Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:13
https://doi.org/10.1007/s12044-018-0379-2

Sharp Adams-type inequality invoking Hardy inequalities

MOHAMED KHALIL ZGHAL

Laboratoire Équations aux dérivées partielles (LR03ES04), Faculté des Sciences de Tunis, Université de Tunis El Manar, 2092 Tunis, Tunisia
E-mail: zghal-khalil@hotmail.fr

MS received 27 May 2015; revised 16 March 2016; accepted 27 May 2016; published online 22 March 2018

Abstract. We establish a sharp Trudinger–Moser type inequality invoking a Hardy inequality for any even dimension. This leads to a non compact Sobolev embedding in some Orlicz space. We also give a description of the lack of compactness of this embedding in the spirit of [8].

Keywords. Trudinger–Moser inequalities; Hardy inequalities; Orlicz space; lack of compactness.

Mathematics Subject Classification. 46E35, 35B33, 46E30.

1. Introduction

1.1 *Setting of the problem*

The Trudinger–Moser type inequalities have a long history beginning with the works of Pohozaev [23] and Trudinger [30]. Letting $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$, the authors looked at these pioneering works for the maximal growth function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{u \in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n} \leq 1} \int_\Omega g(u)\, \mathrm{d}x < +\infty ,$$

and they proved independently that the maximal growth is of exponential type. Thereafter, Moser improved these works by establishing a sharp result known as the Trudinger–Moser inequality (see [22]) and since then, this subject has continued to interest researchers and Trudinger–Moser inequality has been extended in various directions [1,2,21,25,26] generating several applications. Among the results obtained concerning Trudinger–Moser type inequalities, we recall the so-called Adams' inequality in $\mathbb{R}^{2N}$.

PROPOSITION 1.1 [18,26]

There exists a finite constant $\kappa > 0$ such that

$$\sup_{u \in H^N(\mathbb{R}^{2N}),\ \|u\|_{H^N(\mathbb{R}^{2N})} \leq 1} \int_{\mathbb{R}^{2N}} (\mathrm{e}^{\beta_N |u(x)|^2} - 1)\, \mathrm{d}x := \kappa , \tag{1.1}$$

where $\beta_N = N!\,\pi^N 2^{2N}$, *and for any* $\beta > \beta_N$,

$$\sup_{u\in H^N(\mathbb{R}^{2N}),\|u\|_{H^N(\mathbb{R}^{2N})}\leq 1} \int_{\mathbb{R}^{2N}} (\mathrm{e}^{\beta|u(x)|^2} - 1)\,\mathrm{d}x = +\infty\,. \tag{1.2}$$

Remark 1.2. In the above proposition, the norm $\|\,.\,\|_{H^N}$ designates the following Sobolev norm

$$\|u\|^2_{H^N(\mathbb{R}^{2N})} := \|u\|^2_{L^2(\mathbb{R}^{2N})} + \sum_{j=1}^{N} \|\nabla^j u\|^2_{L^2(\mathbb{R}^{2N})}\,,$$

where $\nabla^j u$ denotes the j-th order gradient of u, namely

$$\nabla^j u = \begin{cases} \Delta^{\frac{j}{2}} u & \text{if } j \text{ is even,} \\ \nabla\Delta^{\frac{j-1}{2}} u & \text{if } j \text{ is odd.} \end{cases}$$

The proof of Proposition 1.1, treated firstly in the radial case and then generalized by symmetrization arguments, is based on the following Trudinger–Moser inequality in a bounded domain.

PROPOSITION 1.3 ([2], Theorem 1)

Let Ω *be a bounded domain in* $\mathbb{R}^{2N}$. *There exists a positive constant* C_N *such that*

$$\sup_{u\in H^N_0(\Omega),\ \|\nabla^N u\|_{L^2}\leq 1} \int_{\Omega} \mathrm{e}^{\beta_N|u(x)|^2}\,\mathrm{d}x \leq C_N|\Omega|\,,$$

where $|\Omega|$ *denotes the Lebesgue measure of* Ω. *Furthermore, this inequality is sharp.*

As emphasized above, Proposition 1.1 has been at the origin of numerous applications. Among others, one can mention the description of the lack of compactness of Sobolev embedding involving Orlicz spaces in [8–12], the analysis of some elliptic and biharmonic equations in [27–29] and the study of global well-posedness and the asymptotic completeness for evolution equations with exponential nonlinearity in dimension two in [3,4,7,8,13,16,17].

Sobolev embedding inferred by Proposition 1.1 states as follows:

$$H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N})\,, \tag{1.3}$$

where $\mathcal{L}$ is the so-called Orlicz space associated to the function $\phi(s) := \mathrm{e}^{s^2} - 1$ and defined as follows (for a complete presentation and more details, we refer the reader to [24] and references therein).

DEFINITION 1.4

We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to $\mathcal{L}(\mathbb{R}^d)$ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1 \right) \, \mathrm{d}x < \infty \, .$$

We then denote as follows:

$$\|u\|_{\mathcal{L}(\mathbb{R}^d)} = \inf \left\{ \lambda > 0, \int_{\mathbb{R}^d} \left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1 \right) \, \mathrm{d}x \leq 1 \right\} . \tag{1.4}$$

Remark 1.5.

- It is easy to check that $\| \, . \, \|_{\mathcal{L}}$ is a norm on the $\mathbb{C}$-vector space $\mathcal{L}$ which is invariant under translations and oscillations.
- One can also verify that the number 1 in (1.4) may be replaced by any positive constant. This changes the norm $\| \, . \, \|_{\mathcal{L}}$ to an equivalent one.
- In the sequel, we shall endow the space $\mathcal{L}(\mathbb{R}^{2N})$ with the norm $\| \cdot \|_{\mathcal{L}(\mathbb{R}^{2N})}$ where the number 1 is replaced by the constant κ involved in Identity (1.1). The Sobolev embedding (1.3) then states as follows:

$$\|u\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq \frac{1}{\sqrt{\beta_N}} \|u\|_{H^N(\mathbb{R}^{2N})} , \tag{1.5}$$

where the Sobolev constant $1/\sqrt{\beta_N}$ is sharp.

- Denoting by L^{ϕ_p} the Orlicz space associated to $\phi_p(s) := \mathrm{e}^{s^2} - \sum_{k=0}^{p-1} \frac{s^{2k}}{k!}$, with p an integer larger than 1, we deduce from Proposition 1.1 the more general Sobolev imbedding

$$H^N(\mathbb{R}^{2N}) \hookrightarrow L^{\phi_p}(\mathbb{R}^{2N}) \, . \tag{1.6}$$

- Let us finally observe that $\mathcal{L} \hookrightarrow L^p$ for every $2 \leq p < \infty$.

In this article, our goal is two-fold. Firstly, we obtain an analogue of Proposition 1.1 in the radial framework of a functional space $\mathcal{H}(\mathbb{R}^{2N})$ closely related to Hardy inequalities, which will easily lead to the following Sobolev imbedding:

$$\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N}) \, . \tag{1.7}$$

Secondly, we describe the lack of compactness of (1.7), which could be at the origin of several applications as it has been the case by previous characterizations of defect of compactness of various Sobolev embeddings.

