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LEVEL SETS OF CLASSES OF MAPPINGS OF TWO-STEP CARNOT GROUPS IN A NONHOLONOMIC INTERPRETATION

M. B. Karmanova UDC 514.7:517.4

Abstract: We obtain the metric properties of level sets of certain sufficiently smooth mappings of two-step Carnot groups which are Hölder in the sub-Riemannian sense. In particular, we calculate the Hausdorff dimension of these sets and prove a coarea formula of a new type.

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The goal of this article is to obtain the metric properties for the model cases of classes of Hölder mappings in the sub-Riemannian sense which are defined and take values on two-step nilpotent graded groups, in particular, Carnot groups. The main class of interest consists of the mappings that are sufficiently smooth in the classical sense but belong to the class of Hölder mappings on nonholonomic structures. The basic results are the description of a parametrization of level sets as well as the calculation of the measure of the intersections of nonholonomic balls with the level sets passing through their centers. As application, we establish a coarea formula of a new type when integration over the level set is done with respect to the Hausdorff $\mathscr{H}^{\mu}$-measure, where μ is not the same as the difference of (Hausdorff) dimensions of the preimage and the image. We emphasize that the coarea formula is important in the theory of exterior forms and currents, in the problems concerning some fine properties of minimal surfaces, and so on; see the survey of applications and results as well as references in [1].

The level sets of classes of contact mappings of Carnot–Carathéodory spaces are studied in [1], but to construct examples of these mappings is a complicated particular problem (see [2–5] for instance), while the well-known smooth (in the classical sense) but not contact mappings remain little studied as regards their "sub-Riemannian" properties. Recent results in this direction appeared in the author's articles [6, 7]. They amount to descriptions of methods for approximating the classes of Hölder mappings in the sub-Riemannian sense and derivations of the metric properties of the corresponding image surfaces in the case that the topological dimension of the preimage is at most that of the image. The resulting properties have already found applications in developing the foundations for the theory of extremal surfaces on nonholonomic structures; see [8] for instance.

This article addresses the reverse situation: The topological dimension of the image is less than that of the preimage. The greatest attention is paid to the model case of noncontact mappings of two-step groups. The distinguishing feature of the statement of the problem solved here is that for the image $\widetilde{\mathbb{G}}$ and the preimage $\mathbb{G}$ under study it is impossible to construct any contact mappings $\psi : \mathbb{G} \to \widetilde{\mathbb{G}}$ with the (sub-Riemannian) differential of maximal rank.

1. The Main Definitions and Results

Let us recall some necessary definitions and properties, as well as state the requirements on the mappings in question. We present the definitions of Carnot groups and their generalizations, nilpotent graded groups, in the form used in this study; for instance, see the definition of stratified homogeneous group in [9] and other sources containing main definitions.

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Definition 1.1. A *two-step nilpotent graded* (*Lie*) *group* is a connected simply-connected Lie group $\mathbb{G}$ whose Lie algebra V can be expressed as the direct sum such that

$$V = V_1 \oplus V_2;\ [V_1, V_1] \subset V_2,\ [V_1, V_2] = \{0\}. \tag{1.1}$$

If the condition $[V_1, V_1] \subset V_2$ here is replaced by $[V_1, V_1] = V_2$ then $\mathbb{G}$ is called a *two-step Carnot group*.

The group operations are defined via the Baker–Campbell–Hausdorff formula.

Definition 1.2. Denote the topological dimension of a group $\mathbb{G}$ by N and take left-invariant vector fields $X_1, X_2, \dots, X_N$ on $\mathbb{G}$ constituting a basis for the Lie algebra V such that

$$\begin{cases} X_1, \dots, X_{\dim V_1} \text{ is a basis of } V_1, \\ X_{\dim V_1 + 1}, \dots, X_N \text{ is a basis of } V_2. \end{cases}$$

Here $\dim V_1$ stands for the dimension of V_1 at each point. If $X_j \in V_k$ for $k = 1, 2$ then k is called the *degree* of the field X_j and denoted by $\deg X_j$, for $j = 1, \dots, N$. The vector fields of degree 1 are called *horizontal*. In the case under consideration the *depth* of $\mathbb{G}$ equals 2. Put $\dim V_2 = N - \dim V_1$.

To derive the sought metric properties, we use the following definition of distance which agrees with the structure of nilpotent graded groups.

