to A_n \tag{7}$$

and the identity morphisms $\varphi_{n,n} = \mathrm{id}_{A_n}$, where $m, n \in \mathbb{N}$. Obviously, the equalities $\varphi_{m,\infty} = \varphi_{n,\infty} \circ \varphi_{m,n}$ hold.

§ 2. Sequences of Toeplitz Algebras, Their Limits, and Morphisms

In this section, we consider the inductive sequences of Toeplitz algebras defined by sequences of primes, their inductive limits, and morphisms.

In the literature, the universality property of homomorphism (2) for the semigroup $\Gamma^+ = \mathbb{Z}^+$ is known also as Coburn's Theorem and formulated as follows (see, for example, [2, Theorem 3.5.18]).

Theorem (Coburn). *Let V be an isometry in a unital C^*-algebra B. Then there exists a unique unital $*$-homomorphism $\varphi : \mathscr{T} \to B$ such that $\varphi(T) = V$. Moreover, if $VV^* \neq 1$ then φ is isometric.*

Henceforth, we abbreviate the homomorphism defined in Coburn's Theorem as $\varphi : \mathscr{T} \longrightarrow B : T \mapsto V$.

Coburn's Theorem readily implies

Lemma 1. *For each $n \in \mathbb{N}$, there exists a unique unital $*$-homomorphism $\varphi : \mathscr{T} \longrightarrow \mathscr{T} : T \mapsto T^n$. Moreover, φ is isometric.*

Using Lemma 1, introduce the inductive sequences of Toeplitz algebras defined by sequences of primes.

Definition 1. Let $P = \{p_1, p_2, \dots\}$ be an arbitrary sequence of primes. The sequence

$$\mathscr{T} \xrightarrow{\varphi_1} \mathscr{T} \xrightarrow{\varphi_2} \mathscr{T} \xrightarrow{\varphi_3} \cdots \tag{8}$$

with connecting $*$-homomorphisms $\varphi_n : \mathscr{T} \longrightarrow \mathscr{T} : T \mapsto T^{p_n}$, $n \in \mathbb{N}$, is called the *inductive sequence of Toeplitz algebras defined by P*.

Proposition 1. *Let $\{\mathscr{T}, \varphi_n\}_{n=1}^{+\infty}$ be the inductive sequence of Toeplitz algebras defined by a sequence $P = \{p_1, p_2, \dots\}$ of primes. Then there exists an isomorphism of C^*-algebras:*

$$\varinjlim\{\mathscr{T}, \varphi_n\} \cong C_r^*(\mathbb{Q}_P^+).$$

This assertion is an analog of Theorem 6 in [10] and a particular case of Theorem 1.6 in [6] on the continuity of the functor assigning to groups their Toeplitz algebras. Indeed, it is not hard to verify (see, for example, [7, Proposition 1]) that the group $\mathbb{Q}_P$ is the inductive limit of the inductive sequence of groups

$$\mathbb{Z} \xrightarrow{\tau_1} \mathbb{Z} \xrightarrow{\tau_2} \mathbb{Z} \xrightarrow{\tau_3} \cdots,$$

where the connecting morphisms are defined by $\tau_n(m) = p_n m$, $m \in \mathbb{Z}$.

Let us give an independent proof of Proposition 1. To this end, recall some facts about the algebra $C_r^*(\Gamma^+)$ and fix the notations. In $C_r^*(\Gamma^+)$, consider the dense $*$-subalgebra $\mathscr{P}(\Gamma^+)$ generated by the set of isometries $\{V_a \mid a \in \Gamma^+\}$. In the proof of Proposition 1 and the subsequent arguments, we will need the following familiar assertion, which is straightforward with the use of obvious relations for compositions of operators.

Lemma 2. *For every $Q \in \mathscr{P}(\Gamma^+)$, there exist complex numbers $\lambda_i \in \mathbb{C}$ and distinct pairs of numbers $(r_i, q_i) \in \Gamma^+ \times \Gamma^+$, $i = 1, \dots, n$, such that*

$$Q = \lambda_1 V_{r_1} V_{q_1}^* + \cdots + \lambda_n V_{r_n} V_{q_n}^*. \tag{9}$$

Remark 1. It is easy to see that $\{V_a V_b^* \mid (a, b) \in \Gamma^+ \times \Gamma^+\}$ is linearly independent and expansion (9) is unique for each $Q \in \mathscr{P}(\Gamma^+)$ up to the order of the summands. This observation can be used for proving Lemma 1. Indeed, firstly, consider the $*$-homomorphism τ_0 between the $*$-algebras $\mathscr{P}(\mathbb{Z}^+)$ and $\mathscr{T}$, well defined by the formula