More precisely, for any integer $N \geq 2$, the space we will consider in this paper is defined as follows:

$$\mathcal{H}(\mathbb{R}^{2N}) := \left\{ u \in H^1(\mathbb{R}^{2N}); \, \frac{\nabla u}{| \, . \, |^{N-1}} \in L^2(\mathbb{R}^{2N}) \right\} . \tag{1.8}$$

In view of the well-known Hardy inequalities (see for instance [5,6,14,15]),

$$\left\| \frac{u}{|\,.\,|^s} \right\|_{L^2(\mathbb{R}^d)} \leq C_{d,s} \|u\|_{\dot{H}^s(\mathbb{R}^d)}\,, \quad \forall\, s \in \left[0, \frac{d}{2}\right[, \tag{1.9}$$

the Sobolev space $H^N(\mathbb{R}^{2N})$ continuously embeds in the functional space $\mathcal{H}(\mathbb{R}^{2N})$ endowed with the norm

$$\|u\|^2_{\mathcal{H}(\mathbb{R}^{2N})} = \|u\|^2_{H^1(\mathbb{R}^{2N})} + \left\| \frac{\nabla u}{|\,.\,|^{N-1}} \right\|^2_{L^2(\mathbb{R}^{2N})}\,.$$

Actually, as shown by the example of function

$$x \longmapsto \log(1 - \log|x|)\, \mathbf{1}_{B_1(0)}(x)\,,$$

with $B_1(0)$ the unit ball of $\mathbb{R}^{2N}$, the embedding of $H^N(\mathbb{R}^{2N})$ into $\mathcal{H}(\mathbb{R}^{2N})$ is strict for every $N \geq 2$.
For the convenience of the reader, the following diagram recapitulates the different embeddings including the spaces involved in this work.

$$\begin{tikzcd}
H^N(\mathbb{R}^{2N}) \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] & \mathcal{L}(\mathbb{R}^{2N}) \\
H^1(\mathbb{R}^{2N}) \arrow[ur, hook, "N=1"] & \mathcal{H}(\mathbb{R}^{2N}) \arrow[l, hook'] \arrow[u, hook, "\text{radial case}"']
\end{tikzcd}$$

The interest we take towards the space $\mathcal{H}$ is motivated by the importance of Hardy inequalities in analysis (among others, we can mention blow-up methods or the study of pseudo-differential operators with singular coefficients).

1.2 *Main results*

The result we obtained concerning the sharp Trudinger–Moser type inequality in the framework of the space $\mathcal{H}(\mathbb{R}^{2N})$ takes the following form:

Theorem 1.6. *For any integer N greater than* 2, *there exists a finite constant $\kappa' > 0$ such that*

$$\sup_{u \in \mathcal{H}_{rad}(\mathbb{R}^{2N}),\ \|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1} \int_{\mathbb{R}^{2N}} (\mathrm{e}^{\gamma_N |u(x)|^2} - 1)\, \mathrm{d}x := \kappa'\,, \tag{1.10}$$

where $\gamma_N := \dfrac{4\pi^N N}{(N-1)!}$, *and for any* $\gamma > \gamma_N$,

$$\sup_{u \in \mathcal{H}_{rad}(\mathbb{R}^{2N}),\ \|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1} \int_{\mathbb{R}^{2N}} (\mathrm{e}^{\gamma |u(x)|^2} - 1)\, \mathrm{d}x = +\infty\,. \tag{1.11}$$

Remark 1.7.

• Usually, the proofs of Trudinger–Moser inequalities reduce to the radial framework under symmetrization arguments. In particular, in dimension two this question is achieved by means of Schwarz symmetrization (see [1]). The key point in that process is the preservation of Lebesgue norms and the minimization of energy.

Unfortunately, the quantity $\left\| \frac{\nabla u}{|\,.\,|^{N-1}} \right\|_{L^2(\mathbb{R}^{2N})}$ cannot be minimized under Schwarz symmetrization as shown by the example $u_k(x) := \varphi(|x|+k)$, where $\varphi \neq 0$ is a smooth compactly supported function. The fact that $u_k^* = \varphi$ shows that the control of $\left\| \frac{\nabla u_k^*}{|\,.\,|^{N-1}} \right\|_{L^2(\mathbb{R}^{2N})}$ by $\left\| \frac{\nabla u_k}{|\,.\,|^{N-1}} \right\|_{L^2(\mathbb{R}^{2N})}$ fails.

• It is clear that, when the constant 1 in (1.4) is replaced by κ', Theorem 1.6 implies the following radial continuous embedding:

$$\|u\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq \frac{1}{\sqrt{\gamma_N}} \|u\|_{\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})},$$

where the Sobolev constant $\frac{1}{\sqrt{\gamma_N}}$ is optimal.

• Observe that due to the continuous embedding

$$H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{H}(\mathbb{R}^{2N}),$$

Theorem 1.6 can be viewed as a generalization of Proposition 1.1 in the radial framework.

As mentioned above, our second aim in this paper is to describe the lack of compactness of the Sobolev embedding (1.7). Actually, this embedding is non compact. This is due to the following example derived by Lions [19,20]:

$$f_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ -\sqrt{\frac{2N}{k\gamma_N}} \log|x| & \text{if } \mathrm{e}^{-k} \leq |x| < 1, \\ \sqrt{\frac{2Nk}{\gamma_N}} & \text{if } \ |x| < \mathrm{e}^{-k}. \end{cases} \tag{1.12}$$

Indeed, we have the following proposition, the proof of which is given in §4 for the convenience of the reader.

PROPOSITION 1.8

The sequence $(f_k)_{k\geq 0}$ defined above converges weakly to 0 in $\mathcal{H}(\mathbb{R}^{2N})$ and satisfies

$$\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{k\to\infty}{\longrightarrow} \frac{1}{\sqrt{\gamma_N}}.$$

It will be useful later on to emphasize that f_k can be recast under the following form:

$$f_k(x) = \sqrt{\frac{2Nk}{\gamma_N}}\, \mathbf{L}\Big(-\frac{\log|x|}{k}\Big), \tag{1.13}$$

where

$$\mathbf{L}(t) = \begin{cases} 1 & \text{if } t \geq 1\,, \\ t & \text{if } 0 \leq t < 1\,, \\ 0 & \text{if } t < 0\,, \end{cases}$$

and that

$$\|f_k\|_{H^1(\mathbb{R}^{2N})} \stackrel{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \left\| \frac{\nabla f_k}{|\,.\,|^{N-1}} \right\|_{L^2(\mathbb{R}^{2N})} = \|\mathbf{L}'\|_{L^2(\mathbb{R})} = 1\,. \tag{1.14}$$

In order to state our second result in a clear way, let us introduce some objects as in [8].

DEFINITION 1.9

We shall designate by a scale any sequence $\underline{\alpha} := (\alpha_n)_{n\geq 0}$ of positive real numbers going to infinity and by a profile any function ψ belonging to the set

$$\mathcal{P} := \{\psi \in L^2(\mathbb{R}, \mathrm{e}^{-2Ns}\mathrm{d}s); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\}\,.$$

Two scales $\underline{\alpha}$, $\underline{\beta}$ are said to be orthogonal if

$$\left|\log\left(\frac{\beta_n}{\alpha_n}\right)\right| \stackrel{n\to\infty}{\longrightarrow} \infty\,.$$

Remark 1.10. Recall that each profile $\psi \in \mathcal{P}$ belongs to the Hölder space $C^{\frac{1}{2}}(\mathbb{R})$, and satisfies

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad \text{as } s \to 0\,. \tag{1.15}$$

Indeed taking advantage of the fact that $\psi' \in L^2(\mathbb{R})$, we get for any $s_2 > s_1$,

$$|\psi(s_2) - \psi(s_1)| = \left| \int_{s_1}^{s_2} \psi'(\tau)\,\mathrm{d}\tau \right| \leq \sqrt{s_2 - s_1} \left(\int_{s_1}^{s_2} \psi'^2(\tau)\,\mathrm{d}\tau \right)^{1/2},$$

which ensures that $\psi \in C^{\frac{1}{2}}(\mathbb{R})$ and implies (1.15) by taking $s_1 = 0$.

The result we establish in this paper highlights the fact that the lack of compactness of the Sobolev embedding (1.7) can be described in terms of generalizations of the example by Moser (1.12) as follows.

