Hindawi
Abstract and Applied Analysis
Volume 2018, Article ID 8732457, 7 pages
https://doi.org/10.1155/2018/8732457

Research Article

A Version of Uncertainty Principle for Quaternion Linear Canonical Transform

Mawardi Bahri,[1] **Resnawati**,[2] **and Selvy Musdalifah**[2]

[1] *Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia*
[2] *Department of Mathematics, Tadulako University, Palu 94148, Indonesia*

Correspondence should be addressed to Mawardi Bahri; mawardibahri@gmail.com

Received 2 October 2017; Revised 18 March 2018; Accepted 2 April 2018; Published 7 May 2018

Academic Editor: Feyzi Başar

In recent years, the two-dimensional (2D) quaternion Fourier and quaternion linear canonical transforms have been the focus of many research papers. In the present paper, based on the relationship between the quaternion Fourier transform (QFT) and the quaternion linear canonical transform (QLCT), we derive a version of the uncertainty principle associated with the QLCT. We also discuss the generalization of the Hausdorff-Young inequality in the QLCT domain.

1. Introduction

The quaternion Fourier transform (QFT) is an extension of the classical two-dimensional Fourier transform (FT) [1–4] in the framework of quaternion algebra. It plays an important role in the representation of the two-dimensional quaternion signals. A number of useful properties of the extended transform are generalizations of the corresponding properties of the FT with some modifications (see, e.g., [5–15]). The QFT has found many applications in color image processing and signal analysis; we refer the reader to [16–19] and the references mentioned therein. An extension of the QFT in the framework of the classical linear canonical transform (LCT) (see [20–22]), known as the quaternion linear canonical transform, has received much attention in recent years. Due to the several definitions of the QFT, there are basically three ways of obtaining the quaternion linear canonical transform (QLCT): the right-sided quaternion linear canonical transform, the left-sided quaternion linear canonical transform, and the two-sided quaternion linear canonical transform. The right-sided quaternion linear canonical transform is obtained by substituting the Fourier kernel with the right-sided QFT kernel in the LCT definition, and so on. Recent works related to some important properties of the QLCT such as Parseval's theorem, reconstruction formula, and component-wise uncertainty principles were also published [18, 23–25]. It was found that the properties of the QLCT are extensions of the corresponding version of the QFT with some modifications.

On the other hand, the uncertainty principle plays one important role in signal processing. It describes a function and its FT, which cannot both be simultaneously sharply localized. One example of this is the Heisenberg uncertainty principle concerning position and momentum wave functions in quantum physics. In signal processing, an uncertainty principle states that the product of the variances of the signal in the *time* and *frequency* domains has a lower bound. Up till now, several attempts have been made to extend the uncertainty principles associated with the QFT and QLCT domains. The component-wise and directional uncertainty principles associated with the QFT were proposed in [11]. In [26, 27], the authors established a component-wise uncertainty principle for the QLCT and proved that the equality is achieved for optimal quaternion Gaussian function. Recently, the authors [23] proposed the logarithmic uncertainty principle associated with the QLCT which is the generalization of the logarithmic uncertainty principle for the QFT.

Therefore, the main objective of the present paper is to establish full uncertainty principle for the two-sided QLCT, which is a new general form of component-wise uncertainty principle for the two-sided QLCT. This uncertainty principle is derived using the connection between the QFT and QLCT.

We also obtain full uncertainty principle for the right-sided QLCT using a relation between the right-sided QLCT and two-sided QLCT. We also derive the Hausdorff-Young inequality associated with the two-sided QLCT.

This paper is organized as follows. In Section 2, we briefly review the basic knowledge of quaternion and its split used in the next section. In Section 3, we introduce the definition of the QFT and the QLCT. The relationship between the QFT and the QLCT is also discussed in this section. In Section 4, we derive a new version uncertainty principle associated with the QLCT, which shows that the spread of a quaternion-valued function and its QLCT are inversely proportional. Section 5 concludes this paper.

2. Preliminaries

In this section, let us briefly recall some basic definitions and properties of the quaternions (for more details, see [28]).