$$\tau_0(Q(T, T^*)) = Q(T^n, T^{*n}),$$

where $Q(T, T^*) = \lambda_1 T^{l_1} T^{*m_1} + \cdots + \lambda_k T^{l_k} T^{*m_k}$ is an arbitrary element of $\mathscr{P}(\mathbb{Z}^+)$, $\lambda_i \in \mathbb{C}$, $l_i, m_i, k \in \mathbb{N}$, $T^{*m_i} = (T^*)^{m_i}$, $i = 1, \dots, k$. In other words, firstly the element $T^l T^{*m}$ of $\mathscr{P}(\mathbb{Z}^+)$ is mapped to the element $T^{nl} T^{*nm}$ of $\mathscr{T}$, $l, m \in \mathbb{N}$. Then this mapping extends by linearity to the whole space $\mathscr{P}(\mathbb{Z}^+)$. Secondly, we can show that the homomorphism τ_0 is isometric. Thirdly, extend τ_0 by continuity to a $*$-homomorphism defined on the whole Toeplitz algebra $\mathscr{T}$.

Proof of Proposition 1. Using Coburn's Theorem, define the following sequence of $*$-homomorphisms ($n \geq 2$):

$$\psi_{1,\infty} : \mathscr{T} \to C_r^*(\mathbb{Q}_P^+) : T \mapsto V_1; \quad \psi_{n,\infty} : \mathscr{T} \to C_r^*(\mathbb{Q}_P^+) : T \mapsto V_{\frac{1}{p_1 p_2 \cdots p_{n-1}}}. \tag{10}$$

Further, assert that, for each $n \in \mathbb{N}$, the diagram (see (5))

$$\begin{tikzcd} \mathscr{T} \arrow[rr, "\varphi_n"] \arrow[dr, "\psi_{n,\infty}"'] & & \mathscr{T} \arrow[dl, "\psi_{n+1,\infty}"] \\ & C_r^*(\mathbb{Q}_P^+) & \end{tikzcd}$$

commutes. Indeed, by the uniqueness property of the homomorphism defined in Coburn's Theorem applied to (10), for proving the relation $\psi_{n+1,\infty} \circ \varphi_n = \psi_{n,\infty}$, it suffices to show that $\psi_{n+1,\infty} \circ \varphi_n(T) = \psi_{n,\infty}(T)$. The validity of the last relation follows from the chain of equalities

$$\psi_{n+1,\infty} \circ \varphi_n(T) = \psi_{n+1,\infty}(T^{p_n}) = V_{\frac{p_n}{p_1 \cdots p_n}} = \psi_{n,\infty}(T).$$

By the universality of inductive limits in the category of C^*-algebras and $*$-homomorphisms, there exists a unique $*$-homomorphism $\psi : \varinjlim\{\mathscr{T}, \varphi_n\} \to C_r^*(\mathbb{Q}_P^+)$ such that the diagram (see (6))

$$\begin{tikzcd} & \mathscr{T} \arrow[dl, "\varphi_{n,\infty}"'] \arrow[dr, "\psi_{n,\infty}"] & \\ \varinjlim\{\mathscr{T}, \varphi_n\} \arrow[rr, "\psi"'] & & C_r^*(\mathbb{Q}_P^+) \end{tikzcd}$$

commutes for each $n \in \mathbb{N}$.

We assert that ψ is an isomorphism of C^*-algebras.

The injectivity of ψ stems from the injectivity of the homomorphisms $\psi_{n,\infty}$ (see, for example, [26, Proposition 6.2.4(iv)(b)]).

The surjectivity of ψ is equivalent to the equality

$$C_r^*(\mathbb{Q}_P^+) = \overline{\bigcup_{j=1}^{+\infty} \psi_{j,\infty}(\mathscr{T})} \tag{11}$$

(see [26, Proposition 6.2.4(iv)(c)]). For proving (11), demonstrate that

$$\mathscr{P}(\mathbb{Q}_P^+) \subset \bigcup_{j=1}^{+\infty} \psi_{j,\infty}(\mathscr{T}). \tag{12}$$

To this end, fix $Q \in \mathscr{P}(\mathbb{Q}_P^+)$ and the expansion (9) of Q. Let us first show that the operator $V_{r_1}V_{q_1}^*$ belongs to $\psi_{j,\infty}(\mathscr{T})$ for some $j \in \mathbb{N}$. Let $r_1 = \frac{t}{n_1 \dots n_k}$ and $q_1 = \frac{s}{n_1 \dots n_l}$, where $t, s \in \mathbb{Z}$ and $k, l \in \mathbb{N}$. Then we have the representations for V_{r_1} and $V_{q_1}^*$:

$$V_{r_1} = \psi_{k+1,\infty}(T^t), \quad V_{q_1}^* = \psi_{l+1,\infty}(T^{*s}). \tag{13}$$

Further, for definiteness, put $k \le l$ and consider the commutative diagram

$$\begin{tikzcd} \mathscr{T} \arrow[rr, "\varphi_{k+1,l+1}"] \arrow[dr, "\psi_{k+1,\infty}"'] & & \mathscr{T} \arrow[dl, "\psi_{l+1,\infty}"] \\ & C_r^*(\mathbb{Q}_P^+). & \end{tikzcd} \tag{14}$$