Theorem 1.11. *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$ such that*

$$u_n \rightharpoonup 0\,, \tag{1.16}$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}(\mathbb{R}^{2N})} = A_0 > 0 \quad \textit{and} \tag{1.17}$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n(x)|^2\,\mathrm{d}x = 0\,. \tag{1.18}$$

Then, there exist a sequence of pairwise orthogonal scales $(\underline{\alpha}^{(j)})_{j\geq 1}$ *and a sequence of profiles* $(\psi^{(j)})_{j\geq 1}$ *such that up to a subsequence extraction, we have for all* $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + r_n^{(\ell)}(x)\,, \tag{1.19}$$

with $\limsup_{n\to\infty} \|r_n^{(\ell)}\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{\ell\to\infty}{\longrightarrow} 0$. *Moreover, we have the following stability estimate*

$$\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)'}\|_{L^2(\mathbb{R})}^2 + \left\|\frac{\nabla r_n^{(\ell)}}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}^2 + \circ(1)\,, \quad n\to\infty\,.$$

Remark 1.12.

- The hypothesis of compactness at infinity (1.18) is crucial: it allows to avoid the loss of Orlicz norm at infinity.
- Note that the elementary concentrations

$$g_n^{(j)}(x) := \sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right)\,, \tag{1.20}$$

involved in Decomposition (1.19) are in $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$ whereas *a priori*, they do not belong to $H^N(\mathbb{R}^{2N})$.
- Actually, the lack of compactness of $H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N})$ was characterized in [10] by means of the following type of elementary concentrations:

$$f_n(x) := \frac{C_N}{\sqrt{\alpha_n}} \int_{|\xi|\geq 1} \frac{\mathrm{e}^{i\,(x-x_n)\cdot\xi}}{|\xi|^{2N}}\, \varphi\Big(\frac{\log|\xi|}{\alpha_n}\Big)\, \mathrm{d}\xi\,, \tag{1.21}$$

with $(\alpha_n)_{n\geq 0}$ a scale in the sense of Definition 1.9, $(x_n)_{n\geq 0}$ a sequence of points in $\mathbb{R}^{2N}$ and φ a function in $L^2(\mathbb{R}_+)$. Note that (see Proposition 1.7 in [10])

$$f_n(x) = \widetilde{C}_N\, \sqrt{\alpha_n}\, \psi\Big(\frac{-\log|x|}{\alpha_n}\Big) + \mathrm{t}_n(x)\,,$$

with $\psi(y) = \int_0^y \varphi(t)\,\mathrm{d}t$ and $\|\mathrm{t}_n\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{n\to\infty}{\longrightarrow} 0$.
- Arguing as in [8], we have Proposition 1.13 below.
- Arguing as in Proposition 1.18 in [8], we get

$$\left\|\sum_{j=1}^{\ell} g_n^{(j)}\right\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{n\to\infty}{\longrightarrow} \sup_{1\leq j\leq \ell}\Big(\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}(\mathbb{R}^{2N})}\Big)\,, \tag{1.22}$$

where $g_n^{(j)}$ is defined by (1.20).

PROPOSITION 1.13

Let us consider

$$g_n(x) := \sqrt{\frac{2N\alpha_n}{\gamma_N}}\,\psi\left(\frac{-\log|x|}{\alpha_n}\right),$$

with ψ a profile and $(\alpha_n)_{n\geq 0}$ a scale. Then

$$\|g_n\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{\gamma_N}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}. \tag{1.23}$$

Proof. Setting $L = \liminf_{n\to\infty} \|g_n\|_{\mathcal{L}(\mathbb{R}^{2N})}$, we have for any fixed $\varepsilon > 0$ and any n sufficiently large (up to a subsequence extraction)

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\left|\frac{g_n(x)}{L+\varepsilon}\right|^2} - 1\right) \mathrm{d}x \leq \kappa'.$$

Therefore, there exists a positive constant C such that

$$\alpha_n \int_0^{+\infty} \mathrm{e}^{2N\alpha_n s\left[\frac{1}{\gamma_N (L+\varepsilon)^2}\left|\frac{\psi(s)}{\sqrt{s}}\right|^2 - 1\right]} \mathrm{d}s \leq C.$$

Using the fact that ψ is a continuous function, we deduce that

$$L + \varepsilon \geq \frac{1}{\sqrt{\gamma_N}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}},$$

which ensures that

$$L \geq \frac{1}{\sqrt{\gamma_N}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}.$$

To end the proof of (1.23), it suffices to show that for any positive real number δ, the following estimate holds

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\left|\frac{g_n(x)}{\lambda}\right|^2} - 1\right) \mathrm{d}x \overset{n\to\infty}{\longrightarrow} 0,$$

where $\lambda := \dfrac{1+\delta}{\sqrt{\gamma_N}} \max\limits_{s>0} \dfrac{|\psi(s)|}{\sqrt{s}}$.

Performing the change of variable $r = \mathrm{e}^{-\alpha_n s}$, we easily get

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\left|\frac{g_n(x)}{\lambda}\right|^2} - 1\right) \mathrm{d}x &= \frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\infty \mathrm{e}^{-2N\alpha_n s\left(1 - \frac{1}{\gamma_N \lambda^2}\left|\frac{\psi(s)}{\sqrt{s}}\right|^2\right)} \mathrm{d}s \\ &\quad - \frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\infty \mathrm{e}^{-2N\alpha_n s}\, \mathrm{d}s.\end{aligned} \tag{1.24}$$

Recalling that

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad \text{as } s \to 0\,,$$

we infer that for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\frac{1}{\gamma_N \lambda^2} \left| \frac{\psi(s)}{\sqrt{s}} \right|^2 < \varepsilon \quad \text{for any } 0 \le s < \eta\,.$$

According to (1.24), this gives rise to

$$\begin{aligned} &\frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\eta \mathrm{e}^{-2N\alpha_n s\left(1 - \frac{1}{\gamma_N \lambda^2} \left| \frac{\psi(s)}{\sqrt{s}} \right|^2\right)} \, \mathrm{d}s - \frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\eta \mathrm{e}^{-2N\alpha_n s} \, \mathrm{d}s \\ &\le \frac{\pi^N \varepsilon}{N!(1-\varepsilon)} + \circ(1)\,, \quad n \to \infty\,, \end{aligned}$$

which ensures the desired result. □

1.3 *Layout*

The paper is organized as follows: Section 2 is devoted to the proof of the sharp Trudinger–Moser type inequality in the framework of the space $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$, namely Theorem 1.6. In §3, we establish Theorem 1.11 by describing the algorithm construction of the decomposition of a bounded sequence $(u_n)_{n\ge 0}$ in $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$, up to a subsequence extraction, in terms of asymptotically orthogonal profiles in the spirit of the example by Moser. The last section is devoted to the proof of Proposition 1.8.

Finally, we mention that C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \le CB$ for some absolute constant C. For simplicity, we shall still denote by (u_n) any subsequence of (u_n).

2. Proof of the Theorem 1.6

To establish Estimate (1.10), we shall follow the 2D approach adopted in [25] by setting for a fixed $r_0 > 0$ (to be chosen later on),

$$I_1 := \int_{B(r_0)} (\mathrm{e}^{\gamma_N |u(x)|^2} - 1)\, \mathrm{d}x \quad \text{and} \quad I_2 := \int_{\mathbb{R}^{2N} \setminus B(r_0)} (\mathrm{e}^{\gamma_N |u(x)|^2} - 1)\, \mathrm{d}x\,,$$

where $B(r_0)$ denotes the ball centered at the origin and of radius r_0.

The idea consists in showing that it is possible to choose a suitable $r_0 > 0$ independently of u such that I_1 and I_2 are bounded by a constant only depending on r_0 and N.