2.1. Quaternions. The quaternions, a generalization of complex numbers, are members of a noncommutative division algebra. The set of quaternions is denoted by $\mathbb{H}$. Every element of $\mathbb{H}$ can be written in the following form:

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{1}$$

with the units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, which obey the following:

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \tag{2}$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is simply called the scalar part of q denoted by $\mathrm{Sc}(q)$ and $\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is called the vector part of q denoted by $\mathrm{Vec}(q)$.

Let $p, q \in \mathbb{H}$ and $\mathbf{p}$, $\mathbf{q}$ be their vector parts, respectively. Equation (2) yields the quaternionic multiplication qp as

$$qp = q_0 p_0 + \mathbf{q} \cdot \mathbf{p} + q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}, \tag{3}$$

where

$$\begin{aligned} \mathbf{q} \cdot \mathbf{p} &= -(q_1 p_1 + q_2 p_2 + q_3 p_3), \\ \mathbf{q} \times \mathbf{p} &= \mathbf{i}(q_2 p_3 - q_3 p_2) + \mathbf{j}(q_3 p_1 - q_1 p_3) \\ &\quad + \mathbf{k}(q_1 p_2 - q_2 p_1). \end{aligned} \tag{4}$$

Analogously to the complex case, a quaternionic conjugation $\overline{q}$ is given by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \tag{5}$$

which leads to the anti-involution; that is,

$$\overline{qp} = \overline{p}\,\overline{q}. \tag{6}$$

With the help of (5), we get the norm or modulus of $q \in \mathbb{H}$ as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{7}$$

One can easily verify that

$$\begin{gathered} |qp| = |q|\,|p|, \\ |q + p| \le |q| + |p|, \\ \forall p, q \in \mathbb{H}. \end{gathered} \tag{8}$$

Using conjugate (5) and the modulus of q, we can define the inverse of $q \in \mathbb{H}\backslash\{0\}$ as

$$q^{-1} = \frac{\overline{q}}{|q|^2}, \tag{9}$$

which shows that $\mathbb{H}$ is a normed division algebra.

In quaternionic notation, we may define an inner product for quaternion-valued functions $f, g : \mathbb{R}^2 \to \mathbb{H}$ as follows:

$$(f, g) = \int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g(\mathbf{x})}\,d\mathbf{x}, \quad d\mathbf{x} = dx_1 dx_2 \tag{10}$$

with symmetric real scalar part

$$\begin{aligned} \langle f, g \rangle = (f, g)_0 &= \frac{1}{2}\left[(f, g) + (g, f)\right] \\ &= \int_{\mathbb{R}^2} \mathrm{Sc}\left(f(\mathbf{x})\,\overline{g}(\mathbf{x})\right) d\mathbf{x}. \end{aligned} \tag{11}$$

In particular, for $f = g$, we obtain the $L^2(\mathbb{R}^2; \mathbb{H})$-norm

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}. \tag{12}$$

A quaternion module $L^2(\mathbb{R}^2; \mathbb{H})$ is then defined as

$$L^2(\mathbb{R}^2; \mathbb{H}) = \left\{f \mid f : \mathbb{R}^2 \longrightarrow \mathbb{H},\ \|f\| < \infty\right\}. \tag{13}$$

2.2. Split Quaternion and Properties. In this section, we study some of the basic formulas of split quaternion (see [9]), which will be used to prove the fundamental results in the sequel.

Definition 1. For two quaternion square roots μ, ν such that $\mu^2 = \nu^2 = -1$, one may express a quaternion q as

$$q = q_- + q_+, \quad q_\pm = \frac{1}{2}(q \pm \mu q \nu). \tag{14}$$

For the special case of $\mu = \nu$, any quaternion q may be split up into commuting and anticommuting parts with respect to μ; that is,

$$\begin{aligned} \mu q_- &= q_- \mu \\ \mu q_+ &= -q_+ \mu. \end{aligned} \tag{15}$$

It easily seems that the commuting and anticommuting parts satisfy the interesting properties:

$$\begin{gathered} \mu^2 = \mu_-^2 + \mu_+^2 = -1 \\ \mu_+ \mu_- + \mu_- \mu_+ = 0. \end{gathered} \tag{16}$$

We learn from the above equation that

$$q_{\pm}e^{\mu\theta} = e^{\mp\mu\theta}q_{\pm}, \tag{17}$$

where

$$\cos\theta = \frac{q_0}{|q|}, \qquad \sin\theta = \frac{\sqrt{q_1^2+q_2^2+q_3^2}}{|q|}. \tag{18}$$