Definition 1.3 (cf. [10]). Given a nilpotent graded group $\mathbb{G}$ of topological dimension N, put

$$w = \exp\left(\sum_{i=1}^{N} w_i X_i\right)(v).$$

Define d_2 as

$$d_2(w, v) = \max\left\{\Big(\sum_{j:\deg X_j = 1} |w_j|^2\Big)^{\frac{1}{2}}, \Big(\sum_{j:\deg X_j = 2} |w_j|^2\Big)^{\frac{1}{2\cdot 2}}\right\}.$$

The set $\{w \in \mathbb{G} : d_2(w, v) < r\}$ is called the radius $r > 0$ *ball* with respect to d_2 centered at v, and denoted by $\mathrm{Box}_2(v, r)$.

Property 1.4. *The quantity $d_2(v, w)$ is locally a quasimetric: it is nonnegative* (*and vanishes if and only if $v = w$*), *it is symmetric, and for every neighborhood $D \Subset \mathbb{G}$ there exists a constant $C_D < \infty$ such that the generalized triangle inequality*

$$d_2(v, w) \le C_D(d_2(v, u) + d_2(u, w))$$

holds for all $v, u, w \in D$.

Definition 1.5. Consider $v \in \mathbb{G}$ and $(w_1, \dots, w_N) \in \mathbb{R}^N$. Define $\theta_v : \mathbb{R}^N \to \mathbb{G}$ as

$$\theta_v(w_1, \dots, w_N) = \exp\left(\sum_{i=1}^{N} w_i X_i\right)(v).$$

It is known that θ_v is a smooth diffeomorphism.

Definition 1.6. The tuple $\{w_i\}_{i=1}^N$ is called the *normal coordinates* or *coordinates of the first kind* (with respect to $v \in \mathbb{G}$) of $w = \theta_v(w_1, \dots, w_N)$.

Property 1.7. *On a nilpotent graded group $\mathbb{G}$ the image of $\mathrm{Box}_2(v, r)$ under θ_v^{-1} is the Cartesian product*

$$\mathrm{Box}_2(0, r) = B_2^{\dim V_1}(0, r) \times B_2^{\dim V_2}(0, r^2)$$

of balls of diameters $2r$ and $2r^2$, where B_2^l is the Euclidean ball of dimension l.

This directly implies the next property.

Property 1.8. *The Hausdorff dimension of $\mathbb{G}$ with respect to d_2 equals $\nu = \dim V_1 + 2 \dim V_2$.*

2. The Structure of Levels and the Coarea Formula

Let us describe the classes of mappings whose level sets we will study.

Assumption 2.1 (cf. [6, 7]). *Consider $\varphi : \Omega \to \widetilde{\mathbb{G}}$, where*

(1) $\Omega \subset \mathbb{G}$ *is an open set;*

(2) $\varphi : \Omega \to \widetilde{\mathbb{G}}$ *is a mapping of class C^1;*

(3) $\mathbb{G}$ *is a two-step nilpotent graded group of topological dimension N with smooth basis fields $X_1, \dots, X_N$, the Lie algebra $V = V_1 \oplus V_2$, the quasimetric d_2, and the Hausdorff dimension ν;*

(4) $V_1 = \operatorname{span}\{X_1, \dots, X_{n_1}\}$, $V_2 = \operatorname{span}\{X_{n_1+1}, \dots, X_N\}$, *and $n_2 = N - n_1$;*

(5) $\widetilde{\mathbb{G}}$ *is a two-step nilpotent graded group of topological dimension $\widetilde{N}$ with smooth basis fields $\widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}$, the Lie algebra $\widetilde{V} = \widetilde{V}_1 \oplus \widetilde{V}_2$, the quasimetric $\tilde{d}_2$, and the Hausdorff dimension $\tilde{\nu}$;*

(6) $\widetilde{V}_1 = \operatorname{span}\{\widetilde{X}_1, \dots, \widetilde{X}_{\tilde{n}_1}\}$, $\widetilde{V}_2 = \operatorname{span}\{\widetilde{X}_{\tilde{n}_1+1}, \dots, \widetilde{X}_{\widetilde{N}}\}$, *and $\tilde{n}_2 = \widetilde{N} - \tilde{n}_1$;*

(7) $N > \widetilde{N}$, $n_1 > \tilde{n}_1$, *and $\tilde{n}_2 > n_2$;*

(8) $\operatorname{rank} D\varphi = \widetilde{N}$ *everywhere and the differential is strictly separated from zero on the orthogonal complement to its kernel.*

Remark 2.2. If $n_1 > \widetilde{N}$ then under a certain assumption we impose the additional condition $\varphi \in C^2(\Omega, \widetilde{\mathbb{G}})$; see Assumption 2.10 for more details.

