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Passivity analysis of coupled inertial neural networks with time-varying delays and impulsive effects

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Abstract. This paper is devoted to the passivity analysis of an array model for coupled inertial delayed neural networks (NNs) with impulses under different network structures, namely directed and undirected topologies. Firstly, utilising the information of eigenvectors for the directed coupling matrix, a new Lyapunov functional is constructed, by which, together with the aid of some inequality techniques and network characteristics, the two sets of sufficient criteria are established to, respectively, guarantee the strictly input passivity and strictly output passivity of the impulsive network with directed coupling. Secondly, benefited from the properties of the undirected coupling matrix, some more concise conditions that are easier to be verified for the passivities of the undirected coupled network accompanied by impulsive effects are proposed. Finally, two numerical examples are designed to execute the verification of the derived theoretical results.

Keywords. Inertial neural networks; passivity; impulsive effects; directed and undirected topologies; time-varying delays.

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1. Introduction

Over the past few decades, the artificial neural networks (NNs) have attracted a high degree of research owing to their extensive applications in pattern recognition [1], signal processing [2], optimisation, motion control [3] and so on. Customarily, they are described by a variety of first-order differential equations, such as Hopfield NNs, bidirectional associative memory (BAM) NNs, Cohen–Grossberg NNs, Memristor NNs [4–7], which, however, do not take into consideration the possible influence arising from the second derivatives of the states, also called the inertia item or inductance in physics. The inertial NNs mean the incorporation of the inertial terms into neuron models, which are proposed in [8] and subsequently are applicable to diverse areas. Take biology issue as an example in [9,10], the inductance of the semicircular canals for some animals is used to design an equivalent circuit, by which the electrical tuning or filtering behaviours of the membrane for a hair cell can be successfully modelled. It has been found that when the neurons are of an inertial nature, more complicated dynamic characteristics could be depicted or richer dynamic behaviours could be generated, such as bifurcation and chaos for a system [11,12]. So the inertial NNs have been a highly promising research topic and fruitful findings were reported, including synchronisation, bifurcation and stability analyses of inertial NNs [13–18].

As is well known, the qualitative analysis of system dynamics is an indispensable procedure for the actual application and modelling of NNs. Among them, the passivity, originating from the circuit theory [19], is an important one. It employs the product of input and output as the energy supply and represents an energy attenuation characteristic of the system in that the energy is only burned but not produced. Therefore, a passive system can sustain internal stability referring to the energy-related considerations. From this point of view, it is regarded that passivity can deduce broader and more general results on the dynamic analysis of a system. Recently, it has been successfully used to analyse the stability [20], synchronisation [21], signal processing [22] and chaos control [23] of systems. Meanwhile, the passive analysis of various models is also extensively investigated [24–27]. In [24], by using the Wirtinger-type inequality, the passivity analysis is addressed for memristive NNs with consideration of probabilistic time-varying delays. With the aid of the delay fractioning technique and linear matrix inequality approach, Sakthivel *et al* [25] derived several criteria to guarantee the passivity of the fuzzy Cohen–Grossberg BAM NNs with uncertainties. Wang *et al* [26] investigated the passivity for coupled reaction–diffusion NNs by designing appropriate adaptive coupling strategies. Unfortunately, the passivity of the inertial NNs remains an open problem.

Owing to the complexity of the real world, the established models often have the framework that the multiple NNs simultaneously operate accompanied by the interaction with each other through nodes coupling, which is named coupled NNs. Recently, the coupled NNs have sparked much research interest from different fields [15,16,26,28]. Remarkably, Hu *et al* [15] and Dharani *et al* [16] considered the synchronisation control of coupled inertial NNs, in which the former is for the pinning synchronisation, and the latter is for the sampled-data synchronisation of that with reaction–diffusion terms.

In the circuit implementation of the NNs, it sometimes occurs that the instantaneous perturbations or sharp changes in the voltages come out of electronic components, namely the impulsive phenomena. Besides, in dynamical investigation, the impulses can be added to the system at certain instants for the effective or quick achievement of the desired state behaviour [28–30]. On the other hand, it inevitably brings the signal delays suffering from the limitation of the finite speed of an amplifier switch and signal propagation, whereas these time-delay terms in systems may heavily affect the original performance of the system, leading to instability, bifurcation, oscillation and chaotic attractors [5,31]. Furthermore, generally, the form of delay is a function that varies with time rather than the case of constant delays. Hence, the impulses and time-varying delays deserve consideration in view of the practical applications as well as theoretical analysis of the NNs. It is worth mentioning that in [14], a designed impulsive controller acted on the inertial NNs with time-varying delays for the purpose of exponential stability of systems. However, until now, there has been no study on the delayed inertial NNs both with impulsive and coupling effects, which is exactly the model that we shall consider.

Inspired by the aforementioned statements, this paper is intended to explore the passivity of the coupled inertial NNs with time-varying delays and impulsive effects. From what we know, so far no attempt has been made on this aspect. Compared with the existing relevant literature, the main contributions of our work can be attributed as follows: (i) The features of the inertial terms, coupled nodes and impulsive effects are included when addressing NNs, which is more general and consistent with the reality. (ii) Both directed and undirected coupling topologies are considered, respectively, in which different Lyapunov functions are constructed based on the graph theory. (iii) For the first time, the passivity of the inertial NNs is investigated, which further exploits the performance of the inertial NNs.

The framework of the paper is listed as follows. Section 2 presents the model to be addressed and proposes some preliminaries, including useful conceptions, assumptions and lemmas. The main results of the passivity analyses are, respectively, reported in §3 and 4, in which the former is for impulsive inertial NNs with directed coupling, whereas the latter is for the same network but with an undirected coupling topology. Section 5 exhibits two numerical examples to validate the correction of the theoretical conclusions. At last, conclusions are drawn in §6.

Notations: Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the space of real numbers, and the n-dimensional real Euclidean space, respectively. For $a \in \mathbb{R}$, Res(a) represents the real part of a. For two number sets M and N, $\mathscr{C}(M, N)$ and $\mathscr{C}^1(M, N)$ represent, respectively, the set of all continuous maps and the set of all continuous differentiable maps, both from M and N. For a real symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$ means P is a positive-definite matrix. For a symmetric matrix G, $\lambda_{\max}(G)$ and $\lambda_2(G)$ denote, respectively, the maximum eigenvalue and the second largest eigenvalue of P. I_n is the $n \times n$ identity matrix, A^{T} denotes the transpose of matrix A and $\otimes$ is the Kronecker operation.

2. Problem description and preliminaries

In what follows, we shall show the model to be addressed and display some necessary preliminaries. Let us begin with two critical definitions.

DEFINITION 1 [32,33]
A system is said to be passive if there exists non-negative function $S : \mathbb{R}^+ \to \mathbb{R}^+$, called the storage function, such that

$$\int_{t_0}^{t_p} y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t \geq S(t_p) - S(t_0)$$

for any $t_p, t_0 \in \mathbb{R}^+$ and $t_p \geq t_0$, where $u(t)$ and $y(t) \in \mathbb{R}^n$ are, respectively, the input and output of the system.

DEFINITION 2 [33]
A system is said to be strictly passive if there exists a non-negative function $S : \mathbb{R}^+ \to \mathbb{R}^+$, called the storage function, such that

$$\int_{t_0}^{t_p} y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t \geq S(t_p)-S(t_0)+\varepsilon_1 \int_{t_0}^{t_p} u^{\mathrm{T}}(t)u(t)\,\mathrm{d}t + \varepsilon_2 \int_{t_0}^{t_p} y^{\mathrm{T}}(t)y(t)\,\mathrm{d}t,$$

where $\varepsilon_1, \varepsilon_2 \geq 0$ and $\varepsilon_1 + \varepsilon_2 > 0$, $u(t), y(t) \in \mathbb{R}^n$ represent the same meanings as those in Definition 1.

Remark 1. In Definition 2, if $\varepsilon_1 > 0$, the considered system is specially said to be strictly input passive, and if $\varepsilon_2 > 0$ the system is said to be strictly output passive, both of which are the objectives to be achieved in this paper.

In this paper, we consider an array of linearly coupled delayed inertial NNs with impulsive effects, in which N identical nodes are incorporated. The model of this network is described by

$$\begin{cases} \dfrac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} = -D\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + Af(x_i(t)) \\ \qquad + Bf(x_i(t-\tau(t))) + J \\ \qquad + c\displaystyle\sum_{j=1}^{N} G_{ij}\Gamma\left(\dfrac{\mathrm{d}x_j(t)}{\mathrm{d}t} + x_j(t)\right) + u_i(t), \\ \qquad t \neq t_k, \\ \Delta x_i(t_k) = -\delta_k \cdot x(t_k^-), \quad \Delta\dot{x}_i(t_k) = -\delta_k \cdot \dot{x}(t_k^-), \end{cases} \tag{1}$$

where $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$ corresponds to the state vector of node i, $i = 1, 2, \ldots, N, k = 1, 2, \ldots, N$, is the number of nodes in the network, $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ and $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$ are positive-definite matrices, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ denote, respectively, the connection weight matrices without and with time delays, $J \in \mathbb{R}^n$ is the constant external input; $f(x_i) = (f_1(x_{i1}), \ldots, f_n(x_{in}))^{\mathrm{T}}$ is the activation function, $\tau(t)$ is the time-varying delay with $0 \leq \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \rho < 1$, $u_i(t) \in \mathbb{R}^n$ denotes the control input. $c > 0$ represents the coupling strength, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is the individual coupling between two nodes, in which $\gamma_j > 0$, $j = 1, \ldots, n$; $G = (G_{ij})_{N\times N}$ represents the topological structure of the network, and will be given different forms in the sequel, referring to the directed and undirected networks, respectively. For convenience, we always assume that the configuration coupling matrix G in (1) is irreducible.