In particular, taking $\mu = \mathbf{i}$ and $\nu = \mathbf{j}$, (14) becomes

$$q = q_+ + q_-, \quad q_{\pm} = \frac{1}{2}\left(q \pm \mathbf{i}q\mathbf{j}\right). \tag{19}$$

The above gives

$$\begin{aligned} q_{\pm} &= \{(q_0 \pm q_3) + \mathbf{i}(q_1 \mp q_2)\}\frac{1\pm\mathbf{k}}{2} \\ &= \frac{1\pm\mathbf{k}}{2}\{(q_0 \pm q_3) + \mathbf{j}(q_2 \mp q_1)\}. \end{aligned} \tag{20}$$

This leads to the following modulus identity:

$$|q|^2 = |q_-|^2 + |q_+|^2. \tag{21}$$

Furthermore, one can obtain

$$\mathrm{Sc}\left(p_+\overline{q}_-\right) = 0. \tag{22}$$

3. Relationship between Quaternion Fourier Transform (QFT) and Quaternion Linear Canonical Transform (QLCT)

In this section, we introduce the QFT and its relationship to the QLCT. We begin by introducing different types of the QFT.

Definition 2 (two-sided, right-sided, and left-sided QFTs). The two-sided, right-sided, and left-sided quaternion Fourier transforms (QFTs) of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ are given by, respectively,

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x})\, e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \tag{23}$$

$$\mathcal{F}_q^r\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \tag{24}$$

$$\mathcal{F}_q^l\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} f(\mathbf{x})\, d\mathbf{x}, \tag{25}$$

where $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$, and the quaternion exponential product $e^{-\mathbf{i}\omega_1 x_1}e^{-\mathbf{j}\omega_2 x_2}$ is the quaternion Fourier kernel. From (24), we get the partial right-sided QFT:

$$\begin{aligned} \mathcal{F}_q^{r,\mathbf{i}}\{f\}(\omega_1, x_2) &= \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-\mathbf{i}\omega_1 x_1} dx_1 \\ \mathcal{F}_q^{r,\mathbf{j}}\{f\}(x_1, \omega_2) &= \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-\mathbf{j}\omega_2 x_2} dx_2. \end{aligned} \tag{26}$$

It is not difficult to check that the relationship between the two-sided QFT and the right-sided QFT takes the form

$$\begin{aligned} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \mathcal{F}_q^r\{f_0 + \mathbf{i}f_1\}(\boldsymbol{\omega}) \\ &\quad + \mathcal{F}_q^r\{\mathbf{j}f_2 + \mathbf{k}f_3\}(-\omega_1, \omega_2). \end{aligned} \tag{27}$$

Definition 3 (left-sided, right-sided, and two-sided QLCTs). Suppose that $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ are real matrix parameters satisfying $\det(A_1) = \det(A_2) = 1$. The left-sided, right-sided, and two-sided QLCTs of a quaternion signal $f \in L^1(\mathbb{R}^2;\mathbb{H})$ are defined by, respectively,

$$\begin{aligned} L_{A_1,A_2}^{(l),\mathbb{H}}\{f\}(\boldsymbol{\omega}) \\ = \int_{\mathbb{R}^2} K_{A_1}(x_1,\omega_1)\, K_{A_2}(x_2,\omega_2)\, f(\mathbf{x})\, d\mathbf{x} \end{aligned} \tag{28}$$

$$\begin{aligned} L_{A_1,A_2}^{(r),\mathbb{H}}\{f\}(\boldsymbol{\omega}) \\ = \int_{\mathbb{R}^2} f(\mathbf{x})\, K_{A_1}(x_1,\omega_1)\, K_{A_2}(x_2,\omega_2)\, d\mathbf{x} \end{aligned} \tag{29}$$

$$\begin{aligned} L_{A_1,A_2}^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) \\ = \int_{\mathbb{R}^2} K_{A_1}(x_1,\omega_1)\, f(\mathbf{x})\, K_{A_2}(x_2,\omega_2)\, d\mathbf{x}, \end{aligned} \tag{30}$$