Let us elucidate the features of the chosen image and preimage; namely, display the main difference of this case from [1]. Firstly, introduce the concept of contact mapping. We emphasize that [1] studies sufficiently smooth contact mappings.

Definition 2.3. With $\Omega \subset \mathbb{G}$, a mapping $\varphi : \Omega \to \widetilde{\mathbb{G}}$ of nilpotent graded groups is a *contact mapping* of class C^1_H whenever the horizontal derivatives of φ exist and are continuous, while $\widehat{D}_H\varphi = (X_1\varphi, \dots, X_{\dim V_1}\varphi)$ carries horizontal fields into horizontal fields.

Remark 2.4. The conditions $N > \widetilde{N}$, $n_1 > \tilde{n}_1$, and $\tilde{n}_2 > n_2$ imply that there exist no contact mappings $\psi : \mathbb{G} \to \widetilde{\mathbb{G}}$ with the (sub-Riemannian) differential of maximal rank. Recall (see [11, 12] for instance) that, with $D \subset \mathbb{G}$, a mapping $\psi : D \to \widetilde{\mathbb{G}}$, is called *hc-differentiable*, or *differentiable in the sub-Riemannian sense*, at the (limit) point $x \in D$ whenever there exists a horizontal homomorphism $\mathscr{L}_x : \mathbb{G} \to \widetilde{\mathbb{G}}$ such that

$$d_2(\psi(y), \mathscr{L}_x\langle y\rangle) = o(d_2(x, y)) \quad \text{for } D \ni y \to x.$$

The sub-Riemannian differential is of the form

$$\begin{pmatrix} (\widehat{D}\psi)_{V_1,\widetilde{V}_1} & 0 \\ 0 & (\widehat{D}\psi)_{V_2,\widetilde{V}_2} \end{pmatrix},$$

where $(\widehat{D}\psi)_{V_k,\widetilde{V}_k}$ are blocks of dimension $\tilde{n}_k \times n_k$, for $k = 1, 2$; therefore, the maximal possible rank is $\tilde{n}_1 + n_2 < \widetilde{N}$. The same conclusions hold for the classical differential that for contact mappings of class C^1_H is of the form

$$\begin{pmatrix} (\widehat{D}\psi)_{V_1,\widetilde{V}_1} & * \\ 0 & (\widehat{D}\psi)_{V_2,\widetilde{V}_2} \end{pmatrix}.$$

By Assumption 2.1, the topological dimension of the level sets equals $N - \widetilde{N}$. The measure $\mathscr{H}^{\nu-\tilde{\nu}}$ on the level sets is of interest in [1], where ν and $\tilde{\nu}$ are the Hausdorff dimensions (with respect to sub-Riemannian quasimetrics) of the preimage and image respectively. Observe that, according to the properties of φ established in [1], for the noncharacteristic points of the preimage the minimal possible sum of the degrees of vector fields with linearly independent images equals $\tilde{\nu}$. Noncharacteristic points are useful in applications, for instance, to prove the coarea formula. Here we also consider neighborhoods of points at which the maximal rank is attained on the vector fields with the minimal possible sum of degrees equal to $\min\{n_1, \widetilde{N}\}+2\max\{\widetilde{N}-n_1, 0\}$. Recall that the topological dimension of level sets equals $N-\widetilde{N}$.