Besides, in (1), t_k are the impulsive instants satisfying $0 < t_k < t_{k+1} < \cdots$ for $k = 1, 2, \ldots$, and $\lim_{k\to+\infty} t_k = +\infty$. δ_k is the impulsive gain at instant t_k. $\Delta x(t_k) = x(t_k) - x(t_k^-)$ and $\Delta\dot{x}(t_k) = \dot{x}(t_k) - \dot{x}(t_k^-)$ are the impulses at moments t_k, in which $x(t_k) = x(t_k^+) = \lim_{t\to t_k^+} x(t)$, $x(t_k^-) = \lim_{t\to t_k^-} x(t)$ and $\dot{x}(t_k) = \dot{x}(t_k^+) = \lim_{t\to t_k^+} \dot{x}(t)$, $\dot{x}(t_k^-) = \lim_{t\to t_k^-} \dot{x}(t)$.

The initial conditions with network (1) are

$$x_i(s) = \varphi_i(s), \quad \dot{x}_i(s) = \psi_i(s), \quad s \in [-\tau, 0], \tag{2}$$

where

$$\varphi(s) \in \mathscr{C}^1([-\tau, 0], \mathbb{R}^n),\ \psi(s) \in \mathscr{C}([-\tau, 0], \mathbb{R}^n).$$

Our intention in this paper is to construct a reasonable input-out system based on coupled NNs (1) and exploit its passivity. For this purpose, the following three lemmas are indispensable, in which, the first plays a pivotal role for the construction of both the input-out system and the Lyapunov functional, and the other two are used in the proof of the main results.

Lemma 1 [34]. *Let $G = (G_{ij})$ be an irreducible matrix with non-negative off-diagonal elements, and satisfies $G_{ii} = \sum_{j=1, j\neq i}^{N} G_{ij}$. Then the following items hold*:

(1) *For any non-zero eigenvalues λ of the matrix G, we have $\mathrm{Res}(\lambda) < 0$.*
(2) *G has an eigenvalue 0 with multiplicity 1, and the corresponding right eigenvector is $(1, 1, \ldots, 1)^{\mathrm{T}}$.*
(3) *Suppose that $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^{\mathrm{T}} \in \mathbb{R}^N$, $\sum_{i=1}^{N} \xi_i = 1$ is the normalised left eigenvector of G with respect to eigenvalue 0, then we have $\xi_i > 0$ for all $i = 1, 2, \ldots, N$. Especially, if G is symmetric, then it can be $\xi_i = 1/N$ for $i = 1, 2, \ldots, N$.*

Lemma 2 [35]. *For any vectors $x, y \in \mathbb{R}^n$, and positive-definite matrix $Q \in \mathbb{R}^{n\times n}$, the following inequality holds*:

$$2x^{\mathrm{T}}y \leq x^{\mathrm{T}}Gx + y^{\mathrm{T}}G^{-1}y.$$

Lemma 3 [28]. *Let $\mu \in \mathbb{R}$, and P, Q, R, S be matrices with appropriate dimensions. Then the Kronecker product has the following properties:*

(1) $(P \otimes Q)^{\mathrm{T}} = P^{\mathrm{T}} \otimes Q^{\mathrm{T}}$.
(2) $(\mu P) \otimes Q = P \otimes \mu Q$.
(3) $(P + Q) \otimes R = P \otimes R + Q \otimes R$.
(4) $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$.

Moreover, the following assumptions are also necessary to draw our conclusions.

Assumption 1. For $j = 1, 2, \ldots, n$, suppose the activation functions $f_j(\cdot)$ satisfy the Lipschitz condition, i.e. there exist constants $l_j > 0$ such that

$$|f_j(u) - f_j(v)| \le l_j |u - v|$$

hold for any $u, v \in \mathbb{R}$, $j = 1, 2, \ldots, n$. Denote $L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\}$ for convenience.

Assumption 2. For $k = 1, 2, \ldots$, suppose that the impulsive gains satisfy $0 < \gamma_k < 2$.

3. Passivity of impulsive inertial NNs with coupling via directed topology

In this section, the coupled network under directed topology is considered, and the passivities of the target network are analysed, including strictly input passivity and strictly output passivity.

Let the coupling matrix G of (1) is defined as follows: if there exists a connection from node j to node i, then $G_{ij} > 0$, otherwise, $G_{ij} = 0\ (i \neq j)$, and the diagonal elements of the matrix G are defined by

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij}, \quad i = 1, 2, \ldots, N.$$

It means that the network is directed and the matrix G may be asymmetric. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^{\mathrm{T}}$ is the normalised left eigenvector of G corresponding to eigenvalue 0, i.e. $\xi^{\mathrm{T}} G = 0$ and $\sum_{i=1}^{N} \xi_i = 1$. Then it is seen from Lemma 1 that $\xi_i > 0$ for $i = 1, 2, \ldots, N$. Let $\bar{x}(t) = \sum_{i=1}^{N} \xi_i x_i(t)$ and define $e_i(t) = x_i(t) - \bar{x}(t)$, the dynamics of the error system relating to (1) is given by

$$\begin{cases} \dfrac{\mathrm{d}^2 e_i(t)}{\mathrm{d}t^2} = -D\dfrac{\mathrm{d}e_i(t)}{\mathrm{d}t} - Ce_i(t) \\ \quad + A\tilde{f}(e_i(t)) + B\tilde{f}(e_i(t-\tau(t))) \\ \quad + c\displaystyle\sum_{j=1}^{N} G_{ij}\Gamma\left(\dfrac{\mathrm{d}e_j(t)}{\mathrm{d}t} + e_j(t)\right) + \tilde{u}_i(t), \\ \quad t \neq t_k, \\ \Delta e_i(t_k) = -\delta_k e_i(t_k^-), \quad \Delta \dot{e}_i(t_k) = -\delta_k \dot{e}_i(t_k^-), \end{cases} \tag{3}$$

where $\tilde{f}(e_i(t)) = f(x_i(t)) - \sum_{j=1}^{N} \xi_j f(x_j(t))$ and $\tilde{u}_i(t) = u_i(t) - \sum_{j=1}^{N} \xi_j u_j(t)$.

Next, introduce the variable transformation $z_i(t) = (\mathrm{d}e_i(t)/\mathrm{d}t) + e_i(t)$, then the second-order differential system (3) can be degenerated by

$$\begin{cases} \dfrac{\mathrm{d}e_i(t)}{\mathrm{d}t} = -e_i(t) + z_i(t), \\ \dfrac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -\tilde{C}e_i(t) - \tilde{D}z_i(t) \\ \quad + A\tilde{f}(e_i(t)) + B\tilde{f}(e_i(t-\tau(t))) \\ \quad + c\displaystyle\sum_{j=1}^{N} G_{ij}\Gamma z_j(t) + \tilde{u}_i(t), \ t \neq t_k, \\ \Delta e_i(t_k) = -\delta_k e_i(t_k^-), \quad \Delta z_i(t_k) = -\delta_k z_i(t_k^-), \end{cases} \tag{4}$$

where $\tilde{C} = C + I_n - D$ and $\tilde{D} = D - I_n$.

Further, letting $e(t) = (e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_N^{\mathrm{T}})^{\mathrm{T}}$ and $z(t) = (z_1^{\mathrm{T}}, z_2^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}})^{\mathrm{T}}$ and combining the operator of the Kronecker product, system (4) can be written in the following compact form:

$$\begin{cases} \dfrac{\mathrm{d}e(t)}{\mathrm{d}t} = -e(t) + z(t), \\ \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = -(I_N \otimes \tilde{C})e(t) - (I_N \otimes \tilde{D})z(t) \\ \qquad + (I_N \otimes A)\tilde{F}(e(t)) + c(G \otimes \Gamma)z(t) \\ \qquad + (I_N \otimes B)\tilde{F}(e(t-\tau(t))) + \tilde{U}(t), \\ \qquad t \neq t_k, \\ \Delta e(t_k) = -\delta_k e(t_k^-), \quad \Delta z(t_k) = -\delta_k z(t_k^-), \end{cases} \tag{5}$$

where $\tilde{F}(e(t)) = (\tilde{f}(e_1(t)), \tilde{f}(e_2(t)), \ldots, \tilde{f}(e_N(t)))^{\mathrm{T}}$ and $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)^{\mathrm{T}}$.

For the analysis of the passivity for system (5), the corresponding output vector $y(t)$ is defined as

$$y(t) = (I_N \otimes F)e(t) + (I_N \otimes H)u(t), \tag{6}$$

where $F, H \in \mathbb{R}^{n\times n}$ are known real matrices.