Let us start by studying the part I_2. Using the power series expansion of the exponential, we can write

$$I_2 = \sum_{k=1}^{\infty} \frac{\gamma_N^k}{k!} I_{2,k}\,, \quad \text{where } I_{2,k} := \int_{\mathbb{R}^{2N} \setminus B(r_0)} |u(x)|^{2k} \, \mathrm{d}x\,.$$

In order to estimate $I_{2,k}$, we take advantage of the following radial estimate available for any function u in $H^1_{\rm rad}(\mathbb{R}^{2N})$ (for further details, see [26]):

$$|u(x)| \leq \sqrt{\frac{(N-1)!}{\pi^N}}\, \frac{\|u\|_{H^1(\mathbb{R}^{2N})}}{|x|^{N-\frac{1}{2}}} \quad \text{for a.e. } x \in \mathbb{R}^{2N}\,, \tag{2.1}$$

which for any integer $k \geq 2$, implies that

$$\begin{aligned} I_{2,k} &\leq \left(\frac{(N-1)!}{\pi^N}\right)^k \|u\|^{2k}_{H^1(\mathbb{R}^{2N})} \frac{2\pi^N}{(N-1)!} \int_{r_0}^{\infty} \frac{\mathrm{d}r}{r^{(k-1)(2N-1)}} \\ &\leq \frac{2\pi^N}{(N-1)!} \left(\frac{(N-1)!}{\pi^N}\right)^k \|u\|^{2k}_{H^1(\mathbb{R}^{2N})} \frac{r_0^{k(1-2N)+2N}}{(2N-1)k-2N} \\ &\leq \frac{2\pi^N}{(N-1)!} \frac{r_0^{2N}}{2(N-1)} \left(\frac{(N-1)!}{\pi^N}\right)^k \|u\|^{2k}_{H^1(\mathbb{R}^{2N})} \frac{1}{r_0^{(2N-1)k}}\,. \end{aligned}$$

This gives rise to

$$\begin{aligned} I_2 &\leq \gamma_N \|u\|^2_{L^2(\mathbb{R}^{2N})} + \frac{2\pi^N}{(N-1)!} \frac{r_0^{2N}}{2(N-1)} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{\gamma_N (N-1)!}{\pi^N} \frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{(2N-1)}}\right)^k \\ &\leq \gamma_N + \frac{2\pi^N}{(N-1)!} \frac{r_0^{2N}}{2(N-1)} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\gamma_N \frac{(N-1)!}{\pi^N} \frac{1}{r_0^{(2N-1)}}\right)^k\,, \end{aligned}$$

under the fact that $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1$, which ensures that I_2 is bounded by a constant only dependent of r_0 and N.

In order to estimate I_1, we shall make use of the following Trudinger–Moser type inequality, the proof of which is postponed at the end of the section.

PROPOSITION 2.1

There exists a constant $C_N > 0$ such that for any positive real number R, we have

$$\sup_{u \in \left(\mathcal{H}_{rad} \cap H^1_0\right)(B(R)),\ \left\| \frac{\nabla u}{|\,.\,|^{N-1}} \right\|_{L^2} \leq 1} \int_{B(R)} \mathrm{e}^{\gamma_N |u(x)|^2}\, \mathrm{d}x \leq C_N R^{2N}\,,$$

and this inequality is sharp.

Let us admit this proposition for the time being, and continue the proof of the theorem. The key point consists in associating to a function u in $\mathcal{H}_{\rm rad}(B(r_0))$ with $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1$, an auxiliary function $w \in \left(\mathcal{H}_{\rm rad} \cap H^1_0\right)(B(r_0))$ such that

$$\left\| \frac{\nabla w}{|\,.\,|^{N-1}} \right\|_{L^2(B(r_0))} \leq 1 \quad \text{and} \quad u^2 \leq w^2 + d(r_0)\,,$$

where the function $d(r_0) > 0$ depends only on r_0. To this end, let us first emphasize that if u belongs to $\mathcal{H}_{\rm rad}(B(r_0))$ and satisfies $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1$, then u is continuous far away from the origin. Indeed, for any real numbers $r_2 > r_1 > 0$, writing

$$u(r_2) - u(r_1) = \int_{r_1}^{r_2} u'(s)\, \mathrm{d}s\,,$$

we get by Cauchy–Schwarz inequality

$$|u(r_2)-u(r_1)| \leq \Big(\int_{r_1}^{r_2} |u'(s)|^2 s^{2N-1}\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_{r_1}^{r_2} s^{-(2N-1)}\,\mathrm{d}s\Big)^{\frac{1}{2}}$$
$$\leq C\|\nabla u\|_{L^2(\mathbb{R}^{2N})}\Big(\int_{r_1}^{r_2} s^{-(2N-1)}\,\mathrm{d}s\Big)^{\frac{1}{2}},$$

which leads to the result. Thus, for any $0<r<r_0$, we can define the function

$$v(r) := u(r)-u(r_0)\,,$$

which clearly belongs to $\big(\mathcal{H}_{\mathrm{rad}}\cap H_0^1\big)(B(r_0))$. In light of the radial estimate (2.1), this implies that

$$u^2(r) \leq v^2(r)+v^2(r)u^2(r_0)+1+u^2(r_0)$$
$$\leq v^2(r)+v^2(r)\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}+1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}$$
$$\leq v^2(r)\left(1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\right)+d(r_0)\,,$$

where $d(r_0) := 1+\dfrac{(N-1)!}{\pi^N}\dfrac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}$.

Now by construction, the function

$$w(r) := v(r)\sqrt{1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}}\,,$$

belongs to $\big(\mathcal{H}_{\mathrm{rad}}\cap H_0^1\big)(B(r_0))$, and easily satisfies

$$\int_{B(r_0)}\frac{|\nabla w(x)|^2}{|x|^{2(N-1)}}\,\mathrm{d}x = \left(1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\right)\int_{B(r_0)}\frac{|\nabla u(x)|^2}{|x|^{2(N-1)}}\,\mathrm{d}x$$
$$\leq \left(1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\right)\Big(1-\|u\|^2_{H^1(\mathbb{R}^{2N})}\Big)\leq 1\,,$$

provided that $\dfrac{\pi^N}{(N-1)!}r_0^{2N-1}\geq 1$.

Applying Proposition 2.1 with r_0 fixed so that $\dfrac{\pi^N}{(N-1)!}r_0^{2N-1}\geq 1$, we deduce that

$$I_1 \leq \mathrm{e}^{\gamma_N d(r_0)}\int_{B(r_0)}\mathrm{e}^{\gamma_N|w(x)|^2}\,\mathrm{d}x \leq C_N\,\mathrm{e}^{\gamma_N d(r_0)}\,r_0^{2N}\,,$$

which ensures the desired estimate, up to the proof of Proposition 2.1.

To achieve the proof of Identity (1.10), let us then establish Proposition 2.1. To this end, for a function u in $\big(\mathcal{H}_{\mathrm{rad}}\cap H_0^1\big)(B(R))$ satisfying $\Big\|\frac{\nabla u}{|\,.\,|^{N-1}}\Big\|_{L^2(\mathbb{R}^{2N})}\leq 1$, let us denote by

$$I(R) := \int_{B(R)}\mathrm{e}^{\gamma_N|u(x)|^2}\,\mathrm{d}x\,.$$

Our aim is to show that

$$I(R) \leq C_N R^{2N} \quad \text{whenever } \frac{2\pi^N}{(N-1)!}\int_0^R |v'(r)|^2 r \,\mathrm{d}r \leq 1\,.$$

For that purpose, let us perform the change of variable $s = r^N$, and introduce the function $w(s) = \sqrt{\frac{N\pi^{N-1}}{(N-1)!}}\, v\big(s^{\frac{1}{N}}\big)$. Recalling that $\gamma_N = \frac{4\pi^N N}{(N-1)!}$, we infer that

$$I(R) = \frac{2\pi^N}{(N-1)!}\int_0^R \mathrm{e}^{\gamma_N |v(r)|^2} r^{2N-1}\,\mathrm{d}r = \frac{2\pi^N}{N!}\int_0^{R^N} \mathrm{e}^{4\pi|w(s)|^2} s\,\mathrm{d}s$$

and

$$\frac{2\pi^N}{(N-1)!}\int_0^R |v'(r)|^2 r\,\mathrm{d}r = 2\pi\int_0^{R^N} |w'(s)|^2 s\,\mathrm{d}s\,.$$

The conclusion then stems from the 2D radial framework of Proposition 1.3.

