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ON THE NILPOTENT RESIDUALS OF ALL SUBALGEBRAS OF LIE ALGEBRAS

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Abstract. Let $\mathcal{N}$ denote the class of nilpotent Lie algebras. For any finite-dimensional Lie algebra L over an arbitrary field $\mathbb{F}$, there exists a smallest ideal I of L such that $L/I \in \mathcal{N}$. This uniquely determined ideal of L is called the nilpotent residual of L and is denoted by $L^{\mathcal{N}}$. In this paper, we define the subalgebra $S(L) = \bigcap_{H \leqslant L} I_L(H^{\mathcal{N}})$. Set $S_0(L) = 0$. Define $S_{i+1}(L)/S_i(L) = S(L/S_i(L))$ for $i \geqslant 1$. By $S_\infty(L)$ denote the terminal term of the ascending series. It is proved that $L = S_\infty(L)$ if and only if $L^{\mathcal{N}}$ is nilpotent. In addition, we investigate the basic properties of a Lie algebra L with $S(L) = L$.

Keywords: solvable Lie algebra; nilpotent residual; Frattini ideal

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1. Introduction

Throughout this paper, L is a finite-dimensional Lie algebra over an arbitrary field $\mathbb{F}$. Because of the connection between finite groups and Lie algebras of finite dimension, such investigations were successfully carried out by Barnes (see [1]–[5]), Marshall (see [10]), Schwarck (see [11]), Stitzinger (see [13], [14]), Towers (see [16]–[20]), et al. The intersection of all maximal subgroups (subalgebras) in a group (algebra) is called the Frattini subgroup (subalgebra). The Frattini theory was initiated in the study of finite groups by a paper of Frattini in 1885. Marshall (see [10]) investigated the Frattini subalgebra analogous to that of the Frattini subgroup. Chen and Meng (see [6]) studied the intersection of maximal

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subalgebras and obtained deeper structure theorems by extending and developing the Frattini theory for Lie superalgebras.

It therefore seems natural to study the intersection of other special subalgebras in a Lie algebra. Let $\mathcal{N}$ denote the class of nilpotent Lie algebras. For any finite-dimensional Lie algebra L, there exists a smallest ideal I of L such that $L/I \in \mathcal{N}$. This uniquely determined ideal of L is called the nilpotent residual of L and is denoted by $L^{\mathcal{N}}$. If H is a subalgebra of L, then we write $H \leqslant L$. For any subalgebra H of L, the idealizer $I_L(H)$ of H is the set of all elements x of L such that $[x, H] \subseteq H$, that is, $I_L(H) = \{x \in L\colon\ [x, h] \in H \text{ for all } h \in H\}$.

In this paper, we consider the intersection of the idealizers of the nilpotent residuals of all subalgebras of L and introduce the following notation:

Definition 1.1. Let L be a finite dimensional Lie algebra. By $S(L)$ denote the intersection of the idealisers of the nilpotent residuals of all subalgebras of L. That is

$$S(L) = \bigcap_{H \leqslant L} I_L(H^{\mathcal{N}})$$

where $H^{\mathcal{N}}$ is the nilpotent residual of H.

Obviously, $S(L)$ is an ideal of L, $S(L) = L$ if and only if the nilpotent residual of each subalgebra of L is an ideal of L. In the following, we define an ascending series of ideals of a Lie algebra L in terms of $S(L)$.

Definition 1.2. Let L be a finite dimensional Lie algebra. There exists a series of ideals

$$0 = S_0(L) \subseteq S_1(L) \subseteq S_2(L) \subseteq \ldots \subseteq S_n(L) \subseteq \ldots$$

satisfying $S_{i+1}(L)/S_i(L) = S(L/S_i(L))$ for $i = 0, 1, 2, \ldots$ and $S_n(L) = S_{n+1}(L)$ for some integer $n \geqslant 1$. Write $S_\infty(L)$ for the terminal term of the ascending series.

This is analogous to the concept of $S(G)$-subgroup as introduced by Shen, Shiand and Qian (see [12]); this concept has since been further studied by a number of authors, including Gong and Guo (see [7], [8]), Su and Wang (see [15]).