Theorem 1. *In the light of Assumptions 1 and 2, let* $\Xi = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}$ *and* $\tilde{G} = \Xi G + G^{\mathrm{T}}\Xi$. *Then the coupled inertial system (3) accompanied by the output system (6) is strictly input passive if there exist a scalar* $\gamma > 0$ *and a matrix* $P = \mathrm{diag}(P^1, P^2, \ldots, P^N) > 0$ $(P^i \in \mathbb{R}^{n\times n}, i = 1, 2, \ldots, N)$ *such that*

$$H + H^{\mathrm{T}} - \gamma I_n > 0 \tag{7}$$

and

$$\begin{pmatrix} -2P + \Upsilon_1 & P - \Xi \otimes \tilde{C} \\ P - \Xi \otimes \tilde{C} & \Upsilon_2 \end{pmatrix} \le 0, \tag{8}$$

where

$$\Upsilon_1 = \frac{2-\rho}{1-\rho}(\Xi \otimes L^2) + 2I_N \otimes \big(F^{\mathrm{T}}(H+H^{\mathrm{T}}-\gamma I_n)^{-1}F\big),$$
$$\Upsilon_2 = \Xi \otimes (-2\tilde{D} + AA^{\mathrm{T}} + BB^{\mathrm{T}}) + c(\tilde{G}\otimes\Gamma) + 2\Xi^2 \otimes (H+H^{\mathrm{T}}-\gamma I_n)^{-1}.$$

Proof. Choose the Lyapunov functional as

$$V(t) = V_1(t) + V_2(t), \tag{9}$$

where

$$V_1(t) = e^{\mathrm{T}}(t)Pe(t) + z^{\mathrm{T}}(t)(\Xi\otimes I_n)z(t),$$
$$V_2(t) = \frac{1}{1-\rho}\int_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)(\Xi\otimes L^2)e(s)\,\mathrm{d}s.$$

Firstly, taking the time derivative of $V_1(t)$ along the trajectory of (5) leads to

$$\begin{aligned}\dot{V}_1(t) &= 2e^{\mathrm{T}}(t)P\dot{e}(t) + 2z^{\mathrm{T}}(t)(\Xi\otimes I_n)\dot{z}(t)\\ &= -2e^{\mathrm{T}}(t)Pe(t) + 2e^{\mathrm{T}}(t)Pz(t)\\ &\quad - 2z^{\mathrm{T}}(t)(\Xi\otimes\tilde{C})e(t)\\ &\quad - 2z^{\mathrm{T}}(t)(\Xi\otimes\tilde{D})z(t) + 2z^{\mathrm{T}}(t)(\Xi\otimes A)\tilde{F}(e(t))\\ &\quad + 2z^{\mathrm{T}}(t)(\Xi\otimes B)\tilde{F}(e(t-\tau(t)))\\ &\quad + 2cz^{\mathrm{T}}(t)(\Xi G\otimes\Gamma)z(t) + 2z^{\mathrm{T}}(t)(\Xi\otimes I_n)\tilde{U}(t).\end{aligned} \tag{10}$$

Since $\sum_{i=1}^{N}\xi_i = 1$ and combining with the definitions of $e_i(t)$ and $\bar{x}(t)$, we can derive that

$$\sum_{i=1}^{N}\xi_i e_i(t) = \sum_{i=1}^{N}\xi_i(x_i(t)-\bar{x}(t)) = \sum_{i=1}^{N}\xi_i\left(x_i(t) - \sum_{j=1}^{N}\xi_j x_j(t)\right) = 0 \tag{11}$$

and

$$\sum_{i=1}^{N}\xi_i \dot{e}_i(t) = \sum_{i=1}^{N}\xi_i\left(\dot{x}_i(t) - \sum_{j=1}^{N}\xi_j \dot{x}_j(t)\right) = 0. \tag{12}$$

The combination of (11) and (12) yields

$$\sum_{i=1}^{N}\xi_i z_i(t) = \sum_{i=1}^{N}\xi_i(\dot{e}_i(t)+e_i(t)) = 0. \tag{13}$$

Then, by means of (13), Lemma 2 and Assumption 1, it is found that

$$\begin{aligned}&2z^{\mathrm{T}}(t)(\Xi\otimes A)\tilde{F}(e(t))\\ &= 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\xi_i A\left(f(x_i(t)) - \sum_{j=1}^{N}\xi_j f(x_j(t))\right)\\ &= 2\sum_{i=1}^{N}\xi_i z_i^{\mathrm{T}}(t)A(f(x_i(t)) - f(\bar{x}(t)))\\ &\quad + 2\left(\sum_{i=1}^{N}\xi_i z_i^{\mathrm{T}}(t)\right)A\left(f(\bar{x}(t)) - \sum_{j=1}^{N}\xi_j f(x_j(t)\right)\\ &= 2\sum_{i=1}^{N}\xi_i z_i^{\mathrm{T}}(t)A(f(x_i(t)) - f(\bar{x}(t)))\\ &\le \sum_{i=1}^{N}\xi_i z_i^{\mathrm{T}}(t)AA^{\mathrm{T}}z_i(t) + \sum_{i=1}^{N}\xi_i\,(f(x_i(t))\\ &\quad - f(\bar{x}(t)))^{\mathrm{T}}(f(x_i(t)) - f(\bar{x}(t)))\\ &\le z^{\mathrm{T}}(t)(\Xi\otimes AA^{\mathrm{T}})z(t) + e^{\mathrm{T}}(t)(\Xi\otimes L^2)e(t).\end{aligned} \tag{14}$$

Take the same schemes, we can obtain

$$\begin{aligned}2z^{\mathrm{T}}(t)(\Xi\otimes B)\tilde{F}(e(t-\tau(t))) &\le z^{\mathrm{T}}(t)(\Xi\otimes BB^{\mathrm{T}})z(t)\\ &\quad + e^{\mathrm{T}}(t-\tau(t))(\Xi\otimes L^2)e(t-\tau(t)).\end{aligned} \tag{15}$$

Besides, it is also acquired from (13) that

$$\begin{aligned}&2z^{\mathrm{T}}(t)(\Xi\otimes I_n)\tilde{U}(t)\\ &= 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\xi_i\left(u_i(t) - \sum_{j=1}^{N}\xi_j u_j(t)\right)\\ &= 2\sum_{i=1}^{N} z_i^{\mathrm{T}}(t)\xi_i u_i(t)\\ &= 2z^{\mathrm{T}}(t)(\Xi\otimes I_n)u(t).\end{aligned} \tag{16}$$

Applying (14)–(16) to (10) deduces

$$\begin{aligned}\dot{V}_1(t) \le\; & e^{\mathrm{T}}(t)\{-2P + \Xi\otimes L^2\}e(t)\\ & +2e^{\mathrm{T}}(t)\{P - \Xi\otimes\tilde{C}\}z(t)\\ & +z^{\mathrm{T}}(t)\{\Xi\otimes(-2\tilde{D} + AA^{\mathrm{T}} + BB^{\mathrm{T}})\\ & +c(\tilde{G}\otimes\Gamma)\}z(t)\\ & +e^{\mathrm{T}}(t-\tau(t))(\Xi\otimes L^2)e(t-\tau(t))\\ & +2z^{\mathrm{T}}(t)(\Xi\otimes I_n)u(t).\end{aligned} \tag{17}$$

Next, by taking the time derivative of $V_2(t)$ along the trajectory (3), and noting that $0 < \dot{\tau}(t) < \rho$, we have

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{1-\rho}e^{\mathrm{T}}(t)(\Xi\otimes L^2)e(t)\\ &\quad -\frac{(1-\dot{\tau}(t))}{1-\rho}e^{\mathrm{T}}(t-\tau(t))(\Xi\otimes L^2)e(t-\tau(t))\\ &< \frac{1}{1-\rho}e^{\mathrm{T}}(t)(\Xi\otimes L^2)e(t)\\ &\quad -e^{\mathrm{T}}(t-\tau(t))(\Xi\otimes L^2)e(t-\tau(t)). \qquad (18)\end{aligned}$$

In light of (9), (17) and (18), the estimate of $\dot{V}(t)$ can be expressed by

$$\begin{aligned}\dot{V}(t) &\le e^{\mathrm{T}}(t)\left\{-2P+\frac{2-\rho}{1-\rho}\Xi\otimes L^2\right\}e(t)\\ &\quad +2e^{\mathrm{T}}(t)\{P-\Xi\otimes\tilde{C}\}z(t)\\ &\quad +z^{\mathrm{T}}(t)\{\Xi\otimes(-2\tilde{D}+AA^{\mathrm{T}}+BB^{\mathrm{T}})\\ &\quad +c(\tilde{G}\otimes\Gamma)\}z(t)+2z^{\mathrm{T}}(t)(\Xi\otimes I_n)u(t). \qquad (19)\end{aligned}$$

Owing to the output (6), we have

$$\begin{aligned}&\dot{V}(t)-2y^{\mathrm{T}}(t)u(t)+\gamma u^{\mathrm{T}}(t)u(t)\\ &\quad\le e^{\mathrm{T}}(t)\left\{-2P+\frac{2-\rho}{1-\rho}\Xi\otimes L^2\right\}e(t)\\ &\quad +2z^{\mathrm{T}}(t)(\Xi\otimes I_n)u(t)\\ &\quad +z^{\mathrm{T}}(t)\{\Xi\otimes(-2\tilde{D}+AA^{\mathrm{T}}+BB^{\mathrm{T}})\\ &\quad +c(\tilde{G}\otimes\Gamma)\}z(t)\\ &\quad +2e^{\mathrm{T}}(t)\{P-\Xi\otimes\tilde{C}\}z(t)-2e^{\mathrm{T}}(t)(I_N\otimes F^{\mathrm{T}})u(t)\\ &\quad -u^{\mathrm{T}}(t)(I_N\otimes(H^{\mathrm{T}}+H-\gamma H^{\mathrm{T}}H))u(t). \qquad (20)\end{aligned}$$