Let us now prove a proposition about the Hausdorff dimension (with respect to sub-Riemannian quasimetrics) of the intersections of neighborhoods of such points and level sets. In other words, we establish

Theorem 2.5. *If at some $x \in \Omega$ the sum of degrees of vector fields with linearly independent images equals $\min\{n_1, \widetilde{N}\} + 2\max\{\widetilde{N} - n_1, 0\}$ then the $\mathscr{H}^{N-\widetilde{N}}$-measure of the intersection of the level set $\varphi^{-1}(\varphi(x))$ and $\mathrm{Box}_2(x, r)$ equals $O(r^{\mu})$ for $r > 0$ sufficiently small, where*

$$\mu = \max\{n_1 - \widetilde{N}, 0\} + 2\min\{n_2, N - \widetilde{N}\}.$$

Remark 2.6. Recall the expressions for the Hausdorff dimensions $\nu = n_1 + 2n_2$ and $\tilde{\nu} = \tilde{n}_1 + 2\tilde{n}_2$. In contrast to the contact case, the equality $\mu = \nu - \tilde{\nu}$ fails in the present situation. Indeed, suppose that $n_1 > \widetilde{N}$. Then $n_2 = N - n_1 < N - \widetilde{N}$. Consequently, $\mu = n_1 - \widetilde{N} + 2n_2 = n_1 - \tilde{n}_1 + 2n_2 - \tilde{n}_2 > n_1 - \tilde{n}_1 + 2(n_2 - \tilde{n}_2)$ because $\tilde{n}_2 \neq 0$. If $n_1 \leq \widetilde{N}$ then $\mu = 2(N - \widetilde{N}) = n_1 - \tilde{n}_1 + 2(n_2 - \tilde{n}_2) + n_1 - \tilde{n}_1 > \nu - \tilde{\nu}$ because Assumption 2.1 yields $n_1 - \tilde{n}_1 > 0$.

Proof of Theorem 2.5. Fix $x \in \Omega$ and consider the surface

$$S = \theta_x^{-1}(\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))) \tag{2.1}$$

and its tangent plane T_0S at the origin in the normal coordinates. Denote by $S_0^{\perp}$ the plane that is spanned by the basis vectors with linearly independent images the sum of whose degrees equals $\min\{n_1, \widetilde{N}\} + 2\max\{\widetilde{N} - n_1, 0\}$. Consider the following parametrization of the surface by the tangent plane:

$$\varphi^{-1}(\varphi(x)) \ni y \overset{\pi_x}{\mapsto} T_0S \cap \left(y + S_0^{\perp}\right).$$

By the definition of $S_0^{\perp}$, the intersection $T_0S \cap \left(y + S_0^{\perp}\right)$ consists of one point. Moreover, for all y in the domain we have $|y|/|\pi_x(y)| = 1 + o(1)$, where $o(1) \to 0$ as $|y| \to 0$ uniformly in $x \in \mathscr{U} \Subset \Omega$.

Suppose firstly that $n_1 \leq \widetilde{N}$. Then, by the choice of x, all $N - \widetilde{N}$ linearly independent vectors tangent to this level set have degrees equal to 2 because $n_2 \geq N - \widetilde{N}$. Consequently, applying the arguments of Theorem 3.7 of [1], we infer that the area of intersection of the tangent plane and the ball is minimal possible. Namely, it is determined by the sum of degrees of linearly independent tangent vectors, which is minimal possible. Here the degree of a vector coincides with the largest degree of the constituent basis vectors. Therefore, its value is comparable to $r^{2(N-\widetilde{N})}$ because $2(N - \widetilde{N})$ coincides with the minimal possible sum of degrees of linearly independent tangent vectors. Since the subspace orthogonal to the tangent plane consists of the vectors of degree at most 2, the area of intersection of the sub-Riemannian ball and the surface itself differs from that above by $o(r^{2(N-\widetilde{N})})$. In other words, an $o(r^2)$-neighborhood of the set S includes $T_0S \cap \mathrm{Box}_2(0, r)$. The quantity $o(1)$ is uniform on $\mathscr{U} \Subset \Omega$.

Suppose that $n_1 > \widetilde{N}$. In this case $S_0^{\perp}$ consists of horizontal vectors; therefore, as in the case $n_1 \leq \widetilde{N}$, the areas of the intersections of $\mathrm{Box}_2(0, r)$ with the level set and with its tangent plane coincide up to a factor $1 + o(1)$ because an $o(r)$-neighborhood of S includes $T_0S \cap \mathrm{Box}_2(0, r)$. Moreover, their values are comparable to $r^{n_1 - \widetilde{N} + 2n_2}$ by the arguments of Theorem 3.7 of [1] because the minimum of the sum of degrees of linearly independent tangent vectors equals $n_1 - \widetilde{N} + 2n_2$. The quantity $o(1)$ is uniform on $\mathscr{U} \Subset \Omega$. The proof of Theorem 2.5 is complete. $\square$

Remark 2.7. The condition $\tilde{n}_2 > n_2$ is unnecessary for the validity of Theorem 2.5.