Now in order to prove the sharpness of the exponent γ_N, let us consider the sequence (f_k) defined by (1.12). Since according to (1.14), we have

$$\|f_k\|_{\mathcal{H}(\mathbb{R}^{2N})} = 1 + \circ(1)\,, \quad \text{as } k \to \infty\,,$$

we get for any $\gamma > \gamma_N$,

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\gamma\left|\frac{f_k(x)}{\|f_k\|_{\mathcal{H}(\mathbb{R}^{2N})}}\right|^2} - 1\right)\mathrm{d}x &\geq \frac{2\pi^N}{(N-1)!}\int_0^{\mathrm{e}^{-k}} \left(\mathrm{e}^{\frac{2Nk\gamma}{\gamma_N(1+\circ(1))}} - 1\right) r^{2N-1}\,\mathrm{d}r\\ &\geq \frac{\pi^N}{N!}\left(\mathrm{e}^{2Nk\frac{\gamma-\gamma_N(1+\circ(1))}{\gamma_N(1+\circ(1))}} - \mathrm{e}^{-2Nk}\right) \stackrel{k\to\infty}{\longrightarrow} \infty\,,\end{aligned}$$

which ends the proof of the theorem.

3. Proof of Theorem 1.11

3.1 *Scheme of the proof*

The proof of Theorem 1.11 relies on a diagonal subsequence extraction and uses in a crucial way the radial setting and particularly the fact that we deal with bounded functions far away from the origin. The heart of the matter is reduced to the proof of the following lemma:

Lemma 3.1. *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$ satisfying Assumptions* (1.16), (1.17) *and* (1.18). *Then there exist a scale $(\alpha_n)_{n\geq 0}$ and a profile ψ in the sense of Definition* 1.9, *such that*

$$\|\psi'\|_{L^2(\mathbb{R})} \geq C_N A_0\,, \tag{3.1}$$

where C_N is a constant depending only on N.

Inspired by the strategy developed in [8], the proof is done in three steps. In the first step, according to Lemma 3.1, we extract the first scale and the first profile satisfying inequality (3.1). This reduces the problem to the study of the remainder term. If the limit of its Orlicz norm is null, we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence starts which allow us to extract a second scale and a second profile which verifies the above key property (3.1), by following the lines of reasoning of the first step. Thereafter, we establish the property of orthogonality between the two first scales. Finally, we prove that this process converges.

3.2 *Extraction of the first scale and the first profile*

Let us consider a bounded sequence $(u_n)_{n\geq 0}$ in $\mathcal{H}_{\rm rad}(\mathbb{R}^{2N})$ satisfying the assumptions of Theorem 1.11, and let us set $v_n(s) := u_n(\mathrm{e}^{-s})$. Then, we have the following lemma:

Lemma 3.2. *Under the above assumptions, the sequence $(u_n)_{n\geq 0}$ converges strongly to* 0 *in $L^2(\mathbb{R}^{2N})$, and we have for any real number M,*

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(]-\infty,M])} = 0\,. \tag{3.2}$$

Proof. Let us first observe that for any positive real number R, we have

$$\|u_n\|_{L^2(\mathbb{R}^{2N})} = \|u_n\|_{L^2(|x|\leq R)} + \|u_n\|_{L^2(|x|>R)}\,.$$

Now, invoking Rellich's theorem and the Sobolev embedding of $\mathcal{H}(\mathbb{R}^{2N})$ into $H^1(\mathbb{R}^{2N})$, we infer that the space $\mathcal{H}(|x|<R)$ is compactly embedded in $L^2(|x|<R)$. Therefore,

$$\limsup_{n\to\infty} \|u_n\|_{L^2(|x|<R)} \overset{n\to\infty}{\longrightarrow} 0\,.$$

Taking advantage of the hypothesis of the compactness at infinity (1.18), we deduce the strong convergence of the sequence $(u_n)_{n\geq 0}$ to 0 in $L^2(\mathbb{R}^{2N})$.

Finally, (3.2) stems from the strong convergence to zero of $(u_n)_{n\geq 0}$ in $L^2(\mathbb{R}^{2N})$ and the following well-known radial estimate available for any function u in $H^1_{\rm rad}(\mathbb{R}^{2N})$ is

$$|u(x)| \leq \sqrt{\frac{(N-1)!}{\pi^N}}\,\frac{\|u\|^{\frac12}_{L^2(\mathbb{R}^{2N})}\|\nabla u\|^{\frac12}_{L^2(\mathbb{R}^{2N})}}{|x|^{N-\frac12}}\,, \qquad \text{for a.e. } x\in\mathbb{R}^{2N}\,.$$

□

Now, arguing as in the proof of Proposition 2.3 in [8], we deduce the following result:

PROPOSITION 3.3

For any $\delta>0$, we have

$$\sup_{s\geq 0}\left(\left|\frac{v_n(s)}{A_0-\delta}\right|^2 - (2N-1)s\right) \to \infty\,, \quad n\to\infty\,. \tag{3.3}$$

A by-product of the previous proposition is the following corollary:

COROLLARY 3.4

Under the above notations, there exists a sequence $(\alpha_n^{(1)})_{n\geq 0}$ *in* $\mathbb{R}_+$ *tending to infinity such that*

$$4\left|\frac{v_n\left(\alpha_n^{(1)}\right)}{A_0}\right|^2 - (2N-1)\,\alpha_n^{(1)} \overset{n\to\infty}{\longrightarrow} \infty\,, \tag{3.4}$$

and for n sufficiently large, there exists a positive constant C such that

$$\frac{A_0}{2}\sqrt{(2N-1)\alpha_n^{(1)}} \leq |v_n(\alpha_n^{(1)})| \leq C\sqrt{\alpha_n^{(1)}} + \circ(1)\,, \tag{3.5}$$

where $C = \sqrt{\frac{(N-1)!}{2\pi^N}}\limsup_{n\to\infty}\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$.

Proof. In order to establish (3.4), let us consider the sequences

$$W_n(s) := 4\left|\frac{v_n(s)}{A_0}\right|^2 - (2N-1)s \quad \text{and} \quad a_n := \sup_{s\geq 0} W_n(s)\,.$$

By definition, there exists a positive sequence $(\alpha_n^{(1)})_{n\geq 0}$ such that

$$W_n(\alpha_n^{(1)}) \geq a_n - \frac{1}{n}\,.$$

Now, in view of (3.3), $a_n \overset{n\to\infty}{\longrightarrow} \infty$ and then $W_n(\alpha_n^{(1)}) \overset{n\to\infty}{\longrightarrow} \infty$. It remains to prove that $\alpha_n^{(1)} \overset{n\to\infty}{\longrightarrow} \infty$. If not, up to a subsequence extraction, the sequence $(\alpha_n^{(1)})_{n\geq 0}$ is bounded and so is $(W_n(\alpha_n^{(1)}))_{n\geq 0}$ by (3.2), which yields a contradiction.