In the present paper, the basic properties of $S(L)$ and $S_\infty(L)$ are investigated (see Section 3). Let $\mathcal{F}_n$ denote the class of Lie algebras L such that $L^{\mathcal{N}}$ is nilpotent. We characterize the class $\mathcal{F}_n$ of Lie algebra in terms of $S(L)$ and $S_\infty(L)$ (see Section 4). In addition, L is called an S-Lie algebra if $L = S(L)$, that is, the nilpotent residuals of all subalgebras of L are ideals of L. We establish some basic properties of S-Lie algebras and minimal non-S-Lie algebras (see Section 5). The results and proofs of this paper have analogues in the theory of groups. The proofs are presented here for completeness.

If A and B are subalgebras of L, for which $L = A + B$ and $A \cap B = 0$, we will write $L = A \oplus B$. B_L is the core (with respect to L) of B, that is the largest ideal of L contained in B; $C_L(B) = \{x \in L\colon [x,h] = 0 \text{ for all } h \in H\}$; $Z(L)$ is the centre of L; $\varphi(L)$ is the Frattini subalgebra of L, that is the intresection of all maximal subalgebras of L; $\psi(L)$ is the largest ideal of L that is contained in $\varphi(L)$. All unexplained notation and terminology are standard and can be found in [9], [10], [13].

2. Preliminaries

The *lower central series* (see [9], page 11) of a Lie algebra L is the sequence $\{L^i\}$ of ideals of L,

$$L = L^1 \supseteq L^2 \supseteq \ldots \supseteq L^i \supseteq \ldots$$

satisfying $L^1 = L$, $L^2 = [L, L^1], \ldots, L^i = [L, L^{i-1}]$.

The algebra L is called *nilpotent* if $L^n = 0$ for some n. It is easily shown that

$$L^{\mathcal{N}} = \bigcap_{i=1}^{\infty} L^i.$$

The *upper central series* (see [10], page 419) of a Lie algebra L is the sequence $\{Z_i(L)\}$ of ideals of L

$$0 = Z_0(L) \subseteq Z_1(L) \subseteq \ldots \subseteq Z_n(L) \subseteq \ldots$$

satisfying $Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L))$. Write

$$Z_\infty(L) = \bigcup_{i=0}^{\infty} Z_i(L)$$

for the terminal term of the upper central series of L.

As L is a finite dimensional Lie algebra, there exists n such that $L^{\mathcal{N}} = L^n$ and $Z_\infty(L) = Z_n$.

Lemma 2.1. *Let L be a Lie algebra. Then*

$$L^{\mathcal{N}} = \bigcap \{I\colon\ I \text{ is an ideal of } L \text{ and } L/I \text{ is nilpotent}\}.$$

Proof. Set $K = \bigcap\{I\colon\ I$ is an ideal of L and L/I is nilpotent$\}$. Suppose I is an ideal of L and L/I is nilpotent. Then

$$L/I \supseteq (L^1 + I)/I \supseteq (L^2 + I)/I \supseteq \ldots$$

is a lower central series of L/I. So there exists n such that $L^n \subseteq I$, and thus, $L^{\mathcal{N}} \subseteq I$. Therefore $L^{\mathcal{N}} \subseteq K$.

Conversely, for every L^i we see that

$$L/L^i \supseteq L^1/L^i \supseteq L^2/L^i \supseteq \ldots \supseteq L^i/L^i$$

is a lower central series of L/L^i and hence L/L^i is nilpotent. So we have $K \subseteq L^i$. Furthermore, $K \subseteq L^{\mathcal{N}}$. The proof is completed. $\square$

Lemma 2.2. *Let L be a Lie algebra. Then*

$$Z_\infty(L) = \bigcap\{I\colon\ I \text{ is an ideal of } L \text{ and } Z(L/I) = 0\}.$$

Proof. As L is a finite dimensional Lie algebra, there exists n such that $Z_\infty(L) = Z_n(L) = Z_{n+1}(L) = \ldots$ Consequently, $Z(L/Z_\infty(L)) = Z(L/Z_n(L)) = Z_{n+1}(L)/Z_n(L) = 0$. So

$$Z_\infty(L) = Z_n(L) \supseteq \bigcap\{I\colon\ I \text{ is an ideal of } L \text{ and } Z(L/I) = 0\}.$$

In another words, if I is an ideal of L with $Z(L/I) = 0$, then $Z_\infty(L/I) = 0$.