Let us find an analytical expression for the $\mathscr{H}^{N-\widetilde{N}}$-measure of the intersection of a level set and a sub-Riemannian ball centered on it.

Theorem 2.8. *Suppose that at $x \in \Omega$ the sum of degrees of vector fields with linearly independent images equals $\min\{n_1, \widetilde{N}\} + 2\max\{\widetilde{N} - n_1, 0\}$. Then the $\mathscr{H}^{N-\widetilde{N}}$-measure of the intersection of $\mathrm{Box}_2(x, r)$ with the level set $\varphi^{-1}(\varphi(x))$ passing through x equals*

$$\omega_{n_1-\widetilde{N}}\omega_{n_2} r^{n_1-\widetilde{N}+2n_2} \frac{\sqrt{\det(D\varphi(x)D\varphi(x)^*)}}{\sqrt{\det(D_{\mathrm{diag}}\varphi(x)D_{\mathrm{diag}}\varphi(x)^*)}} |g|_{\ker D\varphi(x)}(x)|(1 + o(1))$$

if $n_1 > \widetilde{N}$, and

$$\omega_{N-\widetilde{N}} r^{2(N-\widetilde{N})} \frac{\sqrt{\det(D\varphi(x)D\varphi(x)^*)}}{\sqrt{\det(D_{\mathrm{diag}}\varphi(x)D_{\mathrm{diag}}\varphi(x)^*)}} |g|_{\ker D\varphi(x)}(x)|(1 + o(1))$$

if $n_1 \leq \widetilde{N}$. Here g is the Riemann tensor on $\mathbb{G}$, the operator D_{diag} is determined by the structure of vector fields, and $o(1) \to 0$ as $r \to 0$ uniformly in $x \in \mathscr{U} \Subset \Omega$.

Proof. Introduce some auxiliary notation. Suppose that at $x \in \Omega$ the sum of degrees of vector fields with linearly independent images equals $\min\{n_1, \widetilde{N}\} + 2\max\{\widetilde{N} - n_1, 0\}$. Rearrange the matrix of the differential $D\varphi(x)$ as follows. If $n_1 > \widetilde{N}$ then nullify all columns starting with column $n_1 + 1$ and denote the resulting matrix by $D_{\mathrm{diag}}\varphi(x)$:

$$D_{\mathrm{diag}}\varphi(x) = (X_1\varphi(x), \dots, X_{n_1}\varphi(x), 0, \dots, 0).$$

This corresponds to projecting the normal space of the level set $\varphi^{-1}(\varphi(x))$ to the horizontal subspace in the orthogonal direction to the horizontal subspace. Then (see the similar arguments in Theorem 3.11 of [1] concerning the distortion coefficient while passing from the kernel of the hc-differential to the kernel of the classical differential), the $\mathscr{H}^{N-\widetilde{N}}$-measure of the intersection of the level set and $\mathrm{Box}_2(x, r)$ by Theorem 2.5 equals

$$\omega_{n_1-\widetilde{N}}\omega_{n_2} r^{n_1-\widetilde{N}+2n_2} \frac{\sqrt{\det(D\varphi(x)D\varphi(x)^*)}}{\sqrt{\det(D_{\mathrm{diag}}\varphi(x)D_{\mathrm{diag}}\varphi(x)^*)}} |g|_{\ker D\varphi(x)}(x)|(1 + o(1)),$$

where g is the Riemann tensor on $\mathbb{G}$, and $o(1) \to 0$ as $r \to 0$ uniformly in $x \in \mathscr{U} \Subset \Omega$.