Concerning estimate (3.5), the left-hand side follows directly from (3.4). Besides, for any positive real number s, we have

$$|v_n(s)| \leq \left|v_n(0) + \int_0^s v_n'(\tau)\,\mathrm{d}\tau\right| \leq |v_n(0)| + s^{\frac{1}{2}}\|v_n'\|_{L^2(\mathbb{R})}\,,$$

which according to (3.2) implies that $v_n(0) \overset{n\to\infty}{\longrightarrow} 0$, and the following straightforward equality

$$\|v_n'\|_{L^2(\mathbb{R})} = \sqrt{\frac{(N-1)!}{2\pi^N}}\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})},$$

gives the right-hand side of inequality (3.5), and thus ends the proof of the result. □

Corollary 3.4 allows to extract the first scale, it remains to extract the first profile. To do so, let us set

$$\psi_n(y) = \sqrt{\frac{\gamma_N}{2N\alpha_n^{(1)}}}\, v_n\left(\alpha_n^{(1)} y\right)\,.$$

It will be useful later on to point out that, in view of Property (3.2), $\psi_n(0) \overset{n\to\infty}{\longrightarrow} 0$. The following result summarizes the main properties of the sequence $(\psi_n)_{n\geq 0}$:

Lemma 3.5. *Under notations of Corollary* 3.4, *there exists a profile* $\psi^{(1)} \in \mathcal{P}$ *such that, up to a subsequence extraction*

$$\psi_n' \rightharpoonup \psi^{(1)'} \text{ in } L^2(\mathbb{R}) \quad \text{and} \quad \|\psi^{(1)'}\|_{L^2} \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}}\,\gamma_N\,.$$

Proof. Noticing that $\|\psi_n'\|_{L^2(\mathbb{R})} = \left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$, we infer that the sequence $(\psi_n')_{n\geq 0}$ is bounded in $L^2(\mathbb{R})$. Thus, up to a subsequence extraction, $(\psi_n')_{n\geq 0}$ converges weakly in $L^2(\mathbb{R})$ to some function g. Let us now introduce the function

$$\psi^{(1)}(s) := \int_0^s g(\tau)\,\mathrm{d}\tau\,.$$

Our aim is then to prove that $\psi^{(1)}$ is a profile and that $\left\|\psi^{(1)'}\right\|_{L^2} \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}}\,\gamma_N\,.$

On the one hand, applying Cauchy–Schwarz inequality, we get

$$\left|\psi^{(1)}(s)\right| = \left|\int_0^s g(\tau)\,\mathrm{d}\tau\right| \leq \sqrt{s}\|g\|_{L^2(\mathbb{R})}\,,$$

which ensures that $\psi^{(1)} \in L^2(\mathbb{R}_+, \mathrm{e}^{-2Ns}\mathrm{d}s)\,.$

On the other hand, we have $\psi^{(1)}(s) = 0$ for all $s \leq 0$. Indeed, using the fact that

$$\|u_n\|^2_{L^2(\mathbb{R}^{2N})} = \left(\alpha_n^{(1)}\right)^2 \int_{\mathbb{R}} |\psi_n(s)|^2 \mathrm{e}^{-2N\alpha_n^{(1)}s}\,\mathrm{d}s\,,$$

we obtain

$$\int_{-\infty}^0 |\psi_n(s)|^2\,\mathrm{d}s \leq \int_{-\infty}^0 |\psi_n(s)|^2 \mathrm{e}^{-2N\alpha_n^{(1)}s}\,\mathrm{d}s \leq \frac{1}{\left(\alpha_n^{(1)}\right)^2}\|u_n\|^2_{L^2(\mathbb{R}^{2N})}\,,$$

which implies that $(\psi_n)_{n\geq 0}$ converges strongly to zero in $L^2(]-\infty, 0[)$, and thus for almost all $s \leq 0$ (still up to the extraction of a subsequence).

But, we have

$$\psi_n(s) - \psi_n(0) = \int_0^s \psi_n'(\tau)\,\mathrm{d}\tau \underset{n\to\infty}{\longrightarrow} \int_0^s g(\tau)\,\mathrm{d}\tau = \psi^{(1)}(s)\,,$$

which, according to the fact that $\psi_n(0) \overset{n\to\infty}{\longrightarrow} 0$, implies that

$$\psi_n(s) \overset{n\to\infty}{\longrightarrow} \psi^{(1)}(s)\,, \quad \forall\, s \in \mathbb{R}\,. \tag{3.6}$$

We deduce that $\psi^{(1)}{}_{|]-\infty,0]} = 0\,,$ which completes the proof of the fact that $\psi^{(1)} \in \mathcal{P}$.

Finally in light of (3.5), we have

$$\left|\psi^{(1)}(1)\right| \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}\,.$$

Since

$$\|\psi^{(1)'}\|_{L^2(\mathbb{R})} \geq \int_0^1 \left|\psi^{(1)'}(\tau)\right| \mathrm{d}\tau = \left|\psi^{(1)}(1)\right|,$$

this gives rise to

$$\|\psi^{(1)'}\|_{L^2} \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}\,,$$

which ends the proof of the key lemma 3.1. □

3.3 *Study of the remainder term and iteration*

The last step of the proof consists in iteratating the previous process and to prove that the algorithmic construction converges. For this purpose, let us first consider the remainder term

$$\mathrm{r}_n^{(1)}(x) = u_n(x) - g_n^{(1)}(x)\,, \tag{3.7}$$

where

$$g_n^{(1)}(x) = \sqrt{\frac{2N\alpha_n^{(1)}}{\gamma_N}}\,\psi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right).$$

It can be easily proved that $(\mathrm{r}_n^{(1)})_{n\geq 0}$ is a bounded sequence in $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$ satisfying (1.16), (1.18) and the following property:

$$\lim_{n\to\infty}\left\|\frac{\nabla \mathrm{r}_n^{(1)}}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} = \lim_{n\to\infty}\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} - \|\psi^{(1)'}\|^2_{L^2(\mathbb{R})}\,. \tag{3.8}$$

Let us now define $A_1 = \limsup_{n\to\infty}\|\mathrm{r}_n^{(1)}\|_{\mathcal{L}(\mathbb{R}^{2N})}$. If $A_1 = 0$, we stop the process. If not, arguing as above, we prove that there exists a constant C such that

$$\frac{A_1}{2}\sqrt{(2N-1)\,\alpha_n^{(2)}} \leq \left|\tilde{\mathrm{r}}_n^{(1)}\big(\alpha_n^{(2)}\big)\right| \leq C\sqrt{\alpha_n^{(2)}} + o(1)\,, \tag{3.9}$$

where $\tilde{\mathrm{r}}_n^{(1)}(s) = \mathrm{r}_n^{(1)}(\mathrm{e}^{-s})$ and that there exist a scale $(\alpha_n^{(2)})$ satisfying the statement of Corollary 3.4 with A_1 instead of A_0 and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$\mathrm{r}_n^{(1)}(x) = \sqrt{\frac{2N\alpha_n^{(2)}}{\gamma_N}}\,\psi^{(2)}\left(\frac{-\log|x|}{\alpha_n^{(2)}}\right) + \mathrm{r}_n^{(2)}(x)\,,$$

with $\|\psi^{(2)\prime}\|_{L^2} \geq \frac{A_1}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}$ and

$$\lim_{n\to\infty}\left\|\frac{\nabla \mathrm{r}_n^{(2)}}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} = \lim_{n\to\infty}\left\|\frac{\nabla \mathrm{r}_n^{(1)}}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} - \|\psi^{(2)\prime}\|^2_{L^2(\mathbb{R})}\,.$$

Moreover, we claim that $(\alpha_n^{(1)})$ and $(\alpha_n^{(2)})$ are orthogonal in the sense of Definition 1.9. Otherwise, there exists a constant C such that

$$\frac{1}{C} \leq \left|\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right| \leq C\,.$$

Making use of equality (3.7), we get

$$\tilde{\mathrm{r}}_n^{(1)}(\alpha_n^{(2)}) = \sqrt{\frac{2N\alpha_n^{(1)}}{\gamma_N}}\left(\psi_n\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right) - \psi^{(1)}\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right)\right)\,.$$

This implies that, up to a subsequence extraction,

$$\lim_{n\to\infty}\sqrt{\frac{\gamma_N}{2N\alpha_n^{(1)}}}\,\tilde{\mathrm{r}}_n^{(1)}(\alpha_n^{(2)}) = \lim_{n\to\infty}\left(\psi_n\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right) - \psi^{(1)}\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right)\right) = 0\,,$$

which is in contradiction with the left-hand side of inequality (3.9). Finally, iterating the process, we get at step ℓ,

$$u_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\,\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x)\,,$$

with

$$\limsup_{n\to\infty}\|\mathrm{r}_n^{(\ell)}\|^2_{\mathcal{H}(\mathbb{R}^{2N})} \lesssim 1 - A_0^2 - A_1^2 - \cdots - A_{\ell-1}^2\,.$$

This implies that $A_\ell \to 0$ as $\ell \to \infty$ and this ends the proof of the theorem.