We claim that $(Z_k(L)+I)/I \subseteq Z_k(L/I)$. Suppose $k = 1$. Since $[Z(L), L] = 0 \subseteq I$, we have $(Z(L) + I)/I \subseteq Z(L/I)$. Suppose $(Z_{k-1}(L) + I)/I \subseteq Z_{k-1}(L/I)$. Since

$$[(Z_k(L) + I)/I, L/I] = ([Z_k(L), L] + I)/I \subseteq (Z_{k-1}(L) + I)/I \subseteq Z_{k-1}(L/I),$$

we get $(Z_k(L) + I)/I \subseteq Z_k(L/I)$.

Therefore $(Z_n(L) + I)/I \subseteq Z_n(L/I) = 0$ and hence $Z_\infty(L) = Z_n(L) \subseteq I$. So $Z_\infty(L) \subseteq \bigcap\{I\colon\ I$ is an ideal of L and $Z(L/I) = 0\}$. The conclusion holds. $\square$

Definition 2.3. The *central series* of a Lie algebra L is the sequence $\{Z_i(L)\}$ of subalgebras of L,

$$L = K_1 \supseteq K_2 \supseteq \ldots \supseteq K_{s+1} = 0$$

satisfying $[K_i, L] \subseteq K_{i+1}$, $i = 1, 2, \ldots, s$.

By Definition 2.3, we see that $[K_i, L] \subseteq K_{i+1} \subseteq K_i$. Hence K_i is an ideal of L. The proof of the following fact is straightforward.

Lemma 2.4. *The following properties of the Lie algebra L are equivalent:*
(i) *L is nilpotent;*
(ii) *$L^{\mathcal{N}} = L^n = 0$ for some n;*
(iii) *$Z_\infty(L) = Z_n(L) = L$ for some n;*
(iv) *L possesses a central series.*

Lemma 2.5.
(i) *Let*

$$L = K_1 \supseteq K_2 \supseteq \ldots \supseteq K_{s+1} = 0$$

be a central series of nilpotent Lie algebra L. Then $[K_i, L^j] \subseteq K_{i+j}$ for all i, j.
(ii) *$[L^i, L^j] \subset L^{i+j}$, $[L^i, Z_j(L)] \subseteq Z_{j-i}(L)$. Clearly $Z_{j-i}(L) = 0$ whenever $j < i$. In particular, $[L^i, Z_i(L)] = 0$.*

Proof. (i) If $j = 1$, then $[K_i, L^1] = [K_i, L] \subseteq K_{i+1}$, and the conclusion holds. Let $j > 1$, suppose the conclusion holds for $l < j$. Since $L^j = [L, L^{j-1}]$, we have

$$\begin{aligned}[K_i, L^j] &= [K_i, [L, L^{j-1}]] = [[K_i, L], L^{j-1}] + [L, [K_i, L^{j-1}]] \\ &\subseteq [K_{i+1}, L^{j-1}] + [L, K_{i+j-1}] \subseteq K_{i+j}.\end{aligned}$$

(ii) This is immediate from (i). □

Lemma 2.6. *Let L be a Lie algebra. Then the following statements hold:*
(i) *If H is a subalgebra of L, then $H^{\mathcal{N}} \subseteq L^{\mathcal{N}}$.*
(ii) *If I is an ideal of L and H is a subalgebra of L with $I \subseteq H$, then $(H/I)^{\mathcal{N}} = (H^{\mathcal{N}} + I)/I$.*

Proof. (i) Let H be a subalgebra of L. Since $H/(H \cap L^{\mathcal{N}}) \cong (H + L^{\mathcal{N}})/L^{\mathcal{N}} \subseteq L/L^{\mathcal{N}}$ we see that $H/(H \cap L^{\mathcal{N}})$ is nilpotent and therefore $H^{\mathcal{N}} \subseteq H \cap L^{\mathcal{N}} \subseteq L^{\mathcal{N}}$.