where the kernel functions of the QLCT above are given by

$$\begin{aligned} &K_{A_1}(x_1,\omega_1) \\ &= \begin{cases} \dfrac{1}{\sqrt{2\pi b_1}} e^{(\mathbf{i}/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega_1 + (d_1/b_1)\omega_1^2 - \pi/2)}, & \text{for } b_1 \neq 0 \\ \sqrt{d_1}\, e^{\mathbf{i}(c_1 d_1/2)\omega_1^2}, & \text{for } b_1 = 0, \end{cases} \\ &K_{A_2}(x_2,\omega_2) \\ &= \begin{cases} \dfrac{1}{\sqrt{2\pi b_2}} e^{(\mathbf{j}/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega_2 + (d_2/b_2)\omega_2^2 - \pi/2)}, & \text{for } b_2 \neq 0 \\ \sqrt{d_2}\, e^{\mathbf{j}(c_2 d_2/2)\omega_2^2}, & \text{for } b_2 = 0. \end{cases} \end{aligned} \tag{31}$$

From the definition of the QLCT, we can easily see that when $b_1 b_2 = 0$ and $b_1 = b_2 = 0$, the QLCT of a signal is essentially a quaternion chirp multiplication. Therefore, in this work, we always assume $b_1 b_2 \neq 0$. As a special case, when $A_1 = A_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$, for $i = 1, 2$, the QLCT definition (30) will lead to the QFT definition; that is,

$$\begin{aligned} &L_{A_1,A_2}^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi}} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x})\, e^{-\mathbf{j}\omega_2 x_2} \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi}} d\mathbf{x} \\ &= \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi}}. \end{aligned} \tag{32}$$

The following lemma describes the general relationship between the two-sided QFT and the two-sided QLCT of 2D quaternion-valued signals.

Lemma 4. *The QLCT of a signal f with matrix parameters $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ can be seen as the QFT of the signal f in the following form:*

$$\begin{aligned} L^{\mathbf{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}} \\ &\cdot e^{(\mathbf{i}d_1/2b_1)\omega_1^2}\mathcal{F}_q\left\{e^{\mathbf{i}(a_1/2b_1)x_1^2}f(\mathbf{x})\,e^{\mathbf{j}(a_2/2b_2)x_2^2}\right\} \\ &\cdot \left(\frac{\omega_1}{b_1},\frac{\omega_2}{b_2}\right)\frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi b_2}}e^{(\mathbf{j}d_2/2b_2)\omega_2^2}. \end{aligned} \tag{33}$$

We introduce

$$h(\mathbf{x}) = e^{\mathbf{i}(a_1/2b_1)x_1^2}\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}}f(\mathbf{x})\frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi b_2}}e^{\mathbf{j}(a_2/2b_2)x_2^2}. \tag{34}$$

This implies that (33) can be rewritten in the form

$$\begin{aligned} &\mathcal{F}_q\{h\}\left(\frac{\omega_1}{b_1},\frac{\omega_2}{b_2}\right) \\ &\quad = e^{-(\mathbf{i}d_1/2b_1)\omega_1^2}L^{\mathbf{H}}_{A_1,A_2}\{h\}(\boldsymbol{\omega})\,e^{-(\mathbf{j}d_2/2b_2)\omega_2^2}. \end{aligned} \tag{35}$$

Further, we have the following lemma which describes a relation between the right-sided QLCT and the two-sided QLCT of 2D quaternion-valued signals.

Lemma 5. *For $f \in L^1(\mathbb{R}^2;\mathbb{H})$, one has*

$$\begin{aligned} L^{(r),\mathbf{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= L^{\mathbf{H}}_{A_1,A_2}\{f_0 + \mathbf{i}f_1\}(\boldsymbol{\omega}) \\ &\quad + L^{\mathbf{H}}_{A_1^*,A_2^*}\{\mathbf{j}f_2 + \mathbf{k}f_3\}(-\omega_1,\omega_2), \end{aligned} \tag{36}$$

$$\begin{aligned} L^{\mathbf{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= L^{(r),\mathbf{H}}_{A_1,A_2}\{f_0 + \mathbf{i}f_1\}(\boldsymbol{\omega}) \\ &\quad + L^{(r),\mathbf{H}}_{A_1^*,A_2^*}\{\mathbf{j}f_2 + \mathbf{k}f_3\}(-\omega_1,\omega_2), \end{aligned} \tag{37}$$

where the matrix parameters $A_1^* = (a_1, -b_1, c_1, d_1)$ and $A_2^* = (a_2, -b_2, c_2, d_2)$.