By Lemma 2 and condition (7), we have

$$\begin{aligned}&2z^{\mathrm{T}}(t)(\Xi\otimes I_n)u(t)\le 2z^{\mathrm{T}}(t)(\Xi\otimes I_n)\\ &\quad\times(I_N\otimes(H+H^{\mathrm{T}}-\gamma I_n)^{-1})(\Xi\otimes I_n)z(t)\\ &\quad+\frac{1}{2}u^{\mathrm{T}}(t)(I_N\otimes(H+H^{\mathrm{T}}-\gamma I_n))u(t)\\ &= 2z^{\mathrm{T}}(t)(\Xi^2\otimes(H+H^{\mathrm{T}}-\gamma I_n)^{-1})z(t)\\ &\quad+\frac{1}{2}u^{\mathrm{T}}(t)(I_N\otimes(H+H^{\mathrm{T}}-\gamma I_n))u(t) \qquad (21)\end{aligned}$$

and

$$\begin{aligned}&-2e^{\mathrm{T}}(t)(I_N\otimes F^{\mathrm{T}})u(t)\\ &\quad\le 2e^{\mathrm{T}}(t)(I_N\otimes F^{\mathrm{T}})(I_N\otimes(H+H^{\mathrm{T}}-\gamma I_n)^{-1})\\ &\quad\times(I_N\otimes F)e(t)+\frac{1}{2}u^{\mathrm{T}}(t)\\ &\quad\times(I_N\otimes(H+H^{\mathrm{T}}-\gamma I_n))u(t)\\ &\quad= 2e^{\mathrm{T}}(t)(I_N\otimes\left(F^{\mathrm{T}}(H+H^{\mathrm{T}}-\gamma I_n)^{-1}F\right))e(t)\\ &\quad+\frac{1}{2}u^{\mathrm{T}}(t)(I_N\otimes(H+H^{\mathrm{T}}-\gamma I_n))u(t). \qquad (22)\end{aligned}$$

Letting $\zeta(t)=(e^{\mathrm{T}}(t),z^{\mathrm{T}}(t))^{\mathrm{T}}$ and then substituting (21) and (22) into (20) we derive

$$\begin{aligned}&\dot{V}(t)-2y^{\mathrm{T}}(t)u(t)+\gamma u^{\mathrm{T}}(t)u(t)\\ &\quad\le e^{\mathrm{T}}(t)\Bigg\{-2P+\frac{2-\rho}{1-\rho}(\Xi\otimes L^2)\\ &\quad+2I_N\otimes\left(F^{\mathrm{T}}(H+H^{\mathrm{T}}-\gamma I_n)^{-1}F\right)\Bigg\}e(t)\\ &\quad+z^{\mathrm{T}}(t)\Bigg\{\Xi\otimes(-2\tilde{D}+AA^{\mathrm{T}}+BB^{\mathrm{T}})\\ &\quad+c(\tilde{G}\otimes\Gamma)+2\Xi^2\otimes\ (H+H^{\mathrm{T}}-\gamma I_n)^{-1}\Bigg\}z(t)\\ &\quad+2e^{\mathrm{T}}(t)\{P-\Xi\otimes\tilde{C}\}z(t)\\ &\quad=\zeta^{\mathrm{T}}(t)\begin{pmatrix}-2P+\Upsilon_1 & P-\Xi\otimes\tilde{C}\\ P-\Xi\otimes\tilde{C} & \Upsilon_2\end{pmatrix}\\ &\quad\times\zeta(t)<0,\quad t\ne t_k. \qquad (23)\end{aligned}$$

For any $t_p>t_0$, there exists $m\in\mathbb{Z}^+$ such that $t_m\le t_p<t_{m+1}$. By integrating (23) with respect to t from t_0 to t_p, one can obtain

$$\begin{aligned}&\int_{t_0}^{t_p}[\dot{V}(t)-2y^{\mathrm{T}}(t)u(t)+\gamma u^{\mathrm{T}}(t)u(t)]\mathrm{d}t\\ &\quad=\sum_{l=1}^{m}\int_{t_{l-1}}^{t_l}\dot{V}(t)\,\mathrm{d}t+\int_{t_m}^{t_p}\dot{V}(t))\,\mathrm{d}t\\ &\quad-\int_{t_0}^{t_p}[2y^{\mathrm{T}}(t)u(t)-\gamma u^{\mathrm{T}}(t)u(t)]\mathrm{d}t\\ &\quad=\sum_{l=1}^{m}[V(t_l^-)-V(t_{l-1})]+V(t_p)-V(t_m)\\ &\quad-\int_{t_0}^{t_p}[2y^{\mathrm{T}}(t)u(t)-\gamma u^{\mathrm{T}}(t)u(t)]\mathrm{d}t\\ &\quad=\sum_{l=1}^{m}[V(t_l^-)-V(t_l)]+V(t_p)-V(t_0)\\ &\quad-\int_{t_0}^{t_p}[2y^{\mathrm{T}}(t)u(t)-\gamma u^{\mathrm{T}}(t)u(t)]\mathrm{d}t. \qquad (24)\end{aligned}$$

On the other hand, it is known from Assumption 2 that $|1-\delta_k|<1$ for all $k\in\mathbb{Z}^+$, and noticing the impulses formed as (5), one yields

$$V(t_k)=e^{\mathrm{T}}(t_k)Pe(t_k)+z^{\mathrm{T}}(t_k)(\Xi\otimes I_n)z(t_k)$$

$$+\frac{1}{1-\rho}\int_{t_k-\tau(t_k)}^{t_k} e^{\mathrm{T}}(s)(\Xi\otimes L^2)e(s)\mathrm{d}s$$
$$=(1-\delta_k)^2 e^{\mathrm{T}}(t_k^-)Pe(t_k^-)$$
$$+(1-\delta_k)^2 z^{\mathrm{T}}(t_k^-)(\Xi\otimes I_n)\cdot$$
$$z(t_k^-)+\frac{1}{1-\rho}\int_{t_k^- -\tau(t_k^-)}^{t_k-} e^{\mathrm{T}}(s)(\Xi\otimes L^2)e(s)\mathrm{d}s$$
$$\le e^{\mathrm{T}}(t_k^-)Pe(t_k^-)+z^{\mathrm{T}}(t_k^-)(\Xi\otimes I_n)z(t_k^-)$$
$$+\frac{1}{1-\rho}\int_{t_k^- -\tau(t_k^-)}^{t_k-} e^{\mathrm{T}}(s)(\Xi\otimes L^2)e(s)\mathrm{d}s$$
$$\le V(t_k^-). \tag{25}$$

The combination of (23)–(25) yields

$$V(t_p)-V(t_0)-\int_{t_0}^{t_p}[2y^{\mathrm{T}}(t)u(t)-\gamma u^{\mathrm{T}}(t)u(t)]\mathrm{d}t$$
$$\le\int_{t_0}^{t_p}[\dot V(t)-2y^{\mathrm{T}}(t)u(t)+\gamma u^{\mathrm{T}}(t)u(t)]\mathrm{d}t\le 0,$$

which implies that

$$\int_{t_0}^{t_p} y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t\ge\frac{V(t_p)}{2}-\frac{V(t_0)}{2}+\frac{\gamma}{2}\int_{t_0}^{t_p} u^{\mathrm{T}}(t)u(t)\,\mathrm{d}t.$$

Thus, from Definition 2, the strictly input passivity of system (1) is proved. □

Theorem 2. *On the basis of the same assumptions and notations as Theorem 1, the coupled inertial system (3) accompanied by output system (6) is strictly output passive if there exist a scalar $\gamma>0$ and a matrix $P>0$ such that*

$$H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H>0 \tag{26}$$

and

$$\begin{pmatrix}-2P+\Theta_1 & P-\Xi\otimes\tilde C\\ P-\Xi\otimes\tilde C & \Theta_2\end{pmatrix}\le 0, \tag{27}$$

where

$$\Theta_1=\frac{2-\rho}{1-\rho}(\Xi\otimes L^2)+I_N\otimes(\gamma F^{\mathrm{T}}F)$$
$$+2I_N\otimes(F^{\mathrm{T}}(\gamma H-I_n)(H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H)^{-1}$$
$$\times(\gamma H^{\mathrm{T}}-I_n)F),$$
$$\Theta_2=\Xi\otimes(-2\tilde D+AA^{\mathrm{T}}+BB^{\mathrm{T}})+c(\tilde G\otimes\Gamma)$$
$$+2\Xi^2\otimes(H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H)^{-1}.$$