Using (13) and (14), we infer

$$V_{r_1}V_{q_1}^* = \psi_{k+1,\infty}(T^t)\psi_{l+1,\infty}(T^{*s}) = (\psi_{l+1,\infty} \circ \varphi_{k+1,l+1})(T^t)\psi_{l+1,\infty}(T^{*s}) = \psi_{l+1,\infty}(\varphi_{k+1,l+1}(T^t)T^{*s}).$$

Hence, we obtain the desired membership $V_{r_1}V_{q_1}^* \in \psi_{l+1,\infty}(\mathscr{T})$. Thus, we have the representations

$$\lambda_i V_{r_i} V_{q_i}^* = \psi_{c_i,\infty}(S_i) \tag{15}$$

for some operators $S_i \in \mathscr{T}$, where $i = 1, 2, \dots, n$, $c_i \in \mathbb{N}$. Put $c = \max\{c_1, \dots, c_n\}$. Reckoning with the commutativity of (14), where $k+1 = c_i$ and $l+1 = c$, $i = 1, \dots, n$,, we obtain the equalities

$$\psi_{c_i,\infty}(S_i) = \psi_{c,\infty} \circ \varphi_{c_i,c}(S_i) = \psi_{c,\infty}(\varphi_{c_i,c}(S_i)).$$

Representations (9), (15) and the additivity of $\psi_{c,\infty}$ imply that

$$Q = \psi_{c,\infty}\left(\sum_{i=1}^{n} \varphi_{c_i,c}(S_i)\right).$$

Consequently, $Q \in \psi_{c,\infty}(\mathscr{T})$. Thus, (12) is proved. Since the algebra $\mathscr{P}(\mathbb{Q}_P^+)$ is dense in $C_r^*(\mathbb{Q}_P^+)$, we obtain (11). Proposition 1 is proved. $\square$

Granted Proposition 1, we say that the pair $(C_r^*(\mathbb{Q}_P^+), \{\psi_{n,\infty}\}_{n=1}^{\infty})$ or the semigroup C^*-algebra $C_r^*(\mathbb{Q}_P^+)$ is the *inductive limit of the inductive sequence of Toeplitz algebras* $\{\mathscr{T}, \varphi_n\}_{n=1}^{+\infty}$ defined by the sequence of primes P.

Next, fix a number $k \in \mathbb{N}$ and consider the isometric homomorphism

$$\rho_k : \mathbb{Q}_P^+ \longrightarrow C_r^*(\mathbb{Q}_P^+) : a \longmapsto V_{ka}. \tag{16}$$

By the universality property of the isometric homomorphism, there exists a unique $*$-endomorphism $\rho_k^*: C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ such that the diagram

$$\begin{tikzcd} & \mathbb{Q}_P^+ \arrow[dl, "\pi"'] \arrow[dr, "\rho_k"] & \\ C_r^*(\mathbb{Q}_P^+) \arrow[rr, "\rho_k^*"'] & & C_r^*(\mathbb{Q}_P^+) \end{tikzcd} \tag{17}$$

commutes.

Show that ρ_k^* is a limit endomorphism. To this end, define a morphism between two copies of inductive sequence (8). Using Coburn's Theorem, introduce the sequence $\{\phi_n^k : n \in \mathbb{N}\}$ of unital $*$-homomorphisms:

$$\phi_n^k : \mathscr{T} \longrightarrow \mathscr{T} : T \longmapsto T^k. \tag{18}$$

For each pair $m, n \in \mathbb{N}$ satisfying the inequality $m \leq n$, define the connecting homomorphism $\varphi_{m,n}$ for (8) by (7) and consider the diagram

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\varphi_{m,n}"] \arrow[d, "\phi_m^k"'] & \mathscr{T} \arrow[d, "\phi_n^k"] \\ \mathscr{T} \arrow[r, "\varphi_{m,n}"'] & \mathscr{T} \end{tikzcd} \tag{19}$$

By Coburn's Theorem, diagram (19) commutes; i.e., $\phi_n^k \circ \varphi_{m,n} = \varphi_{m,n} \circ \phi_m^k$. Recall that, in this case, we have the morphism $\{\phi_n^k : n \in \mathbb{N}\} : \{\mathscr{T}, \varphi_n\} \longrightarrow \{\mathscr{T}, \varphi_n\}$ between two copies of (8). Moreover, there exists a unique $*$-homomorphism $\varinjlim\{\phi_n^k : n \in \mathbb{N}\} : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ such that the diagram

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\psi_{n,\infty}"] \arrow[d, "\phi_n^k"'] & C_r^*(\mathbb{Q}_P^+) \arrow[d, "\varinjlim\{\phi_n^k : n\in\mathbb{N}\}"] \\ \mathscr{T} \arrow[r, "\psi_{n,\infty}"'] & C_r^*(\mathbb{Q}_P^+) \end{tikzcd} \tag{20}$$

commutes for each $n \in \mathbb{N}$. The homomorphism $\varinjlim\{\phi_n^k : n \in \mathbb{N}\}$ will be called the *limit endomorphism* of the semigroup C^*-algebra $C_r^*(\mathbb{Q}_P^+)$. In what follows, for brevity, this endomorphism will be denoted by ϕ_P^k.