(ii) Let $(H/I)^{\mathcal{N}} = R/I$. Since $(H/I)/(H/I)^{\mathcal{N}} = (H/I)/(R/I) \cong H/R$, we see that $H^{\mathcal{N}} + I \subseteq R$. Conversely, it follows from

$$H/(H^{\mathcal{N}} + I) \cong (H/H^{\mathcal{N}})/((H^{\mathcal{N}} + I)/H^{\mathcal{N}})$$

and

$$H/(H^{\mathcal{N}} + I) \cong (H/I)/((H^{\mathcal{N}} + I)/I)$$

that $R/I \subseteq (H^{\mathcal{N}} + I)/I$ and hence $(H/I)^{\mathcal{N}} = (H^{\mathcal{N}} + I)/I$. □

The following proposition shows that $C_L(L^{\mathcal{N}})$ is nilpotent.

Proposition 2.7. *Let L be a Lie algebra. Then $C_L(L^{\mathcal{N}})$ is nilpotent.*

Proof. Write $C = C_L(L^{\mathcal{N}})$. Then $C/(C \cap L^{\mathcal{N}}) \cong (C + L^{\mathcal{N}})/L^{\mathcal{N}} \subseteq L/L^{\mathcal{N}}$ and hence $C/(C \cap L^{\mathcal{N}})$ is nilpotent. Since $[C \cap L^{\mathcal{N}}, C] = 0$ and $C \cap L^{\mathcal{N}} \subseteq Z(C)$, we have $C/Z(C)$ is nilpotent. So C is nilpotent (see Proposition in [9], page 12). $\square$

The following proposition characterizes the nilpotent Lie algebra in terms of $L^{\mathcal{N}}$.

Proposition 2.8. *Let L be a Lie algebra. Then L is nilpotent if and only if the nilpotent residual $L^{\mathcal{N}}$ idealizes every subalgebra of L.*

Proof. If L is nilpotent, then $L^{\mathcal{N}} = 0$ and therefore $L^{\mathcal{N}}$ idealizes every subalgebra of L.

Conversely, suppose that $L^{\mathcal{N}}$ idealizes every subalgebra of L. Suppose M is a maximal subalgebra of L. If $L^{\mathcal{N}} \not\subseteq M$, then $L = M + L^{\mathcal{N}}$. Since $L^{\mathcal{N}} \subseteq I_L(M)$, we get $L = I_L(M)$ and hence M is an ideal of L. If $L^{\mathcal{N}} \subseteq M$, then $M/L^{\mathcal{N}}$ is a maximal subalgebra of $L/L^{\mathcal{N}}$. As $L/L^{\mathcal{N}}$ is nilpotent, we know $M/L^{\mathcal{N}}$ is an ideal of $L/L^{\mathcal{N}}$ by the Theorem of [1]. Thus, M is also an ideal of L. Again applying the Theorem of [1], L is nilpotent. The proof is completed. $\square$

3. Basic properties of $S(L)$ and $S_\infty(L)$

In this section, we prove some basic properties of the subalgebras $S(L)$ and $S_\infty(L)$.

Proposition 3.1. *Let L be a Lie algebra. Then $Z_\infty(L) \subseteq C_L(L^{\mathcal{N}}) \subseteq S(L)$.*

Proof. Since $L/L^{\mathcal{N}}$ and $Z_\infty(L)$ are nilpotent, by Lemma 2.5 (ii) we get

$$[L^{\mathcal{N}}, Z_\infty(L)] = 0.$$

Thus, $Z_\infty(L) \subseteq C_L(L^{\mathcal{N}})$. Let H be a subalgebra of L, then $H^{\mathcal{N}} \subseteq L^{\mathcal{N}}$ by Lemma 2.6 (i). For any $x \in C_L(L^{\mathcal{N}})$, x centralizes $H^{\mathcal{N}}$. So $x \in I_L(H)$ and hence $C_L(L^{\mathcal{N}}) \subseteq S(L)$. The proof is complete. $\square$