Proof. Employ the same Lyapunov functional $V(t)$ as in Theorem 1, and the time derivative of $V(t)$ along system (5) is estimated as in (19), then by means of the output (6), we can derive

$$\dot V(t)-2y^{\mathrm{T}}(t)u(t)+\gamma y^{\mathrm{T}}(t)y(t)$$
$$\le e^{\mathrm{T}}(t)\left\{-2P+\frac{2-\rho}{1-\rho}\Xi\otimes L^2+\gamma I_N\otimes(F^{\mathrm{T}}F)\right\}e(t)$$
$$+z^{\mathrm{T}}(t)\{\Xi\otimes(-2\tilde D+AA^{\mathrm{T}}+BB^{\mathrm{T}})$$
$$+c(\tilde G\otimes\Gamma)\}z(t)+2z^{\mathrm{T}}(t)(\Xi\otimes I_n)u(t)$$
$$+2e^{\mathrm{T}}(t)\{I_N\otimes F^{\mathrm{T}}(\gamma H-I_n)\}u(t)$$
$$+2e^{\mathrm{T}}(t)\{P-\Xi\otimes\tilde C\}z(t)$$
$$-u^{\mathrm{T}}(t)(I_N\otimes(H^{\mathrm{T}}+H-\gamma H^{\mathrm{T}}H))u(t). \tag{28}$$

By Lemma 2 and condition (26), we have

$$2e^{\mathrm{T}}(t)\{I_N\otimes(F^{\mathrm{T}}(\gamma H-I_n))\}u(t)$$
$$\le 2e^{\mathrm{T}}(t)(I_N\otimes(F^{\mathrm{T}}(\gamma H-I_n)))$$
$$\times(I_N\otimes H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H)^{-1})$$
$$\times(I_N\otimes((\gamma H^{\mathrm{T}}-I_n)F))e(t)$$
$$+\frac{1}{2}u^{\mathrm{T}}(t)(I_N\otimes(H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H))u(t)$$
$$=2e^{\mathrm{T}}(t)\{I_N\otimes(F^{\mathrm{T}}(\gamma H-I_n)(H+H^{\mathrm{T}}$$
$$-\gamma H^{\mathrm{T}}H)^{-1}(\gamma H^{\mathrm{T}}-I_n)F)\}e(t)$$
$$+\frac{1}{2}u^{\mathrm{T}}(t)(I_N\otimes(H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H))u(t). \tag{29}$$

Let $\zeta(t)=(e^{\mathrm{T}}(t),z^{\mathrm{T}}(t))^{\mathrm{T}}$, and substitute (29) and (21) into (28), then for $t\ne t_k$ it is obtained that

$$\dot V(t)-2y^{\mathrm{T}}(t)u(t)+\gamma y^{\mathrm{T}}(t)y(t)$$
$$\le e^{\mathrm{T}}(t)\Bigg\{-2P+\frac{2-\rho}{1-\rho}(\Xi\otimes L^2)+I_N\otimes(\gamma F^{\mathrm{T}}F)$$
$$+2I_N\otimes(F^{\mathrm{T}}(\gamma H-I_n)(H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H)^{-1}$$
$$\times(\gamma H^{\mathrm{T}}-I_n)F)\Bigg\}e(t)$$
$$+2e^{\mathrm{T}}(t)\{P-\Xi\otimes\tilde C\}z(t)$$
$$+z^{\mathrm{T}}(t)\{\Xi\otimes(-2\tilde D+AA^{\mathrm{T}}+BB^{\mathrm{T}})$$
$$+c(\tilde G\otimes\Gamma)+2\Xi^2\otimes(H+H^{\mathrm{T}}-\gamma H^{\mathrm{T}}H)^{-1}\}z(t)$$
$$=\zeta^{\mathrm{T}}(t)\begin{pmatrix}-2P+\Theta_1 & P-\Xi\otimes\tilde C\\ P-\Xi\otimes\tilde C & \Theta_2\end{pmatrix}\zeta(t)<0. \tag{30}$$

Then, based on (30) and applying that similar techniques as in (24) and (25), we can conclude that

$$V(t_p)-V(t_0)-\int_{t_0}^{t_p}[2y^{\mathrm{T}}(t)u(t)-\gamma y^{\mathrm{T}}(t)y(t)]\mathrm{d}t$$
$$\le\int_{t_0}^{t_p}[\dot V(t)-2y^{\mathrm{T}}(t)u(t)+\gamma y^{\mathrm{T}}(t)y(t)]\mathrm{d}t\le 0,$$

which implies that

$$\int_{t_0}^{t_p} y^{\mathrm{T}}(t)u(t)\,\mathrm{d}t \geq \frac{V(t_p)}{2} - \frac{V(t_0)}{2} + \frac{\gamma}{2}\int_{t_0}^{t_p} y^{\mathrm{T}}(t)y(t)\,\mathrm{d}t.$$

So the strictly output passivity of system (1) in the sense of Definition 2 holds. □

Remark 2. According to the formation of $\tilde{G}$ and Γ, we can read that $\tilde{G}\otimes\Gamma \leq 0$ from Lemma 1, which implies that $\Upsilon_2 \leq 0$ and $\Theta_2 \leq 0$ are possible. By Schur complement lemma [36], conditions (8) and (27) are reasonable.

Remark 3. The delayed inertial NNs are considered in [37,38], where the former addresses the stabilisation problem via periodically intermittent control, and the latter studies the synchronisation by means of the matrix measure technique. The coupled inertial NNs with time-varying delays are considered in [15] to realise the synchronisation based on the pinning control strategy, while the delayed inertial NN is stabilised in [14] under the designed impulsive control. Howbeit, none of them concerns delayed inertial NNs both with coupling and impulse effects simultaneously. Moreover, none of them involved the passivity analysis of the inertial NNs, which is exactly our aim in this paper. From Theorems 1 and 2, the information of the coupled matrix, which implies the structure of the network topology, is utilised not only to construct the Lyapunov functional but also to establish the sufficient conditions of passivity. Thus, our model is more universal, the method is different and the passivity results fill the gap in the field of the inertial NNs.

4. Passivity of impulsive inertial NNs with coupling via undirected topology

In this section, we consider the case when network (1) is undirected, which implies the coupling matrix G is symmetric and is defined as for $i \neq j$, if there exists a connection between node i and node j, then $G_{ij} = G_{ji} > 0$, otherwise, $G_{ij} = G_{ji} = 0$, and the diagonal elements of matrix G are defined by

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij}, \quad i = 1, 2, \ldots, N.$$

Let $\bar{x}(t) = (1/N)\sum_{i=1}^{N} x_i(t)$, define $e(t) = (x_1(t) - \bar{x}(t), \ldots, x_N(t) - \bar{x}(t))^{\mathrm{T}}$ and $z(t) = (\mathrm{d}e(t)/\mathrm{d}t) + e(t)$, we can derive the error system of (1) with an undirected topology by

$$\begin{cases} \dfrac{\mathrm{d}e(t)}{\mathrm{d}t} = -e(t) + z(t), \\ \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = -(I_N\otimes\tilde{C})e(t) - (I_N\otimes\tilde{D})z(t) \\ \quad + (I_N\otimes A)\tilde{F}(e(t)) + c(G\otimes\Gamma)z(t) \\ \quad + (I_N\otimes B)\tilde{F}(e(t-\tau(t))) + \tilde{U}(t),\ t\neq t_k, \\ \Delta e(t_k) = -\delta_k e(t_k^-),\ \Delta z(t_k) = -\delta_k z(t_k^-), \end{cases} \tag{31}$$

where the coefficient matrices are the same as (5). Besides, $\tilde{F}(e(t)) = (\tilde{f}(e_1(t)), \ldots, \tilde{f}(e_N(t)))^{\mathrm{T}}$ with $\tilde{f}(e_i(t)) = f(e_i(t)) - (1/N)\sum_{j=1}^{N} f(e_j(t))$, and $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_N)^{\mathrm{T}}$ with $\tilde{u}_i = u_i(t) - (1/N)\sum_{j=1}^{N} u_j(t)$.

Theorem 3. *Suppose that Assumptions 1 and 2 hold, then the undirected system (31) and the corresponding output system (6) are strictly input passive, if there exist a scalar $\gamma > 0$ and a positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ such that*

$$H + H^{\mathrm{T}} - \gamma I_n > 0 \tag{32}$$

and

$$\begin{pmatrix} -2Q + \Psi_1 & Q - \tilde{C} \\ Q - \tilde{C} & \Psi_2 \end{pmatrix} \leq 0, \tag{33}$$

where

$$\Psi_1 = \frac{2-\rho}{1-\rho}L^2 + 2F^{\mathrm{T}}(H + H^{\mathrm{T}} - \gamma I_n)^{-1}F,$$
$$\Psi_2 = -2\tilde{D} + AA^{\mathrm{T}} + BB^{\mathrm{T}} + 2c\lambda_2(G)\Gamma + 2(H + H^{\mathrm{T}} - \gamma I_n)^{-1}.$$

Proof. Construct the Lyapunov functional as follows:

$$V(t) = e^{\mathrm{T}}(t)(I_N\otimes Q)e(t) + z^{\mathrm{T}}(t)z(t) + \frac{1}{1-\rho}\int_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)(I_N\otimes L^2)e(s)\,\mathrm{d}s.$$

Similar to the derivation of (17), the time derivative of $V(t)$ along the trajectory of (31) is estimated by

$$\begin{aligned} \dot{V}(t) \leq{}& e^{\mathrm{T}}(t)\left\{I_N\otimes\left(-2Q + \frac{2-\rho}{1-\rho}L^2\right)\right\}e(t) \\ &+ 2e^{\mathrm{T}}(t)\{I_N\otimes(Q-\tilde{C})\}z(t) \\ &+ z^{\mathrm{T}}(t)\{I_N\otimes(-2\tilde{D} + AA^{\mathrm{T}} + BB^{\mathrm{T}})\}z(t) \\ &+ 2cz^{\mathrm{T}}(t)(G\otimes\Gamma)z(t) + 2z^{\mathrm{T}}(t)u(t). \end{aligned} \tag{34}$$

Now we focus on the estimation of the coupling term in (34). Since G is an irreducible symmetric matrix with non-negative off-diagonal elements, zero row sum and zero column sum, then by Lemma 1, we have $0 = \lambda_1(G) > \lambda_2(G) > \cdots > \lambda_N(G)$. Furthermore, there exists a unitary matrix $V = (v_1, \ldots, v_N) \in \mathbb{R}^{N\times N}$ such that $G = V\Lambda V^{\mathrm{T}}$, where Λ = diag

$\{0, \lambda_2(G), \ldots, \lambda_N(G)\}$ and $v_1 = (1/\sqrt{N}, 1/\sqrt{N} \ldots, 1/\sqrt{N})^{\mathrm{T}}$.