Proposition 2. $\phi_P^k = \rho_k^*$ *for all* $k \in \mathbb{N}$.

Proof. Let us first show that, for each $a \in \mathbb{Q}_P^+$, the endomorphism ϕ_P^k maps V_a to V_{ka}. To this end, fix an arbitrary element $a = \frac{m}{p_1 \ldots p_l}$ in $\mathbb{Q}_P^+$, where $m \in \mathbb{Z}$, $l \in \mathbb{N}$. By the definition of $\psi_{l+1,\infty}$ (see (10)), we have

$$V_a = \psi_{l+1,\infty}(T^m). \tag{21}$$

Consider diagram (20) with $n = l+1$. Using (21), the commutativity of (20), and also (18) and (10), we obtain the desired property:

$$\phi_P^k(V_a) = \phi_P^k \circ \psi_{l+1,\infty}(T^m) = \psi_{l+1,\infty} \circ \phi_{l+1}^k(T^m) = \psi_{l+1,\infty}(T^{km}) = V_{ka}.$$

Now, the coincidence of the $*$-homomorphism ρ_k^* of (17) with the limit endomorphism ϕ_P^k follows from the uniqueness of the $*$-homomorphism in the universality property of isometric homomorphism (2) for $\Gamma^+ = \mathbb{Q}_P^+$. $\square$

§ 3. Limit $*$-Automorphisms

The section is devoted to the set-theoretic properties of the limit endomorphisms of the semigroup C^*-algebra $C_r^*(\mathbb{Q}_P^+)$. More exactly, here we discuss necessary and sufficient conditions for these endomorphisms to be automorphisms, i.e, $*$-isomorphisms into itself.

So, suppose we are given a sequence of primes $P = (p_1, p_2, \dots)$. Consider the limit endomorphism ϕ_P^k, $k \in \mathbb{N}$, defined by the morphism $\{\phi_n^k : n \in \mathbb{N}\}$ between two copies of the same inductive sequence of Toeplitz algebras defined by P:

$$
\begin{tikzcd}
\mathscr{T} \arrow[r, "\varphi_1"] \arrow[d, "\phi_1^k"'] & \mathscr{T} \arrow[r, "\varphi_2"] \arrow[d, "\phi_2^k"] & \mathscr{T} \arrow[r, "\varphi_3"] \arrow[d, "\phi_3^k"] & \cdots & C_r^*(\mathbb{Q}_P^+) \arrow[d, "\phi_P^k"] \\
\mathscr{T} \arrow[r, "\varphi_1"'] & \mathscr{T} \arrow[r, "\varphi_2"'] & \mathscr{T} \arrow[r, "\varphi_3"'] & \cdots & C_r^*(\mathbb{Q}_P^+)\,.
\end{tikzcd}
\tag{22}
$$

Begin with formulating an assertion that is an immediate corollary to Proposition 2 and the injectivity of the $*$-homomorphism ρ_k^* induced by homomorphism (16).

Proposition 3. *The homomorphism ϕ_P^k is injective for each $k \in \mathbb{N}$.*

Remark 2. Proposition 3 can be proved without appealing to Proposition 2. Namely, we can use the properties of the homomorphisms involved in the definition of ϕ_P^k and [26, Proposition 6.2.4(iv)(b)].

Let us discuss the conditions that must be satisfied by a number k for the limit endomorphism ϕ_P^k to be an automorphism. Clearly, ϕ_P^1 is the identity homomorphism. Therefore, in what follows, we assume that $k \geq 2$.

For formulating the next assertion we will need

Definition 2. We say that a prime p *occurs infinitely often* in a sequence $P = (p_1, p_2, \dots)$ if $p = p_n$ for infinitely many p_n's, $n \in \mathbb{N}$.

Proposition 4. *Suppose that each prime divisor of k occurs infinitely often in a sequence P. Then the limit endomorphism $\phi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism.*

Proof. It is easy to see that if k satisfies the condition given in the statement then the semigroup homomorphism

$$\mathbb{Q}_P^+ \longrightarrow \mathbb{Q}_P^+ : q \longmapsto kq$$

is an isomorphism. Consequently, the image of the $*$-homomorphism ρ_k^* of diagram (17) contains the involutive algebra $\mathscr{P}(\mathbb{Q}_P^+)$, which is dense in $C_r^*(\mathbb{Q}_P^+)$. Hence, ρ_k^* is an automorphism. By Proposition 2, the limit endomorphism ϕ_P^k is an automorphism too. □

Remark 3. The previous assertion can be proved by using the technique of inductive sequences by analogy with the proof of Proposition 2. This is done as follows.