Proposition 3.2. *Let L be a Lie algebra and M a subalgebra of L. Then*

$$M \cap S(L) \subseteq S(M).$$

P r o o f. By definition, we have

$$S(L) = \bigcap_{H \leqslant L} I_L(H^{\mathcal{N}}) \subseteq \bigcap_{H \leqslant L} I_L(H^{\mathcal{N}}).$$

So

$$M \cap S(L) = M \bigcap_{H \leqslant L} I_L(H^{\mathcal{N}}) \subseteq \bigcap_{H \leqslant M} (M \cap I_L(H^{\mathcal{N}})) = \bigcap_{H \leqslant M} I_M(H^{\mathcal{N}}) = S(M).$$

The conclusion holds. □

Proposition 3.3. *Let L be a Lie algebra and I an ideal of L. Then*

$$(S(L) + I)/I \subseteq S(L/I).$$

P r o o f. Let H/I be a subalgebra of L/I. Then $(H/I)^{\mathcal{N}} = (H^{\mathcal{N}} + I)/I$ by Lemma 2.6 (ii). For any element $x \in S(L)$, by definition, $x \in I_L(H^{\mathcal{N}})$. It follows that $x + I \in I_{L/I}((H^{\mathcal{N}} + I)/I) = (H/I)^{\mathcal{N}}$. Thus $(S(G) + I)/I \subseteq I_{L/I}((H/I)^{\mathcal{N}})$ for every subalgebra H/I of L/I, so $(S(G)+I)/I \subseteq S(L/I)$. The proof is completed. □

Proposition 3.4. *Let L be a Lie algebra and I an ideal of L. If $I \subseteq S_\infty(G)$, then $S_\infty(L/I) = S_\infty(L)/I$.*

P r o o f. As $I \subseteq S_\infty(L)$, $I \subseteq S_i(L)$ for some i. Set $S^1(L)/I = S(L/I)$ and by $S^\infty(L)/I$ denote the terminal term of the ascending series of L/I. We claim that $S^1(L) \subseteq S_{i+1}(L)$. For any subalgebra $H/S_i(L)$ of $L/S_i(L)$, H/I is a subalgebra of L/I. By definition, for any element $x \in S^1(L)$, we have $x + I \in I_{L/I}((H/I)^{\mathcal{N}}) = I_{L/I}((H^{\mathcal{N}} + I)/I)$, namely $((H^{\mathcal{N}})^x + I)/I = (H^{\mathcal{N}} + I)/I$. As $I \subseteq S_i(L)$, of course, we have $((H^{\mathcal{N}})^x + S_i(L))/S_i(L) = (H^{\mathcal{N}} + S_i(L))/S_i(L)$, so $x + S_i(L) \in I_{L/S_i(L)}((H/S_i(L))^{\mathcal{N}})$. Therefore $x \in S_{i+1}(L)$. The claim holds. Now, by induction, we have $S^\infty(L) \subseteq S_\infty(L)$. Conversely, clearly $S(L) \subseteq S^1(L)$, by induction we have $S_\infty(L) \subseteq S^\infty(L)$. Consequently, $S_\infty(L/I) = S_\infty(L)/I$. The proof is completed. □

Proposition 3.5. *For any Lie algebra L, $S(L)$ is solvable or $S(L)$ is a minimal non-nilpotent Lie algebra.*

P r o o f. Write $H = S(L)$. Then H has the property: the nilpotent residual of every subalgebra of H is an ideal of H. Let M be a maximal subalgebra of H. If $M^{\mathcal{N}} > 0$, then $M^{\mathcal{N}}$ is an ideal of H. By Propositions 3.2, 3.3 and induction, $H/M^{\mathcal{N}}$ and $M^{\mathcal{N}}$ are solvable, hence H is solvable. Suppose $M^{\mathcal{N}} = 0$ for every maximal subalgebra M of L, then M is nilpotent, and therefore L is a minimal non-nilpotent Lie algebra. □