The following lemma allows us to represent the right-sided QLCT to the single right-sided QFT of 2D quaternion-valued signals.

Lemma 6. *For $f \in L^1(\mathbb{R}^2;\mathbb{H})$, one has*

$$\begin{aligned} L^{(r),\mathbf{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \mathcal{F}^{r,\mathbf{j}}_q\left\{\mathcal{F}^{r,\mathbf{i}}_q\{g(\mathbf{x})\}\left(\frac{\omega_1}{b_1},x_2\right)\right. \\ &\quad \left.\cdot e^{\mathbf{i}(d_1/2b_1)\omega_1^2}\frac{e^{\mathbf{j}((a_2/2b_2)x_2^2-\pi/4)}}{\sqrt{2\pi b_2}}\right\}\left(\frac{\omega_1}{b_1},\frac{\omega_2}{b_2}\right). \end{aligned} \tag{38}$$

Proof. Proofs of Lemmas 4, 5, and 6 are straightforward and are therefore omitted for brevity. □

4. A Version of Uncertainty Principle Associated with QLCT

The classical uncertainty principle of harmonic analysis describes that a nontrivial function and its Fourier transform cannot be sharply localized simultaneously. In quantum mechanics, the uncertainty principle asserts that one cannot at the same time be certain of the position and of the velocity of an electron (or any particle). Let us now establish a version of the uncertainty principle associated with the QLCT. However, before proceeding the statement of this main result, we need to introduce a modified uncertainty principle for the QFT as follows (see [29] for more details).

Theorem 7 (the two-sided QFT uncertainty principle). *Let $\mathcal{S}(\mathbb{R}^2;\mathbb{H})$ be the quaternion Schwartz space. If the quaternion-valued function $f \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$, then the following inequality holds:*

$$\begin{aligned} &\int_{\mathbb{R}^2} x_1^2|f(\mathbf{x})|^2x_2^2d\mathbf{x}\int_{\mathbb{R}^2}\omega_1^2\left|\mathcal{F}_q\{f\}(\boldsymbol{\omega})\right|^2\omega_2^2d\boldsymbol{\omega} \\ &\quad \geq (2\pi)^2 \\ &\quad \cdot\left|\frac{1}{2}\int_{\mathbb{R}^2}|f(\mathbf{x})|^2d\mathbf{x} - \int_{\mathbb{R}^2}\left|x_1x_2\overline{\frac{\partial f(\mathbf{x})}{\partial x_2}}\frac{\partial f(\mathbf{x})}{\partial x_1}\right|d\mathbf{x}\right|^2. \end{aligned} \tag{39}$$

It is shown that the quaternionic Gabor filters minimize the above uncertainty. Analogously, we get the uncertainty principle associated with the right-sided QFT in the following lemma.

Lemma 8 (the right-sided QFT uncertainty principle). *Under the hypotheses of Theorem 7, one has the following inequality:*

$$\begin{aligned} &\int_{\mathbb{R}^2}x_1^2|f(\mathbf{x})|^2x_2^2d\mathbf{x}\int_{\mathbb{R}^2}\omega_1^2\left|\mathcal{F}^r_q\{f_0+\mathbf{i}f_1\}(\boldsymbol{\omega})\right. \\ &\quad \left.+\mathcal{F}^r_q\{\mathbf{j}f_2+\mathbf{k}f_3\}(-\omega_1,\omega_2)\right|^2\omega_2^2d\boldsymbol{\omega} \geq (2\pi)^2\left|\frac{1}{2}\right. \\ &\quad \left.\cdot\int_{\mathbb{R}^2}|f(\mathbf{x})|^2d\mathbf{x} - \int_{\mathbb{R}^2}\left|x_1x_2\overline{\frac{\partial f(\mathbf{x})}{\partial x_2}}\frac{\partial f(\mathbf{x})}{\partial x_1}\right|d\mathbf{x}\right|^2. \end{aligned} \tag{40}$$

Theorem 7 has been recently generalized in the context of the QLCT by the authors of [30]. Our generalization is given by the following theorem.