By Proposition 3, it suffices to prove the surjectivity of ϕ_P^k or, equivalently, the validity of (11) and (12) with $\psi_{j,\infty} \circ \varphi_j^k(\mathscr{T})$ instead of $\psi_{j,\infty}^k(\mathscr{T})$. To this end, fix $Q \in \mathscr{P}(\mathbb{Q}_P^+)$ and consider the representation (9) of Q. Let $r_1 = \frac{t}{p_1 \cdots p_m}$, where $t \in \mathbb{Z}_+$ and $m \in \mathbb{N}$. Since all prime factors of k occur infinitely often in P, there exist naturals a and $s > m$ such that $r_1 = \frac{ak}{p_1 \cdots p_s}$. Further, it is readily checked that

$$V_{r_1} = \psi_{s+1,\infty} \circ \phi_{s+1}^k(T^a). \tag{23}$$

Similarly, there exist two numbers $b, l \in \mathbb{N}$ such that

$$V_{q_1}^* = \psi_{l,\infty} \circ \phi_l^k(T^{*b}). \tag{24}$$

Putting $c_1 = \max\{s+1, l\}$ and using (23), (24), (14), and (19), we have

$$\begin{aligned}\lambda_1 V_{r_1} V_{q_1}^* &= \lambda_1 \psi_{s+1,\infty}\big(\phi_{s+1}^k(T^a)\big)\psi_{l,\infty}\big(\phi_l^k(T^{*b})\big)\\ &= \lambda_1 \psi_{c_1,\infty} \circ \varphi_{s+1,c_1}\big(\phi_{s+1}^k(T^a)\big)\psi_{c_1,\infty} \circ \varphi_{l,c_1}\big(\phi_l^k(T^{*b})\big)\\ &= \lambda_1 \psi_{c_1,\infty}\big(\big(\varphi_{s+1,c_1} \circ \phi_{s+1}^k\big)(T^a)\big(\varphi_{l,c_1} \circ \phi_l^k\big)(T^{*b})\big)\\ &= \lambda_1 \psi_{c_1,\infty}\big(\big(\phi_{c_1}^k \circ \varphi_{s+1,c_1}\big)(T^a)\big(\phi_{c_1}^k \circ \varphi_{l,c_1}\big)(T^{*b})\big)\\ &= \psi_{c_1,\infty} \circ \phi_{c_1}^k(\lambda_1 \varphi_{s+1,c_1}(T^a)\varphi_{l,c_1}(T^{*b})).\end{aligned}$$

Put $S_1 = \lambda_1 \varphi_{s+1,c_1}(T^a)\varphi_{l,c_1}(T^{*b})$. Then $\lambda_1 V_{r_1} V_{q_1}^* = \psi_{c_1,\infty} \circ \phi_{c_1}^k(S_1)$. In the same manner, for the other summands in (9), we obtain the representations $\lambda_i V_{r_i} V_{q_i}^* = \psi_{c_i,\infty} \circ \phi_{c_i}^k(S_i)$, in which $S_i \in \mathscr{T}$. Next, put $c = \max\{c_1, \dots, c_n\}$. Then, for each $i = 1, \dots, n$, using (14) and (19), we get the equalities

$$\begin{aligned}\lambda_i V_{r_i} V_{q_i}^* &= \psi_{c_i,\infty} \circ \phi_{c_i}^k(S_i) = \psi_{c,\infty} \circ \varphi_{c_i,c}\big(\phi_{c_i}^k(S_i)\big) = \psi_{c,\infty}\big(\varphi_{c_i,c} \circ \phi_{c_i}^k(S_i)\big)\\ &= \psi_{c,\infty}\big(\phi_c^k \circ \varphi_{c_i,c}(S_i)\big) = \psi_{c,\infty} \circ \phi_c^k(\varphi_{c_i,c}(S_i)).\end{aligned}$$

Inserting this in (9), we obtain

$$Q = \psi_{c,\infty} \circ \phi_c^k\bigg(\sum_{i=1}^n \varphi_{c_i,c}(S_i)\bigg),$$

which guarantees that $Q \in \psi_{c,\infty} \circ \phi_c^k(\mathscr{T})$. The rest is clear.

Let us now prove the converse to Proposition 4. To this end, we start with the case when k is a prime coinciding only with finitely many terms in the sequence P.