Let $\eta(t) = (V^{\mathrm{T}} \otimes I_n)z(t)$ and since $\sum_{i=1}^{N} z_i(t) = 0$, one has

$$\eta_1(t) = (v_1^{\mathrm{T}} \otimes I_n)z(t) = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} z_i(t) = 0. \tag{35}$$

Noting that $\lambda_1(G) = 0$, together with Lemma 3 and (35), we have

$$\begin{aligned} 2cz^{\mathrm{T}}(t)\big(G \otimes \Gamma\big)z(t) &= 2cz^{\mathrm{T}}(t)((V\Lambda V^{\mathrm{T}}) \otimes \Gamma)z(t) \\ &= 2c\eta^{\mathrm{T}}(t)(\Lambda \otimes \Gamma)\eta(t) \\ &\leq 2c\lambda_2(G) \sum_{i=2}^{N} \eta_i^{\mathrm{T}}(t)\Gamma\eta_i(t) \\ &= 2c\lambda_2(G) \sum_{i=1}^{N} \eta_i^{\mathrm{T}}(t)\Gamma\eta_i(t) \\ &= 2c\lambda_2(G)z^{\mathrm{T}}(t)(I_N \otimes \Gamma)z(t). \end{aligned} \tag{36}$$

Applying (36) to (34) we get

$$\begin{aligned} \dot{V}(t) \leq{}& e^{\mathrm{T}}(t) \left\{ I_N \otimes \left(-2Q + \frac{2-\rho}{1-\rho} L^2 \right) \right\} e(t) \\ &+ z^{\mathrm{T}}(t)\{I_N \otimes (-2\tilde{D} + AA^{\mathrm{T}} \\ &+ BB^{\mathrm{T}} + 2c\lambda_2(G)\Gamma)\}z(t) \\ &+ 2e^{\mathrm{T}}(t)\{I_N \otimes (Q - \tilde{C})\}z(t) + 2z^{\mathrm{T}}(t)u(t). \end{aligned} \tag{37}$$

Note the estimation (37) and the expression of output (6), as well as employ (21) and (22) by replacing Ξ with I_N, we can obtain

$$\begin{aligned} &\dot{V}(t) - 2y^{\mathrm{T}}(t)u(t) + \gamma u^{\mathrm{T}}(t)u(t) \\ &\leq e^{\mathrm{T}}(t)\bigg\{ I_N \otimes \bigg(-2Q + \frac{2-\rho}{1-\rho} L^2 + 2F^{\mathrm{T}} \\ &\quad \times (H + H^{\mathrm{T}} - \gamma I_n)^{-1} F \bigg) \bigg\} e(t) \\ &\quad + 2e^{\mathrm{T}}(t)\{I_N \otimes (Q - \tilde{C})\}z(t) \\ &\quad + z^{\mathrm{T}}(t)\{I_N \otimes (-2\tilde{D} + AA^{\mathrm{T}} + BB^{\mathrm{T}} + 2c\lambda_2(G)\Gamma \\ &\quad + 2(H + H^{\mathrm{T}} - \gamma I_n)^{-1}\}z(t) \\ &= \zeta^{\mathrm{T}}(t) \left\{ I_N \otimes \begin{pmatrix} -2Q + \Psi_1 & Q - \tilde{C} \\ Q - \tilde{C} & \Psi_2 \end{pmatrix} \right\} \\ &\quad \times \zeta(t) < 0, \quad t \neq t_k, \end{aligned}$$

where $\zeta(t) = (e^{\mathrm{T}}(t), z^{\mathrm{T}}(t))^{\mathrm{T}}$.

Then the rest of the proof matches *mutatis mutandis* to a similar proof in Theorem 1 and thus is omitted. So the strictly input passivity of (31) under the output (6) is obtained. □

Utilising (36), and making some slight alterations for the proof of Theorem 2, we can easily gain the strictly output passivity under the case of undirected topology, which is exhibited below without proof.

Theorem 4. *Under Assumptions 1 and 2, then the undirected system (31) is strictly output passive from input $u(t)$ to output vector described by (6), if there exist a scalar $\gamma > 0$ and a positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ such that*

$$H + H^{\mathrm{T}} - \gamma H^{\mathrm{T}} H > 0 \tag{38}$$

and

$$\begin{pmatrix} -2Q + \Phi_1 & Q - \tilde{C} \\ Q - \tilde{C} & \Phi_2 \end{pmatrix} \leq 0, \tag{39}$$

where

$$\begin{aligned} \Phi_1 ={}& \frac{2-\rho}{1-\rho} L^2 + \gamma F^{\mathrm{T}} F + 2F^{\mathrm{T}}(\gamma H - I_n) \\ &\times (H + H^{\mathrm{T}} - \gamma H^{\mathrm{T}} H)^{-1}(\gamma H^{\mathrm{T}} - I_n)F, \\ \Phi_2 ={}& -2\tilde{D} + AA^{\mathrm{T}} + BB^{\mathrm{T}} + 2c\lambda_2(G)\Gamma \\ &+ 2(H + H^{\mathrm{T}} - \gamma H^{\mathrm{T}} H)^{-1}. \end{aligned}$$

Remark 4. It is seen that under directed topology, the dimension of the matrix needed to be chosen in Lyapunov functional, namely P, reaches $Nn \times Nn$, and the dimension under LMI condition (8) or (27) is $2Nn \times 2Nn$. Comparatively, the corresponding magnitudes in the undirected network are, respectively, $n \times n$ and $2n \times 2n$, which greatly reduced the computations. So, Theorems 3 and 4 have the unique advantages during the modelling of the coupled inertial network on account of the passivity.

5. Numerical examples

Example 1. Consider a complex dynamical network including five identical nodes with impulsive effects, in which each node is a 3D NN modelled by

$$\begin{cases} \dfrac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} = -D\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + Af(x_i(t)) \\ \qquad + Bf(x_i(t - \tau(t))) + J + u_i(t) \\ \qquad + c\displaystyle\sum_{j=1}^{N} G_{ij}\Gamma\bigg(\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} + x_j(t)\bigg), \\ \qquad t \neq t_k, \\ \Delta x_i(t_k) = -\delta_k x(t_k^-), \ \ \Delta\dot{x}_i(t_k) = -\delta_k \dot{x}(t_k^-), \end{cases} \tag{40}$$

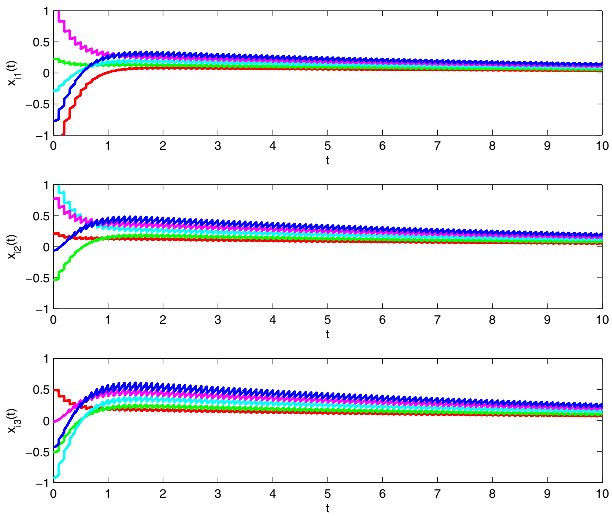

Figure 1. The states of systems (40) with non-zero control input.

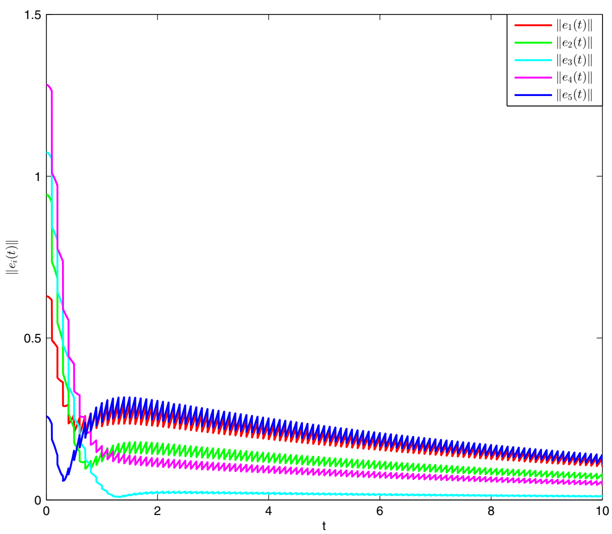

Figure 2. Norm evolutions of error vectors for (40) with non-zero control input.