Consider the case that $n_1 \leq \widetilde{N}$. The images of n_1 horizontal vectors are linearly independent. Hence, there exists an orthogonal transformation O reducing the matrix $D\varphi(x)$ to a block upper-triangular form, where the first block is of size $n_1 \times n_1$, while the parts of rows below the block vanish. Denote this $n_1 \times n_1$ block by $(D_H\varphi)_{n_1}(x)$, and the block consisting of the columns of $O \cdot D\varphi(x)$ starting with column $n_1 + 1$, by

$$(D_{H^\perp}\varphi)(x) = \begin{pmatrix} (D_{H^\perp}\varphi)_{n_1}(x) \\ (D_{H^\perp}\varphi)_{\widetilde{N},n_1}(x) \end{pmatrix}.$$

For the rows $1, \dots, n_1$ nullify all elements starting with column $n_1 + 1$ and denote the resulting matrix by $D_{\mathrm{diag}}\varphi(x)$:

$$\begin{aligned} O \cdot D\varphi(x) &= \begin{pmatrix} (D_H\varphi)_{n_1}(x) & (D_{H^\perp}\varphi)_{n_1}(x) \\ 0 & (D_{H^\perp}\varphi)_{\widetilde{N},n_1}(x) \end{pmatrix} \\ &\mapsto \begin{pmatrix} (D_H\varphi)_{n_1}(x) & 0 \\ 0 & (D_{H^\perp}\varphi)_{\widetilde{N},n_1}(x) \end{pmatrix} = D_{\mathrm{diag}}\varphi(x). \end{aligned}$$

Since O acts on the rows of $D\varphi(x)$, this corresponds to projecting the normal space of the level set $\varphi^{-1}(\varphi(x))$ to the direct product of the horizontal subspace and a part of V_2 of dimension $\widetilde{N} - n_1$. Then see the similar arguments in Theorem 3.11 of [1] and [7], the $\mathscr{H}^{N-\widetilde{N}}$-measure of the intersection of the level set and $\mathrm{Box}_2(x,r)$ by Theorem 2.5 equals

$$\omega_{N-\widetilde{N}} r^{2(N-\widetilde{N})} \frac{\sqrt{\det(D\varphi(x)D\varphi(x)^*)}}{\sqrt{\det(D_{\mathrm{diag}}\varphi(x)D_{\mathrm{diag}}\varphi(x)^*)}} \big|g|_{\ker D\varphi(x)}(x)\big|(1+o(1)),$$

where g is the Riemann tensor on $\mathbb{G}$ and $o(1)\to 0$ as $r\to 0$ uniformly in $x\in\mathscr{U}\Subset\Omega$. $\square$

REMARK 2.9. If the sum of degrees of the vector fields with linearly independent images is strictly greater than $\min\{n_1,\widetilde{N}\}+2\max\{\widetilde{N}-n_1,0\}$ then in both cases we have

$$\sqrt{\det(D_{\mathrm{diag}}\varphi(x)D_{\mathrm{diag}}\varphi(x)^*)}=0.$$

Consider the behavior of the $\mathscr{H}^{\mu}$-measure near the points at which the sum of degrees of vector fields with linearly independent images is strictly greater than $\min\{n_1,\widetilde{N}\}+2\max\{\widetilde{N}-n_1,0\}$. Denote the set of these points by χ.

Assumption 2.10. *If $n_1>\widetilde{N}$ and $\chi\neq\varnothing$ then assume moreover that $\varphi\in C^2(\Omega,\widetilde{\mathbb{G}})$, while $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ are Carnot groups.*

Theorem 2.11. *$\mathscr{H}^{\mu}(\chi\cap\varphi^{-1}(t))=0$ for all $t\in\widetilde{\mathbb{G}}$ if $n_1\le\widetilde{N}$ and almost all $t\in\widetilde{\mathbb{G}}$ if $n_1>\widetilde{N}$.*

PROOF. With $\mathscr{U}\Subset\Omega$, take $x\in\chi\cap\mathscr{U}$, consider $\varphi^{-1}(\varphi(x))$, and estimate

$$\mathscr{H}^{N-\widetilde{N}}(\varphi^{-1}(\varphi(x))\cap\mathrm{Box}_2(x,r))$$