To this end, consider the two subalgebras in the group C^*-algebra $C_r^*(\mathbb{Q}_P^+)$ generated by the same set of isometries $\mathscr{S} := \{V_{kq} : q \in \mathbb{Q}_P^+\}$. These are the $*$-subalgebra $\mathscr{P}(\mathscr{S})$ and the C^*-subalgebra $C^*(\mathscr{S})$ generated by $\mathscr{S}$. The algebra $\mathscr{P}(\mathscr{S})$ is a dense subset in $C^*(\mathscr{S})$ with respect to the topology defined by the norm, i.e.

$$\overline{\mathscr{P}(\mathscr{S})} = C^*(\mathscr{S}). \tag{25}$$

Clearly, the image $\operatorname{Im} \phi_P^k$ of ϕ_P^k satisfies the equality

$$\operatorname{Im} \phi_P^k = C^*(\mathscr{S}). \tag{26}$$

Proposition 5. *Let k be a prime such that, for some $m \in \mathbb{N}$, the condition $k \neq p_n$ is fulfilled for every $n \geq m$. Then the limit endomorphism $\phi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is not surjective.*

Proof. Consider the two cases:

Case 1. Let $m = 1$. In other words, the prime k is not a term in P.

We assert that $V_{\frac{1}{p_1}}$ does not belong to $\operatorname{Im} \phi_P^k$.

Indeed, take an arbitrary $Q \in \mathscr{P}(\mathscr{S})$. It is not hard to see that, by analogy with representation (9) holding for the elements of the $*$-algebra $\mathscr{P}(\Gamma^+)$, this element Q is representable as

$$Q = \lambda_1 V_{kr_1} V_{kq_1}^* + \dots + \lambda_n V_{kr_n} V_{kq_n}^*,$$

where $\lambda_i \in \mathbb{C}$, $r_i, q_i \in \mathbb{Q}_P^+$, $i = 1, \dots, n$. Here the pairs of numbers (r_i, q_i) are distinct and satisfy the inequalities $q_1 \leq q_2 \leq \dots \leq q_n$.

Using (1), we arrive at the following estimate for the norms of the elements of the spaces $\mathscr{B}(l^2(\mathbb{Q}_P^+))$ and $l^2(\mathbb{Q}_P^+)$:

$$\|V_{\frac{1}{p_1}} - Q\| \geq \|(V_{\frac{1}{p_1}} - Q)e_{kq_n}\| = \bigg\|e_{kq_n + \frac{1}{p_1}} - \sum_{i=1}^n \lambda_i e_{k(q_n - q_i + r_i)}\bigg\|. \tag{27}$$

The following conditions are obviously fulfilled for the indices of the basis vectors:

$$kq_n + \frac{1}{p_1} \neq k(q_n - q_i + r_i), \quad i = 1, \dots, n,$$

which imply the orthogonality of the vectors in the Hilbert space $l^2(\mathbb{Q}_P^+)$:

$$\left\langle e_{kq_n+\frac{1}{p_1}}, \sum_{i=1}^{n} \lambda_i e_{k(q_n-q_i+r_i)} \right\rangle = 0. \tag{28}$$

Using (27) and (28), deduce the lower estimate $\|V_{\frac{1}{p_1}} - Q\| \geq 1$.

Consequently, $V_{\frac{1}{p_1}}$ is not an adherent point of $\mathscr{P}(\mathscr{S})$ with respect to the topology defined by the norm. Thus, in view of (25) and (26), we have obtained the condition $V_{\frac{1}{p_1}} \notin \operatorname{Im} \phi_P^k$, as was asserted.

Case 2. Let $m \geq 2$ be the greatest natural l such that $k = p_l$. Put $q = p_1 \cdot \ldots \cdot p_m$.

For reducing our arguments to the first case of the proof, consider a sequence of primes $P_m = (p_{m+1}, p_{m+2}, \dots)$ for which k is not a term. Then define the isometric homomorphism $\beta : \mathbb{Q}_P^+ \longrightarrow C_r^*(\mathbb{Q}_{P_m}^+) : a \longmapsto V_{qa}$. Here V_{qa} stands for the isometry in $\mathscr{B}(l^2(\mathbb{Q}_{P_m}^+))$ defined by (1).

By the universality property of an isometric homomorphism, there exists a unique injective $*$-homomorphism β^* such that the diagram

$$\begin{tikzcd} & \mathbb{Q}_P^+ \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & \\ C_r^*(\mathbb{Q}_P^+) \arrow[rr, "\beta^*"'] & & C_r^*(\mathbb{Q}_{P_m}^+) \end{tikzcd}$$

commutes.

Since the subgroup $q\mathbb{Q}_P := \{qa \mid a \in \mathbb{Q}_P\}$ coincides with the group $\mathbb{Q}_{P_m}$, the $*$-homomorphism β^* is an isomorphism of C^*-algebras.

Further, define $\varphi_{P_m} : C_r^*(\mathbb{Q}_{P_m}^+) \to C_r^*(\mathbb{Q}_{P_m}^+)$ as the homomorphism making the following diagram commute:

$$\begin{tikzcd} C_r^*(\mathbb{Q}_P^+) \arrow[r, "\beta^*"] \arrow[d, "\phi_P^k"'] & C_r^*(\mathbb{Q}_{P_m}^+) \arrow[d, "\varphi_{P_m}"] \\ C_r^*(\mathbb{Q}_P^+) \arrow[r, "\beta^*"'] & C_r^*(\mathbb{Q}_{P_m}^+). \end{tikzcd} \tag{29}$$

Clearly, this $*$-homomorphism is defined as the composition of homomorphisms: $\varphi_{P_m} := \beta^* \circ \phi_P^k \circ (\beta^*)^{-1}$.