Proposition 3.6. *Let L be a Lie algebra. Then*

$$S_\infty(L) = \bigcap\{I\colon\ I \text{ is an ideal of } L \text{ and } S(L/I) = 0\}.$$

Proof. As L is a finite dimensional Lie algebra, there exists an integer n such that

$$S_\infty(L) = S_n(L) = S_{n+1}(L) = \ldots$$

By the definition of the series, we have

$$S(L/S_\infty(L)) = S(L/S_n(L)) = S_{n+1}(L)/S_n(L) = 0$$

and therefore $\bigcap\{I\colon\ I$ is an ideal of L and $S(L/I) = 0\} \subseteq S_\infty(L)$.

Conversely, suppose $S(L/I) = 0$ for an ideal I of L. Then by the definition of the series and induction, $S_n(L/I) = 0$ for any positive integer n. Proposition 3.3 implies that $S_n(L) \subseteq I$ and so $S_\infty(L) \subseteq \bigcap\{I\colon\ I$ is an ideal of L and $S(L/I) = 0\}$. This completes the proof. $\square$

Proposition 3.7. *Let L be a Lie algebra. Then $Z_\infty(L^{\mathcal{N}}) \subseteq S_\infty(L)$.*

Proof. Use induction on $\dim_{\mathbb{F}}(L)$. Since $Z(L^{\mathcal{N}}) \subseteq C_L(L^{\mathcal{N}}) \subseteq S(L)$, we get

$$Z_\infty(L^{\mathcal{N}}/Z(L^{\mathcal{N}})) = Z_\infty((L/Z(L^{\mathcal{N}}))^{\mathcal{N}}) \subseteq S_\infty(L/Z(L^{\mathcal{N}})).$$

The conclusion follows from

$$Z_\infty(L^{\mathcal{N}}/Z(L^{\mathcal{N}})) = Z_\infty(L^{\mathcal{N}})/Z(L^{\mathcal{N}}) \text{ and } S_\infty(L/Z(L^{\mathcal{N}})).$$

$\square$

4. $\mathcal{F}_n$-Lie algebra

In this section, let $\mathcal{F}_n$ denote the class of Lie algebras such that $L \in \mathcal{F}_n$ if and only if $L^{\mathcal{N}}$ is nilpotent.

Theorem 4.1. *The following properties of the Lie algebra L are equivalent:*

(i) $L \in \mathcal{F}_n$;

(ii) $L/\psi(L) \in \mathcal{F}_n$.

P r o o f. (i) $\Rightarrow$ (ii): $L \in \mathcal{F}_n$ implies $L^{\mathcal{N}}$ is nilpotent. By Lemma 2.6 (ii), $(L/\psi(L))^{\mathcal{N}} = (L^{\mathcal{N}} + \psi(L))/\psi(L)$. As $(L^{\mathcal{N}} + \psi(L))/\psi(L) \cong L^{\mathcal{N}}/(L^{\mathcal{N}} \cap \psi(L))$, we have $(L/\psi(L))^{\mathcal{N}}$ is nilpotent and hence $L/\psi(L) \in \mathcal{F}_n$.

(ii) $\Rightarrow$ (i): Since $L/\psi(L) \in \mathcal{F}_n$, we have $(L/\psi(L))^{\mathcal{N}}$ is nilpotent. Thus, $L^{\mathcal{N}}/(L^{\mathcal{N}} \cap \psi(L)) \cong (L^{\mathcal{N}} + \psi(L))/\psi(L) = (L/\psi(L))^{\mathcal{N}}$ is nilpotent. By Barnes' theorem (see [2], Theorem 5), $L^{\mathcal{N}}$ is nilpotent and hence $L \in \mathcal{F}_n$. □

Theorem 4.2. *Let L be a finite dimensional Lie algebra. Then the following statements are equivalent:*

(i) $L \in \mathcal{F}_n$;

(ii) $L/S(L) \in \mathcal{F}_n$.