for $r>0$ sufficiently small. Pass to the normal coordinates using the mapping θ_x^{-1}. Since the minimality condition for the sum of degrees fails, there exists at least one horizontal vector on which the differential $D\varphi(x)$ degenerates; i.e., lying in the tangent space to the level set. Denote by $q(x)$ the size of the set of these horizontal vectors. By the choice of the structure, the orthogonal complement to the tangent plane consists of vectors of degree at most 2. By the properties of the parametrization π_x, its local distortion equals $1+o(1)$, where $o(1)$ is uniform on $\mathscr{U}\Subset\Omega$. Then for all $y\in\partial\,\mathrm{Box}_2(0,r)\cap\varphi^{-1}(\varphi(x))$ such that the image $\pi_x(y)$ lies in the intersection S_H of the plane T_0S (see (2.1)) and the horizontal plane, we have $|\pi_x(y)|\ge r^2/o(1)$ because the Euclidean distance from 0 to y is at least $r^2/o(1)$. Otherwise, the distance from 0 to $\pi_x(y)$ would be comparable to r^2, and since by the choice of y the image $\pi_x(y)$ lies in a horizontal direction, the distance from y to the boundary of the ball in this case could be bounded below by $O(r^2)$; this contradicts the property that y is a boundary point of the ball. Put

$$\min_y\{|\pi_x(y)|\}=\xi(x),$$

where the minimum is taken over all $y\in\partial\,\mathrm{Box}_2(0,r)\cap\varphi^{-1}(\varphi(x))$ for which the image $\pi_x(y)$ lies in the intersection S_H. Then $\xi(x)\ge r^2/o(1)$. In this case, if $n_1\le\widetilde{N}$ then

(1) in the cross-section of the sub-Riemannian ball by a tangent plane to the level set spanned by degree 2 vectors, several degree 2 vectors will be replaced by degree 1 vectors;

(2) since the normal space consists of vectors of degree at most 2, the existing order of tangency does not change the ratio $|\pi_x(y)|/|y|$ for the $N-\widetilde{N}-q(x)$ degree 2 vectors corresponding to y and remaining in the tangent plane.

Therefore, the $\mathscr{H}^{N-\widetilde{N}}$-measure of $\pi_x(\varphi^{-1}(\varphi(x))\cap\mathrm{Box}_2(x,r))$ is bounded below by

$$O(r^{2(N-\widetilde{N}-q(x))})\cdot\xi(x)^{q(x)}=\frac{1}{o(1)}r^{2(N-\widetilde{N})}=\frac{1}{o(1)}r^{\mu},$$

where $o(1)\to 0$ as $r\to 0$ uniformly on $\mathscr{U}\Subset\Omega$. If $n_1>\widetilde{N}$ then the plane $S_0^{\perp}$, see the beginning of the proof of Theorem 2.5, contains not only horizontal vectors, but also degree 2 vectors. Consequently, each factor r^1 corresponding to a tangent horizontal field is replaced by $\xi(x) \geq r^2/o(1)$, which is insufficient for the required estimates. Therefore, we apply here with obvious modifications the arguments of [13]; see also Theorem 4.1 of [1], for C^2-smooth mappings. They imply that the $\mathscr{H}^{N-\widetilde{N}}$-measure of the image $\pi_x(\varphi^{-1}(\varphi(x)) \cap \mathrm{Box}_2(x,r))$ is comparable to $O(r^{\mu-1}) = \frac{1}{o(1)} r^{\mu}$, where $o(1)$ is uniform on $\mathscr{U} \Subset \Omega$ for all $x \in \chi \setminus \Sigma$, with $\mathscr{H}^{\nu}(\Sigma) = 0$. By the definition of Hausdorff measure, this yields

$$\mathscr{H}^{\mu}(\varphi^{-1}(\varphi(x)) \cap (\chi \setminus \Sigma) \cap \mathscr{U}) = 0$$

because for each $\delta > 0$ we obtain

$$\mathscr{H}^{\mu}_{\delta}(\varphi^{-1}(\varphi(x)) \cap (\chi \setminus \Sigma) \cap \mathscr{U}) = o(1) \cdot \mathscr{H}^{N-\widetilde{N}}(\varphi^{-1}(\varphi(x)) \cap (\chi \setminus \Sigma) \cap \mathscr{U}).$$

For the remaining set $\Sigma \subset \chi$ of zero $\mathscr{H}^{\nu}$-measure, using the arguments of Theorem 2.10.25 of [14] and the existence of metrics on the preimage and image which are bi-Lipschitz equivalent respectively to d_2 and $\tilde{d}_2$, we infer that

$$0 \leq \int\limits_{\widetilde{\mathbb{G}}} d\mathscr{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)\cap\Sigma} d\mathscr{H}^{\mu}(u) \leq C(\mathrm{Lip}(\varphi), \mu, \tilde{\nu}) \mathscr{H}^{\mu+\tilde{\nu}}(\Sigma) = 0$$

because $\mu + \tilde{\nu} \geq \nu$ and $\mathscr{H}^{\nu}(\Sigma) = 0$. Thus, $\mathscr{H}^{\mu}(\chi \cap \varphi^{-1}(t)) = 0$ for almost all $t \in \widetilde{\mathbb{G}}$. The proof of Theorem 2.11 is complete. $\square$