Theorem 9 (the two-sided QLCT uncertainty principle). *Under the assumptions of Theorem 7, one has*

$$\begin{aligned} &\int_{\mathbb{R}^2}x_1^2|f(\mathbf{x})|^2x_2^2d\mathbf{x}\int_{\mathbb{R}^2}\omega_1^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2\omega_2^2d\boldsymbol{\omega} \\ &\quad \geq |b_1b_2|^2\left|\frac{1}{2}\int_{\mathbb{R}^2}|f(\mathbf{x})|^2d\mathbf{x}\right. \\ &\quad \left.-\int_{\mathbb{R}^2}\left|x_1x_2\overline{\frac{\partial\tilde{f}(\mathbf{x})}{\partial x_2}}\frac{\partial\tilde{f}(\mathbf{x})}{\partial x_1}\right|d\mathbf{x}\right|^2, \end{aligned} \tag{41}$$

where $\tilde{f}(\mathbf{x})$ is defined by

$$\tilde{f}(\mathbf{x}) = e^{\mathbf{i}(a_1/2b_1)x_1^2}f(\mathbf{x})\,e^{\mathbf{j}(a_2/2b_2)x_2^2}. \tag{42}$$

It is worth noting here that if $\int_{\mathbb{R}^2} |x_1 x_2 \overline{\partial \tilde{f}(\mathbf{x})/\partial x_2} (\partial \tilde{f}(\mathbf{x})/\partial x_1) | d\mathbf{x} = 0$, *(41) can be reduced to*

$$\int_{\mathbb{R}^2} x_1^2 |f(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left| L_{A_1,A_2}^{\mathbb{H}} \{f\}(\boldsymbol{\omega}) \right|^2 \omega_2^2 d\boldsymbol{\omega} \geq \frac{|b_1 b_2|^2}{4} \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} \right)^2. \tag{43}$$

Proof. By replacing f by h defined by (34) on both sides of (39), we immediately get

$$\int_{\mathbb{R}^2} x_1^2 |h(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left| \mathcal{F}_q \{h\}(\boldsymbol{\omega}) \right|^2 \omega_2^2 d\boldsymbol{\omega} \geq (2\pi)^2 \cdot \left| \frac{1}{2} \int_{\mathbb{R}^2} |h(\mathbf{x})|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial h(\mathbf{x})}{\partial x_2}} \frac{\partial h(\mathbf{x})}{\partial x_1} \right| d\mathbf{x} \right|^2. \tag{44}$$

Let $\boldsymbol{\omega} = \boldsymbol{\omega}/\mathbf{b}$; then, (44) becomes

$$\int_{\mathbb{R}^2} x_1^2 |h(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \frac{\omega_1^2}{b_1^2} \left| \mathcal{F}_q \{h\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right) \right|^2 \frac{\omega_2^2}{b_2^2} d\frac{\boldsymbol{\omega}}{\mathbf{b}} \geq (2\pi)^2 \cdot \left| \frac{1}{2} \int_{\mathbb{R}^2} |h(\mathbf{x})|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial h(\mathbf{x})}{\partial x_2}} \frac{\partial h(\mathbf{x})}{\partial x_1} \right| d\mathbf{x} \right|^2. \tag{45}$$

In view of (34), we obtain

$$\int_{\mathbb{R}^2} \frac{x_1^2}{(2\pi)^2 |b_1 b_2|^2} |f(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \frac{\omega_1^2}{b_1^2} \left| \mathcal{F}_q \{h\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right) \right|^2 \frac{\omega_2^2}{b_2^2} d\boldsymbol{\omega} \geq \frac{1}{\left|(2\pi)^2 b_1 b_2\right|^2} \left| \frac{1}{2} \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}\right)}{\partial x_2}} \frac{\partial \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}\right)}{\partial x_1} \right| d\mathbf{x} \right|^2. \tag{46}$$