P r o o f. (i) $\Rightarrow$ (ii): $L \in \mathcal{F}_n$ implies $L^{\mathcal{N}}$ is nilpotent and hence $L^{\mathcal{N}}/(L^{\mathcal{N}} \cap S(G))$ is nilpotent. By Lemma 2.6 (ii), we know $(L/S(L))^{\mathcal{N}} = (L^{\mathcal{N}} + S(G))/S(G)$. Since $(L^{\mathcal{N}} + S(G))/S(G) \cong L^{\mathcal{N}}/(L^{\mathcal{N}} \cap S(G))$, we have $(L/S(L))^{\mathcal{N}}$ is nilpotent and hence $L/S(L) \in \mathcal{F}_n$.

(ii) $\Rightarrow$ (i): We use induction on the dimension of L. If $S(L) = 0$, the result is trivial. Suppose that $S(L) > 0$, so that we can choose a minimal ideal A of L such that $A \subseteq S(L)$.

First suppose $A \subseteq \psi(L)$, the Frattini ideal of L. By Proposition 3.3, $S(L)/A \subseteq S(L/A)$. It follows that $(L/A)/S(L/A) \in \mathcal{F}_n$ since $L/S(L) \in \mathcal{F}_n$. Thus, L/A satisfies the condition of the theorem. By induction, $(L/A)^{\mathcal{N}} = (L^{\mathcal{N}} + A)/A$ is nilpotent. As $A \subseteq \psi(L)$, by Barnes' theorem, $L^{\mathcal{N}} + A$ is nilpotent and hence $L^{\mathcal{N}}$ is also nilpotent, which gives $L \in \mathcal{F}_n$ as desired.

Next, let $A \not\subseteq \psi(L)$. Then there is a maximal subalgebra M of L such that $L = A + M$ with $A \cap M = 0$. By Proposition 3.2, $M \cap S(L) \subseteq S(M)$. Thus, by the hypothesis that $L/S(L) \in \mathcal{F}_n$, and as $L/S(L) = (A + M)/S(L) \cong M/(M \cap S(L))$, we have $M/S(M) \in \mathcal{F}_n$. Hence M satisfies the condition. By induction, $M^{\mathcal{N}}$ is nilpotent. Now, as $A \subseteq S(L)$ and $S(L)$ idealizes the nilpotent residuals of all subalgebras of L, thus $M^{\mathcal{N}}$ is an ideal of L and it follows that $A + M^{\mathcal{N}} = A \oplus M^{\mathcal{N}}$. Since $M^{\mathcal{N}}$ is nilpotent, we conclude that $L^{\mathcal{N}}$ is nilpotent, as desired. □

Theorem 4.3. *Let L be a finite dimensional Lie algebra. Then the following statements are equivalent:*

(i) $L \in \mathcal{F}_n$;

(ii) $L/S_\infty(L) \in \mathcal{F}_n$;

(iii) $L = S_\infty(L)$;

(iv) $S(L/I) > 0$ *for any proper ideal I of L.*

P r o o f. (i) $\Rightarrow$ (ii): The proof is similar to that of Theorem 4.2, so we omit it.

(ii) $\Rightarrow$ (iii): We first observe the following simple fact: If $X > 0$ is an $\mathcal{F}_n$-Lie algebra, then $S(X) > 0$. In fact, $X^{\mathcal{N}}$ is nilpotent, so $C_X(X^{\mathcal{N}}) > 0$. But since $C_X(X^{\mathcal{N}}) \subseteq S(X)$, we have $S(X) > 0$. Using this fact and noting that $S(L/S_\infty(L)) = 0$, we deduce $L = S_\infty(L)$.

(iii) $\Rightarrow$ (i): As $S_\infty(L/S(L)) = S_\infty(L)/S(L)$, by induction, $L/S(L) \in \mathcal{F}_n$. It follows that $L \in \mathcal{F}_n$ by Proposition 3.2.

(i) $\Rightarrow$ (iv): See the argument of (ii).