It is easy to check that $\varphi_{P_m}(V_a) = V_{ka}$ for each $a \in \mathbb{Q}_{P_m}^+$. Reckoning with the uniqueness of the $*$-homomorphism in the universality property of isometric homomorphism (2) for $\Gamma^+ = \mathbb{Q}_{P_m}^+$, we obtain $\varphi_{P_m} = \phi_{P_m}^k$. Here $\phi_{P_m}^k : C_r^*(\mathbb{Q}_{P_m}^+) \longrightarrow C_r^*(\mathbb{Q}_{P_m}^+)$ is the limit endomorphism defined by (22), for which we put $P = P_m$ and the homomorphisms φ_n are defined by the formula $\varphi_n : \mathscr{T} \longrightarrow \mathscr{T} : T \longmapsto T^{p_{m+n}}$, $n \in \mathbb{N}$.

By the first case of the proof, since the prime k is not a term of P_m, we conclude that φ_{P_m} is not surjective. This fact and the commutativity of (29) imply immediately that the limit endomorphism ϕ_P^k is not surjective. $\square$

Using the uniqueness of the $*$-homomorphism in the universality property of an isometric homomorphism, we get the following:

Lemma 3. *Let $k = l \cdot m$ for some naturals l and m. Then the diagram*

$$\begin{tikzcd}
 & C_r^*(\mathbb{Q}_P^+) \arrow[dr, "\phi_P^m"] & \\
C_r^*(\mathbb{Q}_P^+) \arrow[ur, "\phi_P^l"] \arrow[rr, "\phi_P^k"'] & & C_r^*(\mathbb{Q}_P^+)
\end{tikzcd}$$

commutes.

Lemma 3 and Proposition 3 imply

Proposition 6. *If there is a prime divisor k' of a number $k \in \mathbb{N}$ such that, for some $m \in \mathbb{N}$, the condition $k' \neq p_n$ is fulfilled for each $n \geq m$ then the limit endomorphism $\phi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is not surjective.*

We summarize up the previous results in the following

Theorem 1. *For the limit endomorphism $\phi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ to be an automorphism, it is necessary and sufficient that each prime divisor of k occurs infinitely often in the sequence P.*

Remark 4. The necessity of the condition in Theorem 1 can be proved by means of the results on Toeplitz algebras that are represented in diagram (30) below and the results on P-adic solenoids. For details of these results the reader is referred to [5, 6, 8] and [12, 13, 15] respectively. Below we give a sketch of such a proof.

Let $\phi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ be the given automorphism defined by diagram (22). We have the commutative diagram

$$\begin{array}{ccc}
C_r^*(\mathbb{Q}_P^+) & \xrightarrow{\phi_P^k} & C_r^*(\mathbb{Q}_P^+) \\
\psi \big\downarrow & & \big\downarrow \psi \\
C_r^*(\mathbb{Q}_P^+)/K & \xrightarrow[\varphi]{} & C_r^*(\mathbb{Q}_P^+)/K \\
\iota \big\downarrow & & \big\downarrow \iota \\
C(\widehat{\mathbb{Q}}_P) & \xrightarrow[\widetilde{\varphi}]{} & C(\widehat{\mathbb{Q}}_P).
\end{array} \tag{30}$$

Here K is the commutator ideal of $C_r^*(\mathbb{Q}_P^+)$, while ψ is the natural $*$-homomorphism. The mappings φ, ι, and $\widetilde{\varphi}$ are isomorphisms of C^*-algebras, $\widehat{\mathbb{Q}}_P$ is the compact group dual to the discrete group $\mathbb{Q}_P$, and $C(\widehat{\mathbb{Q}}_P)$ is the commutative C^*-algebra of all complex-valued continuous functions on $\widehat{\mathbb{Q}}_P$.

Consider the spectrum functor $\mathscr{UCBA} \longrightarrow \mathscr{CT}op$ (see, for example, [1, Chapter 4, § 1]), i.e. the contravariant functor from the category of unital commutative Banach algebras and their continuous unital homomorphisms into the category of compact topological spaces and their continuous mappings. It assigns to $C(\widehat{\mathbb{Q}}_P)$ its spectrum, i.e. the space of multiplicative functionals, or, equivalently, the space of maximal ideals which are homeomorphic to the compact space $\widehat{\mathbb{Q}}_P$.