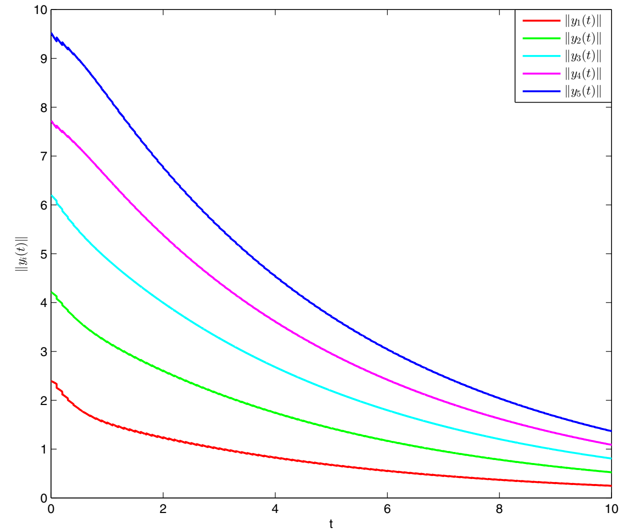

Figure 3. Norm evolutions of output vectors for (40) with non-zero control input.

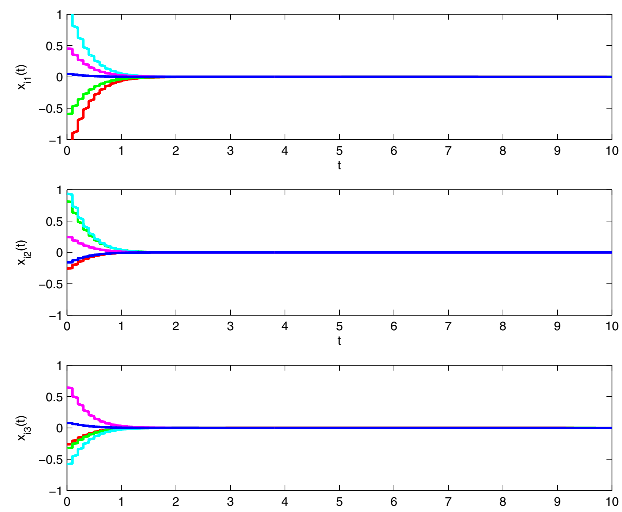

Figure 4. The states of systems (40) without control input.

where $i = 1, 2, \ldots, 5, k = 1, 2, \ldots, x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}}$, $f(x_i) = 0.2(\tanh(x_{i1}), \tanh(x_{i2}), \tanh(x_{i3}))^{\mathrm{T}}$, and the time delay is given by $\tau(t) = 0.3 - 0.3\mathrm{e}^{-t}$. So it is easy to obtain that $l_i = 0.2$, $\tau = 0.3$ and $\rho = 0.3$. Setting $D = \mathrm{diag}\{2.8, 3.0, 3.2\}$, $C = \mathrm{diag}\{2.8, 3.2, 3.5\}$, $J = (0, 0)^{\mathrm{T}}$ and $c = 0.6$, the other coefficient matrices and coupling matrix are given as

$$A = \begin{pmatrix} 0.35 & 0.75 & -0.4 \\ -0.55 & 0.6 & 0.65 \\ 0.5 & 0.3 & 0.7 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.5 & -0.7 & 0.7 \\ -0.6 & -0.35 & 0.4 \\ -0.55 & 0.75 & 0.45 \end{pmatrix},$$

$$G = \begin{pmatrix} -3.5 & 1.5 & 0 & 1 & 1 \\ 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -4 & 2 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Moreover, we take $\delta_k = 0.2$, $t_k - t_{k-1} = 0.1$. By employing the MATLAB function NULL, it is derived that the normalised left eigenvector of G is $\xi = (0.2222, 0.1667, 0.1667, 0.1667, 0.2778)^{\mathrm{T}}$. Choose

$$F = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}, H = \begin{pmatrix} 0.75 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}.$$

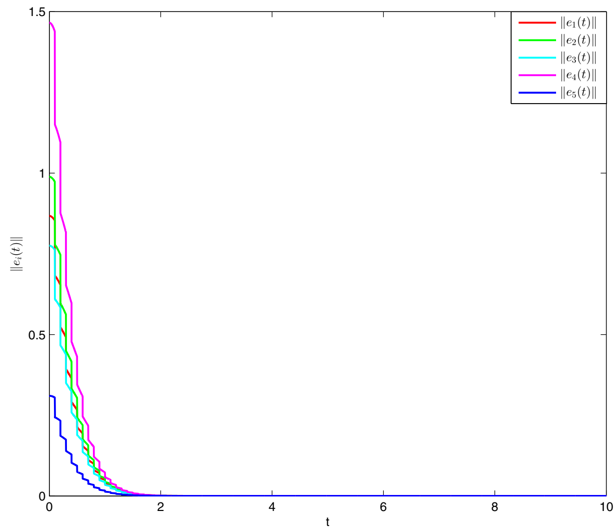

Figure 5. Norm evolutions of error vectors for (40) without control input.

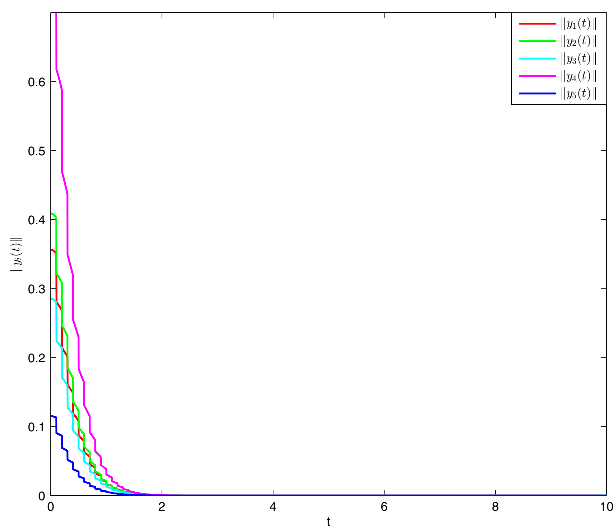

Figure 6. Norm evolutions of output vectors for (40) without control input.

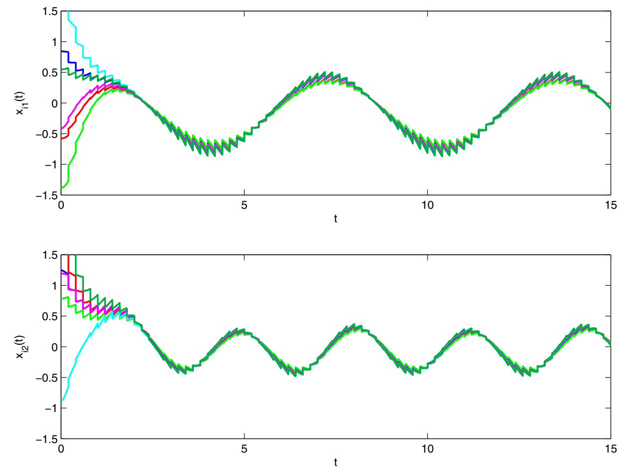

Figure 7. The states of systems (41) with non-zero control input.

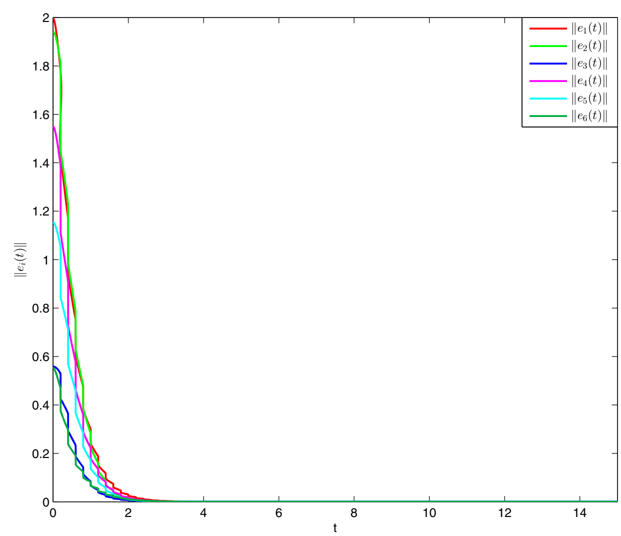

Figure 8. Norm evolutions of error vectors for (41) with non-zero control input.

Employing the YALMIP Toolbox of MATLAB, the feasible solution $P^1 = \text{diag}\{0.3612, 0.4985, 0.4249\}$, $P^2 = P^3 = P^4 = \text{diag}\{0.3438, 0.4780, 0.4027\}$ and $P^5 = \text{diag}\{0.3786, 0.5191, 0.4471\}$ can be acquired by referring to (8) with $\gamma = 0.4$. Thus, in light of Theorem 1, system (3) derived by (40) is strictly input passive under output system (6).