Let us obtain an expression for the Hausdorff $\mathscr{H}^{\mu}$-measure of level sets and prove a coarea formula of new type. Observe that the $\mathscr{H}^{\mu}$-measure is additive on distant balls. Moreover, it is absolutely continuous with respect to $\mathscr{H}^{N-\widetilde{N}}$ on each level set if $n_1 \leq \widetilde{N}$, and on almost each level set otherwise. Indeed, take t such that $\mathscr{H}^{\mu}(\chi \cap \varphi^{-1}(t)) = 0$ and $\Sigma \subset \varphi^{-1}(t)$ of $\mathscr{H}^{N-\widetilde{N}}$-measure zero. Without loss of generality we may consider $\mathscr{U} \Subset \Omega$ instead of Ω. Theorem 2.11 shows that $\Sigma \cap \chi$ has a covering by Box_2-balls such that the corresponding sum of the radii raised to power μ is at most $\varepsilon > 0$. Since χ, being the complement to an open set, is closed, $\Sigma \setminus \chi = \bigcup_{k\in\mathbb{N}} \Sigma_k$, where the d_2-distance from Σ_k to χ is at least $1/k$. Since each of these sets lies in the compact preimage of the continuous function $\psi(x) = d(x,\chi)$, on each of them the values of the differential on the vectors with the minimal possible sum of degrees are strictly separated from zero. Cover Σ_k by the "balls" $\mathrm{Box}_2(x_i, r_i) \cap \varphi^{-1}(\varphi(t))$ of radius at most $\delta < 1/k$ the sum of whose $\mathscr{H}^{N-\widetilde{N}}$-measures is at most $\frac{\varepsilon}{2^k}$. By the generalized triangle inequality and the doubling conditions, without loss of generality we may assume that $x_i \in \Sigma_k$. Then it remains to apply Theorem 2.8 to each ball in the covering and obtain an upper bound for $\mathscr{H}^{\mu}_{\delta}$.

Straightforward calculations show that for $r > 0$ sufficiently small and $x \notin \chi$ we have

$$\mathscr{H}^{\mu}(\varphi^{-1}(\varphi(x)) \cap \mathrm{Box}_2(x,r)) = \omega_{\mu} r^{\mu},$$

see the sketch in Theorem 3.17 of [1]. This leads to the following coarea formula:

Theorem 2.12. *Under Assumption* 2.1, *the coarea formula holds:*

$$\int\limits_{\Omega} \sqrt{\det(D_{\mathrm{diag}}\varphi(x) D_{\mathrm{diag}}\varphi(x)^*)}\, d\mathscr{H}^{\nu}(x) = \kappa(\mathbb{G}, \widetilde{\mathbb{G}}) \int\limits_{\varphi(\Omega)} d\mathscr{H}^{\tilde{\nu}}(z) \int\limits_{\varphi^{-1}(z)} d\mathscr{H}^{\mu}(u),$$

where

$$\kappa(\mathbb{G}, \widetilde{\mathbb{G}}) = \begin{cases} \frac{\omega_{\nu}}{\omega_{\tilde{\nu}}} \frac{\omega_{\widetilde{N}}}{\omega_N} \frac{\omega_{N-\widetilde{N}}}{\omega_{\mu}} & \text{if } n_1 \leq \widetilde{N}, \\ \frac{\omega_{\nu}}{\omega_{\tilde{\nu}}} \frac{\omega_{\widetilde{N}}}{\omega_N} \frac{\omega_{n_1-\widetilde{N}}\omega_{n_2}}{\omega_{\mu}} & \text{if } n_1 > \widetilde{N}. \end{cases}$$

Proof. With the results and special features mentioned above, the claim follows from the classical coarea formula; see the proof outline for the particular case of contact mappings in Theorem 3.21 of [1]. $\square$

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M. B. Karmanova
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: `maryka@math.nsc.ru; maryka84@gmail.com`