It is well known that the compact group $\widehat{\mathbb{Q}}_P$ is topologically isomorphic to the P-adic solenoid Σ_P (see, for example, [27, (25.3)]). The solenoid Σ_P is the inverse limit of the inverse sequence dual to the inductive sequence giving the group $\mathbb{Q}_P$:

$$\mathbb{S}^1 \xleftarrow{f_1} \mathbb{S}^1 \xleftarrow{f_2} \mathbb{S}^1 \xleftarrow{f_3} \cdots,$$

where $\mathbb{S}^1$ is the unit circle regarded as a subspace in the space of complex numbers $\mathbb{C}$ with the natural topology, and the connecting mapping f_n, $n \in \mathbb{N}$, is the operation of taking the power p_n, i.e., $f_n(z) = z^{p_n}$ for each $z \in \mathbb{S}^1$. The P-adic solenoid is a connected compact abelian group with respect to the coordinatewise multiplication with the unit $(1, 1, \dots)$.

In our situation, the isomorphism $\widetilde{\varphi}$ induces the isomorphism of the topological groups defined by the formula

$$\Sigma_P \longrightarrow \Sigma_P : g \longmapsto g^k. \tag{31}$$

Finally, consider the following contravariant functor from the category of compact groups into the category of discrete groups:

$$G \longmapsto \widehat{G}; \quad \{\sigma : G_1 \to G_2\} \longmapsto \{\hat{\sigma} : \widehat{G}_2 \to \widehat{G}_1 : \chi \mapsto \chi \circ \sigma\}.$$

Here $\widehat{G}$ is the character group of a compact group G, σ is a morphism of compact groups, and $\chi : G_2 \longrightarrow \mathbb{S}^1$ is a character of the compact group G_2. To isomorphism (31), this functor assigns the isomorphism

$$\mathbb{Q}_P \longrightarrow \mathbb{Q}_P : q \longmapsto kq$$

in the category of discrete groups. Either at this stage of the discussion or at the previous stage, working with isomorphism (31), we conclude that each prime divisor of k occurs infinitely often in the sequence P.

Consider some examples corresponding to various sequences of primes.

Example 1. Take n different primes $p_1, \dots, p_n$, $n \in \mathbb{N}$. Consider the periodic sequence $P = (p_1, \dots, p_n, p_1, \dots, p_n, \dots)$ and the group $\mathbb{Q}_P$. The limit endomorphism ϕ_P^k is an automorphism if and only if there exist nonnegative integers $m_1, \dots, m_n$ such that $k = p_1^{m_1} \cdot \dots \cdot p_n^{m_n}$.

Moreover, if $n = 1$ and $p_1 = 2$ then we have the constant sequence $P = (2, 2, \dots)$ and the group of dyadic rationals

$$\mathbb{Q}_{(2,2,\dots)} = \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z}^+, \ n \in \mathbb{N} \right\}.$$

In this case, the endomorphism ϕ_P^k is an automorphism if and only if k equals some power of 2.

Example 2. Let $P = (2, 2, 3, 2, 3, 5, 2, 3, 5, 7, \dots)$. Then $\mathbb{Q}_P$ coincides with the group of all rationals $\mathbb{Q}$. In this case, each limit endomorphism ϕ_P^k is an automorphism.

Example 3. Let $P = (2, 3, 5, 7, \dots)$ be the sequence of all primes. Then, for each $k \geq 2$, the limit endomorphism ϕ_P^k is not an automorphism.

Give some interesting conditions equivalent to the properties of Theorem 1. For this we first recall some definitions.

Definition 3. Given $n \in \mathbb{N}$, an additive abelian group G is called *n-divisible* if for every $g \in G$ there is $h \in G$ with $nh = g$.

It is not hard to verify that, given a natural $k \geq 2$, the group of rationals $\mathbb{Q}_P$ is k-divisible if and only if each prime divisor of k occurs infinitely often in P.

Let G be a connected compact abelian group. There is a number of necessary and sufficient conditions for the characters group $\widehat{G}$ to be k-divisible (see, for example, [20, 28]). It is known (see [20, Theorem 1.1]) that $\widehat{G}$ is k-divisible if and only if G admits a k-mean.

Definition 4. Given a natural $k \geq 2$ and a topological space X, a continuous mapping $\mu : X \times \dots \times X \longrightarrow X$ from the Cartesian product of k copies of X is called a *k-mean on X* if $\mu(x, x, \dots, x) = \mu(x)$ and $\mu(x_1, x_2, \dots, x_k) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)})$ for all $x, x_1, x_2, \dots, x_k \in X$ and any permutation σ of the number set $\{1, 2, \dots, k\}$.

A vast literature exists about means on topological spaces and groups (see, for example, [29] and the references therein).

The above facts and Theorem 1 imply

Theorem 2. *The following are equivalent for a sequence of primes P and a natural $k \geq 2$:*
(1) *the $*$-homomorphism $\phi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism;*
(2) *the group of rationals $\mathbb{Q}_P$ is a k-divisible group;*
(3) *the P-adic solenoid Σ_P admits a k-mean.*

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R. N. Gumerov
Kazan (Volga Region) Federal University, Kazan, Russia
E-mail address: `Renat.Gumerov@kpfu.ru`