In addition, by the same technique, we can also find the matrices $P^1 = \text{diag}\{0.3565, 0.4473, 0.4016\}$, $P^2 = P^3 = P^4 = \text{diag}\{0.3391, 0.4267, 0.3794\}$ and $P^5 = \text{diag}\{0.3739, 0.4679, 0.4238\}$ asserting (27) accompanied by $\gamma = 0.7$. From Theorem 2, system (3) derived from (40) is strictly output passive with the output (6).

Pick the control input as $u_{i1} = i\mathrm{e}^{-0.1t}$, $u_{i2} = 1.5i\mathrm{e}^{-0.1t}$ and $u_{i3} = 2i\mathrm{e}^{-0.1t}$, $i = 1, 2, \ldots, 5$, the initial values are randomly selected within interval $[-1.2, 1.2]$, the results of numerical simulations are described in figures 1–3, Meanwhile, the simulation results without control input are also exhibited in figures 4–6 for comparison. It can be directly observed that the numerical conclusions affirm Theorems 1 and 2.

Example 2. Consider a coupled network with undirected topology described by

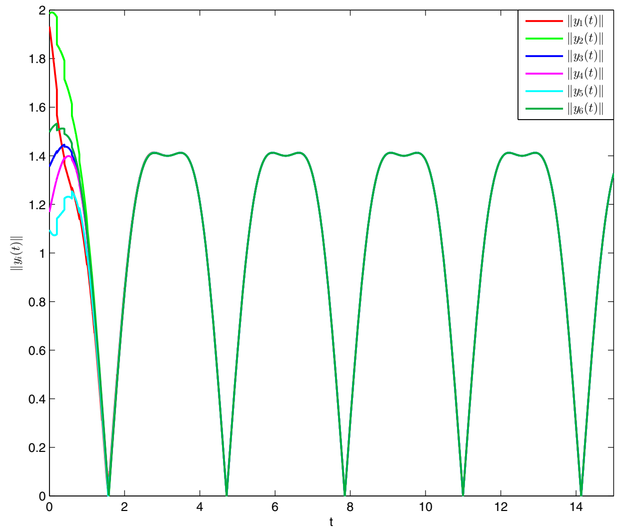

Figure 9. Norm evolutions of output vectors for (41) with non-zero control input.

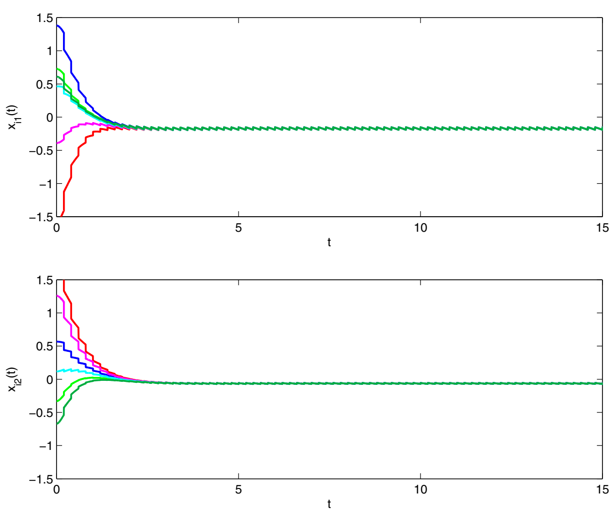

Figure 10. Norm evolutions of output vectors for (41) without control input.

$$\begin{cases} \dfrac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} = -D\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + Af(x_i(t)) \\ \qquad\qquad + Bf(x_i(t-\tau(t))) + J \\ \qquad\qquad + c\displaystyle\sum_{j=1}^{N} G_{ij}\Gamma\left(\dfrac{\mathrm{d}x_j(t)}{\mathrm{d}t} + x_j(t)\right) + u_i(t), \\ \Delta x_i(t_k) = -\delta_k \cdot x(t_k^-),\ \ \Delta \dot{x}_i(t_k) = -\delta_k \cdot \dot{x}(t_k^-), \end{cases} \tag{41}$$

where $i = 1, 2, \ldots, 6$, $k = 1, 2, \ldots$, $x_i = (x_{i1}, x_{i2})^{\mathrm{T}}$, $f(x_i) = 0.5(\sin(x_{i1}), \cos(x_{i2}))^{\mathrm{T}}$, and the time delay is given by $\tau(t) = 0.3e^t/(1+\mathrm{e}^t)$. So it is easy to obtain that $l_i = 0.5$, $\tau = 0.3$ and $\rho = 0.0750$. Setting $D =$

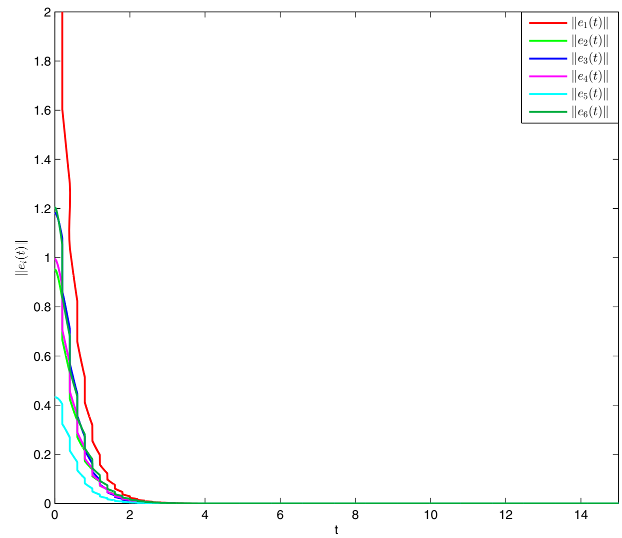

Figure 11. Norm evolutions of error vectors for (41) without control input.

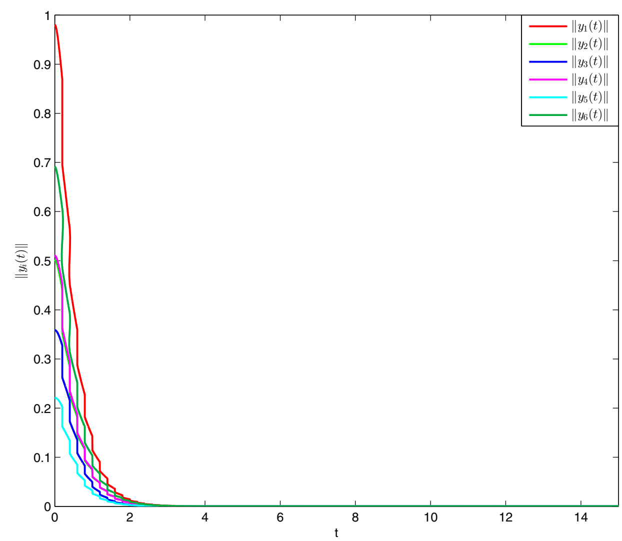

Figure 12. Norm evolutions of output vectors for (41) without control input.

$\mathrm{diag}\{0.4, 0.3\}$, $C = \mathrm{diag}\{1.2, 0.4\}$, $J = (0, 0)^{\mathrm{T}}$ and $c = 0.6$, the other coefficient matrices and coupling matrix are given as

$$A = \begin{pmatrix} 0.2 & -0.6 \\ 0.5 & 0.3 \end{pmatrix},\quad B = \begin{pmatrix} 0.3 & -0.3 \\ -0.2 & -0.5 \end{pmatrix},$$

$$G = \begin{pmatrix} -3.5 & 1.5 & 0 & 1 & 0 & 1 \\ 1.5 & -2 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & -3 & 1 & 1.5 & 0 \\ 1 & 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 1.5 & 1 & -4 & 1.5 \\ 1 & 0 & 0 & 1 & 1.5 & -3.5 \end{pmatrix}.$$

Moreover, we take $\delta_k = 0.2$, $t_k - t_{k-1} = 0.2$. Choose

$$F = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad H = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.8 \end{pmatrix}.$$

By means of MATLAB, the feasible solutions of (32) and (33) can be obtained by $\gamma = 0.4$ and $Q = \mathrm{diag}\{2.0683, 1.9883\}$. According to Theorem 3, system (3) derived from (41) is strictly input passive under output (6).

On the other hand, we can also find that when $\gamma = 0.7$ and $Q = \mathrm{diag}\{2.0911, 1.9670\}$, inequalities (38) and (39) hold. So, based on Theorem 4, this coupled network is strictly output passive.

For the numerical simulations, take the control input as those in Example 1, select the initial values randomly within interval $[-1.8, 1.8]$, then the simulation results for this undirected network are depicted in figures 7–9. The simulation results without control input are also depicted in figures 10–12 for comparison. From these simulations, it is read that the conclusions of Theorems 3 and 4 hold.

6. Conclusions

This paper has formulated and investigated the passivity issues for impulsive inertial delayed NNs with different coupled structures, namely directed topology and undirected topology. The normalised left eigenvector for a coupling matrix with respect to the eigenvalue 0 plays a critical role for the achievement of passivities for the directed network, while the combination of the properties for the undirected coupled matrix and the characters of the network helps to derive some more condensed conditions for the passivities of an undirected coupled network. Two numerical examples have been furnished to validate the correctness and merits of the theoretical findings.

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