4. Proof of Proposition 1.8

This section is devoted to the proof of Proposition 1.8. Actually, the fact that the sequence $(f_k)_{k\geq 0}$ converges weakly to 0 in $\mathcal{H}_{\mathrm{rad}}(\mathbb{R}^{2N})$ stems from straightforward computations, and the heart of the matter consists in showing that

$$\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{k\to\infty}{\longrightarrow} \frac{1}{\sqrt{\gamma_N}}\,. \tag{4.1}$$

Firstly, let us prove that $\liminf_{k\to\infty}\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \geq \frac{1}{\sqrt{\gamma_N}}$. For this purpose, let us consider $\lambda > 0$ such that

$$\int_{\mathbb{R}^{2N}}\left(\mathrm{e}^{\left|\frac{f_k(x)}{\lambda}\right|^2} - 1\right)\mathrm{d}x \leq \kappa'\,.$$

By definition, this gives rise to

$$\int_{|x|\leq e^{-k}} \left(e^{\left|\frac{f_k(x)}{\lambda}\right|^2} - 1\right) dx \leq \kappa' ,$$

and thus consequently

$$\frac{\pi^N}{N!}\left(e^{\frac{2Nk}{\gamma_N \lambda^2}} - 1\right)e^{-2Nk} \leq \kappa' .$$

We deduce that

$$\lambda^2 \geq \frac{2Nk}{\gamma_N \log(1 + \frac{N!}{\pi^N}\kappa' e^{2Nk})} \underset{k\to\infty}{\longrightarrow} \frac{1}{\gamma_N} ,$$

which ensures that

$$\liminf_{k\to\infty} \|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \geq \frac{1}{\sqrt{\gamma_N}} \cdot$$

Now the fact that $\limsup_{k\to\infty} \|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq \dfrac{1}{\sqrt{\gamma_N}}$ derives from the following proposition the proof of which is postponed at the end of this section:

PROPOSITION 4.1

Let $\gamma \in]0, \gamma_N[$. A positive constant $C_{\gamma,N}$ exists such that

$$\int_{\mathbb{R}^{2N}} (e^{\gamma|u(x)|^2} - 1)\, dx \leq C_{\gamma,N}\|u\|^2_{L^2(\mathbb{R}^{2N})}, \tag{4.2}$$

for any non-negative function u belonging to $\mathcal{H}_{\rm rad}(\mathbb{R}^{2N})$, compactly supported and satisfying $u(|x|) : [0, \infty[\to \mathbb{R}$ is decreasing and $\left\|\dfrac{\nabla u}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})} \leq 1$. Besides, inequality (4.2) is sharp.

Assume indeed for the time being that the above proposition is true. Then, for any fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^{2N}} (e^{(\gamma_N-\varepsilon)|f_k(x)|^2} - 1)\, dx \leq C_{\varepsilon,N}\|f_k\|^2_{L^2(\mathbb{R}^{2N})},$$

which leads to the desired result, by virtue of the convergence of (f_k) to zero in $L^2(\mathbb{R}^{2N})$.

To end the proof of Proposition 1.8, it remains to establish Proposition 4.1 the proof of which is inspired from Theorem 0.1 in [1].

Proof. Let u satisfy the assumptions of Proposition 4.1. Then there exists a function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &u(x) = v(r)\,, \quad |x| = r\,,\\ &v'(r) \leq 0\,, \quad \forall\, r \geq 0\,, \text{ and}\\ &\exists\, r_0 > 0 \text{ such that } v(r) = 0 \quad \forall\, r \geq r_0\,. \end{aligned}$$

Setting $w(t)=\sqrt{\gamma_N}\, v\big(e^{-\frac{t}{2}}\big)$, we can notice that w satisfies the following properties:

$$w(t)\geq 0\,,\quad \forall\, t\in\mathbb{R}\,, \tag{4.3}$$
$$w'(t)\geq 0\,,\quad \forall\, t\in\mathbb{R}\,,\ \text{and} \tag{4.4}$$
$$\exists\, t_0\in\mathbb{R}\ \text{such that}\ w(t)=0\quad \forall\, t\leq t_0\,. \tag{4.5}$$

Besides, we obtain by straightforward computations that

$$\|w'\|_{L^2(\mathbb{R})}=\sqrt{N}\,\Big\|\frac{\nabla u}{|\,.\,|^{N-1}}\Big\|_{L^2(\mathbb{R}^{2N})}\leq\sqrt{N}\,, \tag{4.6}$$
$$\int_{\mathbb{R}}|w(t)|^2\,e^{-Nt}\,dt=4N\|u\|^2_{L^2(\mathbb{R}^{2N})}\,, \tag{4.7}$$

and

$$\int_{\mathbb{R}}\Big(e^{\frac{\gamma}{\gamma_N}|w(t)|^2}-1\Big)\,e^{-Nt}\,dt=\frac{(N-1)!}{\pi^N}\int_{\mathbb{R}^{2N}}\Big(e^{\gamma|u(x)|^2}-1\Big)\,dx\,. \tag{4.8}$$

Thus to prove (4.2), it suffices to show that for any β belonging to $]0,1[$, there exists a positive constant C_β such that

$$\int_{\mathbb{R}}(e^{\beta|w(t)|^2}-1)\,e^{-Nt}\,dt\leq C_\beta\int_{\mathbb{R}}|w(t)|^2\,e^{-Nt}\,dt\,, \tag{4.9}$$

where w satisfies (4.3), (4.4), (4.5) and (4.6). For that purpose, let us set

$$T_0:=\sup\{t\in\mathbb{R};\ w(t)\leq 1\}\in\,]-\infty,+\infty]$$

and write

$$\int_{\mathbb{R}}(e^{\beta|w(t)|^2}-1)\,e^{-Nt}\,dt=I_1+I_2\,,$$

where

$$I_1:=\int_{-\infty}^{T_0}(e^{\beta|w(t)|^2}-1)\,e^{-Nt}\,dt\quad\text{and}\quad I_2:=\int_{T_0}^{+\infty}(e^{\beta|w(t)|^2}-1)\,e^{-Nt}\,dt\,.$$

In order to estimate I_1, let us notice that for any $t\leq T_0$, $w(t)$ belongs to $[0,1]$. Using the fact that there exists a positive constant M such that

$$e^x-1\leq Mx\,,\quad \forall\, x\in[0,1]\,,$$

we deduce that

$$I_1\leq M\,\beta\int_{-\infty}^{T_0}|w(t)|^2e^{-Nt}\,dt\,.$$

Let us now estimate I_2. By virtue of Cauchy–Schwarz inequality, we get for any $t \geq T_0$,

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^{t} w'(\tau)\, \mathrm{d}\tau \\ &\leq 1 + \sqrt{t - T_0}\, \|w'\|_{L^2(\mathbb{R})}\,. \end{aligned}$$

This implies, in view of (4.6), that

$$w(t) \leq 1 + \sqrt{(t - T_0)N}\,.$$

In addition, using the fact that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$1 + \sqrt{s} \leq \sqrt{(1+\varepsilon)s + C_\varepsilon}\,.$$