(iv) $\Rightarrow$ (iii): By definition, $S(L/S_i(L)) = S_{i+1}(L)/S_i(L)$. As $S(L/S_i(L)) > 0$ by hypothesis, we have $S_{i+1}(L) > S_i(L)$ for $i = 0, 1, 2, \ldots$ So the terminal term $S_\infty(L)$ of the ascending series must be L. $\square$

5. Minimal non-$\mathcal{S}$-Lie algebra

By definition of $S(L)$, we know that $0 \subseteq S(L) \subseteq L$. If $S(L) = 0$, then $Z_\infty(L) = 0$ by Proposition 3.1. In other words, $S(L) = L$ if and only if the nilpotent residuals of all subalgebras of L are ideals of L.

Definition 5.1. A Lie algebra L is called an S-Lie algebra if $L = S(L)$, that is, the nilpotent residuals of all subalgebras of L are ideals of L.

Theorem 5.2.

(i) *The subalgebras of an S-Lie algebra are S-Lie algebras.*

(ii) *The quotient algebras of an S-Lie algebra are S-Lie algebras.*

Proof. (i) Suppose L is an S-Lie algebra and H is a subalgebra of L. We choose a subalgebra K of H, then $K^{\mathcal{N}}$ is an ideal of L and hence $K^{\mathcal{N}}$ is also an ideal of H. Therefore $S(H) = H$, that is, H is an S-Lie algebra.

(ii) Suppose L is an S-Lie algebra and I is an ideal of L. Let H/I be a subgroup of L/I, then H is a subalgebra of L and hence $H^{\mathcal{N}}$ is an ideal of L. By Lemma 2.6 (ii), $(H/I)^{\mathcal{N}} = (H^{\mathcal{N}} + I)/I$. Thus, $(H/I)^{\mathcal{N}}$ is an ideal of L/I. So we have $S(L/I) = L/I$, and L/I is an S-Lie algebra. $\square$

Theorem 5.3. *Let L be a non-nilpotent S-Lie algebra. If there is a maximal subalgebra M of L with $M_G = 0$, then $L = L^{\mathcal{N}} + M$, where $L^{\mathcal{N}}$ is a minimal ideal of L, M is nilpotent and $L^{\mathcal{N}} \cap M = 0$.*

Proof. Since M is a maximal subalgebra of L and $M_L = 0$, $L^{\mathcal{N}} \not\subseteq M$ and hence $L = L^{\mathcal{N}} + M$. Because $C_L(C_L(L^{\mathcal{N}}) \cap M) \supseteq L^N$ and $I_L(C_L(L^{\mathcal{N}}) \cap M) \supseteq M$, we have $L = I_L(C_L(L^{\mathcal{N}}) \cap M)$. It follows that $C_L(L^{\mathcal{N}}) \cap M = 0$. For any nontrivial ideal I of L contained in $C_L(L^{\mathcal{N}})$, we get $L = I + M$ and $C_L(L^{\mathcal{N}}) = I$, which implies $C_L(L^{\mathcal{N}})$ is a minimal ideal of L. $\square$

Definition 5.4. A Lie algebra G is called a minimal non-S-Lie algebra if L is not an S-Lie algebra, but every proper subalgebra of L is an S-Lie algebra.

Theorem 5.5. *Let L be a minimal non-S-Lie algebra and $\psi(L) \neq 0$. Then either $L/\psi(L)$ is a minimal non-S-Lie algebra or it is an S-Lie algebra.*

P r o o f. Let H be a maximal subalgebra of L and K a subalgebra of H. Since L is a minimal non-S-Lie algebra, we know H is a S-Lie algebra, then $K^{\mathcal{N}}$ is an ideal of H. We consider $L/\psi(L)$ and its maximal subalgebra $H/\psi(L)$. It is clear that $((K+\psi(L))/\psi(L))^{\mathcal{N}}$ is an ideal of $H/\psi(L)$, so $H/\psi(L)$ is an S-Lie algebra, and every maximal subalgebra of $L/\psi(L)$ is an S-Lie algebra. Then $L/\psi(L)$ is a minimal non-S-Lie algebra or an S-Lie algebra. $\square$

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