We deduce that for any $t \geq T_0$,

$$w(t)^2 \leq (1+\varepsilon)(t - T_0)N + C_\varepsilon\,.$$

As $\beta \in]0, 1[$, we can choose ε such that $\beta(1+\varepsilon) - 1 < 0$. Hence

$$\begin{aligned} I_2 &\leq \int_{T_0}^{+\infty} \mathrm{e}^{\beta(1+\varepsilon)(t-T_0)N + \beta C_\varepsilon - Nt}\, \mathrm{d}t \\ &\leq \mathrm{e}^{\beta C_\varepsilon - NT_0} \int_{T_0}^{+\infty} \mathrm{e}^{(t-T_0)N[\beta(1+\varepsilon)-1]}\, \mathrm{d}t \\ &\leq \frac{\mathrm{e}^{\beta C_\varepsilon - NT_0}}{N[1 - \beta(1+\varepsilon)]}\,. \end{aligned}$$

Since

$$\int_{T_0}^{+\infty} |w(t)|^2\, \mathrm{e}^{-Nt}\, \mathrm{d}t \geq \int_{T_0}^{+\infty} \mathrm{e}^{-Nt}\, \mathrm{d}t = \frac{\mathrm{e}^{-NT_0}}{N}\,,$$

we infer that

$$I_2 \leq \frac{\mathrm{e}^{\beta C_\varepsilon}}{1 - \beta(1+\varepsilon)} \int_{T_0}^{+\infty} |w(t)|^2\, \mathrm{e}^{-Nt}\, \mathrm{d}t\,.$$

Now, setting $C_\beta = \max\left\{M\beta, \dfrac{\mathrm{e}^{\beta C_\varepsilon}}{1 - \beta(1+\varepsilon)}\right\}$, we get (4.9). This ends the proof of inequality (4.2).

Finally, note that the example by Moser f_k defined by (1.12) illustrates the sharpness of inequality (4.2), since $\|f_k\|_{L^2(\mathbb{R}^{2N})} \overset{k\to\infty}{\longrightarrow} 0$ and

$$\begin{aligned} \int_{\mathbb{R}^{2N}} (\mathrm{e}^{\gamma_N |f_k(x)|^2} - 1)\, \mathrm{d}x &\geq \int_{|x|<\mathrm{e}^{-k}} (\mathrm{e}^{\gamma_N |f_k(x)|^2} - 1)\, \mathrm{d}x \\ &= \frac{\pi^N}{N!}\left(1 - \mathrm{e}^{-2Nk}\right) \underset{k\to\infty}{\longrightarrow} \frac{\pi^N}{N!}\,. \end{aligned}$$

$\square$

Acknowledgements

The author is very grateful to Prof. Hajer Bahouri and Prof. Mohamed Majdoub for interesting discussions and careful reading of the manuscript. A part of this work was done at LAMA-Université Paris-Est Créteil whose hospitality and support is gratefully acknowledged.

References

[1] Adachi S and Tanaka K, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, *Proc. Am. Math. Soc.* **128** (1999) 2051–2057
[2] Adams D R, A sharp inequality of J. Moser for higher order derivates, *Ann. Math.* **128** (1988) 385–398
[3] Bahouri H, Structure theorems for 2D linear and nonliner Schrödinger equations, *Commun. Contemp. Math.* **10** (2015) 1142
[4] Bahouri H, Structure theorems for 2D linear and nonliner Schrödinger equations, *Notes C.R. Acad. Sci. Paris* **353** (2015) 235–240
[5] Bahouri H, Chemin J-Y and Gallagher I, Refined Hardy inequalities, *Ann. Scuola Norm. Pisa*, **5** (2006) 375–391
[6] Bahouri H and Cohen A, Refined Sobolev inequalities in Lorentz spaces, *J. Fourier Anal. Appl.* **17** (2011) 662–673
[7] Bahouri H, Ibrahim S and Perelman G, Scattering for the critical 2-D NLS with exponential growth, *J. Differ. Integral Equ.* **27** (2014) 233–268
[8] Bahouri H, Majdoub M and Masmoudi N, On the lack of compactness in the 2D critical Sobolev embedding, *J. Funct. Anal.* **260** (2011) 208–252
[9] Bahouri H, Majdoub M and Masmoudi N, Lack of compactness in the 2D critical Sobolev embedding, the general case, *J. Math. Pures Appl.* **101** (2014) 415–457
[10] Bahouri H and Perelman G, A Fourier approach to the profile decomposition in Orlicz spaces, *Math. Res. Lett.* **21** (2014) 33–54
[11] Ben Ayed I and Zghal M K, Characterization of the lack of compactness of $H^2_{\mathrm{rad}}(\mathbb{R}^4)$ into the Orlicz space, *Commun. Contemp. Math.* **16** (2014)
[12] Ben Ayed I and Zghal M K, Description of the lack of compactness in Orlicz spaces and applications, *J. Differ. Integral Equ.* **28** (2015) 553–580
[13] Colliander J, Ibrahim S, Majdoub M and Masmoudi N, Energy critical NLS in two space dimensions, *J. Hyperbol. Differ. Equ.* **6(3)** (2009) 549–575
[14] Hardy G H, Note on a theorem of Hilbert, *Math. Z.* **6** (1920) 314–317
[15] Hardy G H, An inequality between integrals, *Messenger of Math.* **54** (1925) 150–156
[16] Ibrahim S, Majdoub M and Masmoudi N, Global solutions for a semilinear, two-dimensional Klein–Gordon equation with exponential-type nonlinearity, *Commun. Pure Appl. Math.* (2006) 1–20
[17] Ibrahim S, Majdoub M, Masmoudi N and Nakanishi K, Scattering for the two-dimentional energy-critical wave equation, *Duke Math. J.* **150** (2009) 287–329
[18] Lam N, and Lu G, Sharp Adams inequalities in Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for arbitrary integer m, *J. Differ. Equ.* **253(4)** (2012) 1143–1171
[19] Lions PL, The concentration-compactness principle in the calculus of variations. The limit case. I., *Rev. Mat. Iberoam.* **1** (1985) 145–201
[20] Lions P-L, The concentration-compactness principle in the calculus of variations. The limit case. II., *Rev. Mat. Iberoam.* **1** (1985) 45–121
[21] Masmoudi N and Sani F, Adams' Inequality with the exact growth condition in $\mathbb{R}^4$, *Commun. Pure Pure Appl. Math.* (2014) 1307–1335
[22] Moser J, A sharp form of an inequality of N. Trudinger, *Indiana Univ. Math. J.* **20** (1971) 1077–1092

[23] Pohozaev S I, The Sobolev embedding in the case $pl = n$, Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964–1965, Mathematics Sections, Moscov. Energet. Inst., Moscow, (1965) 158–170
[24] Rao M M, and Ren Z D, Applications of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics, 250 (2002)
[25] Ruf B, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$, *J. Funct. Anal.* **219** (2005) 340–367
[26] Ruf B, and Sani F, Sharp Adams-type inequalities in $\mathbb{R}^n$, *Trans. Am. Math. Soc.* **365** (2013) 645–670
[27] Ruf B, and Sani F, Ground states for elliptic equations in $\mathbb{R}^2$ with exponential critical growth, *Geom. Prop. Parabol. Elliptic PDE's*, **2** (2013) 251–267
[28] Sani F, A biharmonic equation in $\mathbb{R}^4$ involving nonlinearities with critical exponential growth, *Commun. Pure Appl. Anal.* **12** (2013) 251–267
[29] Sani F, A biharmonic equation in $\mathbb{R}^4$ involving nonlinearities with subcritical exponential growth, *Adv. Nonlinear Stud.* **11** (2011) 889–904
[30] Trudinger N S, On imbeddings into Orlicz spaces and some applications, *J. Appl. Math. Mech.* **17** (1967) 473–484

Communicating Editor: S Kesavan