This leads to

$$\int_{\mathbb{R}^2} \frac{x_1^2}{|b_1 b_2|^4} |f(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left| \mathcal{F}_q \{h\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right) \right|^2 \omega_2^2 d\boldsymbol{\omega} \geq \frac{1}{|b_1 b_2|^2} \left| \frac{1}{2} \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}\right)}{\partial x_2}} \frac{\partial \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}\right)}{\partial x_1} \right| d\mathbf{x} \right|^2. \tag{47}$$

Inserting (35) into (47), we easily obtain

$$\int_{\mathbb{R}^2} \frac{x_1^2}{|b_1 b_2|^4} |f(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left| e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L_{A_1,A_2}^{\mathbb{H}} \{f\}(\boldsymbol{\omega}) e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} \right|^2 \omega_2^2 d\boldsymbol{\omega} \geq \frac{1}{|b_1 b_2|^2} \left| \frac{1}{2} \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}\right)}{\partial x_2}} \frac{\partial \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}\right)}{\partial x_1} \right| d\mathbf{x} \right|^2. \tag{48}$$

The above identity can be further simplified to

$$\int_{\mathbb{R}^2} x_1^2 |f(\mathbf{x})|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left| L_{A_1,A_2}^{\mathbb{H}} \{f\}(\boldsymbol{\omega}) \right|^2 \omega_2^2 d\boldsymbol{\omega} \geq |b_1 b_2|^2 \left| \frac{1}{2} \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial \tilde{f}(\mathbf{x})}{\partial x_2}} \frac{\partial \tilde{f}(\mathbf{x})}{\partial x_1} \right| d\mathbf{x} \right|^2, \tag{49}$$

where $\tilde{f}(\mathbf{x})$ is defined in (42). Therefore, the proof is complete. □

The above theorem is also valid for the right-sided QLCT as described in the following lemma.

Lemma 10 (the right-sided QLCT uncertainty principle). *Under the assumptions of Theorem 7, one has*

$$\int_{\mathbb{R}^2} x_1^2 \left|f(\mathbf{x})\right|^2 x_2^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left| L_{A_1,A_2}^{(r),\mathbf{H}} \{f_0 + \mathbf{i} f_1\}(\boldsymbol{\omega}) + L_{A_1^*,A_2^*}^{(r),\mathbf{H}} \{\mathbf{j} f_2 + \mathbf{k} f_3\}(-\omega_1,\omega_2)\right|^2 \omega_2^2 d\boldsymbol{\omega} \geq \left|b_1 b_2\right|^2 \left| \frac{1}{2} \cdot \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x} - \int_{\mathbb{R}^2} \left| x_1 x_2 \overline{\frac{\partial \widetilde{f}(\mathbf{x})}{\partial x_2}} \frac{\partial \widetilde{f}(\mathbf{x})}{\partial x_1} \right| d\mathbf{x} \right|^2. \quad (50)$$

Proof. The proof follows directly from (37). □

Now, we define a module of $\mathscr{F}_q\{f\}(\boldsymbol{\omega})$ as

$$\left|\mathscr{F}_q\{f\}(\boldsymbol{\omega})\right|_q = \left(\left|\mathscr{F}_q\{f_0\}(\boldsymbol{\omega})\right|^2 + \left|\mathscr{F}_q\{f_1\}(\boldsymbol{\omega})\right|^2 + \left|\mathscr{F}_q\{f_2\}(\boldsymbol{\omega})\right|^2 + \left|\mathscr{F}_q\{f_3\}(\boldsymbol{\omega})\right|^2\right)^{1/2}, \quad (51)$$

where

$$\mathscr{F}_q\{f_i\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f_i(\mathbf{x}) e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x}, \quad i = 0, 1, 2, 3. \quad (52)$$

Furthermore, we obtain the $L^p(\mathbb{R}^2;\mathbb{H})$-norm

$$\left\|\mathscr{F}_q\{f\}\right\|_{q,p} = \left(\int_{\mathbb{R}^2} \left|\mathscr{F}_q\{f\}(\boldsymbol{\omega})\right|_q^p d\mathbf{x}\right)^{1/p}. \quad (53)$$

It should be noticed that if $\mathscr{F}_q\{f_i\}$, $i = 0, 2, 3$, is real-valued function, then (53) reduces to

$$\left\|\mathscr{F}_q\{f\}\right\|_{q,p} = \left\|\mathscr{F}_q\{f\}\right\|_p, \quad (54)$$

where

$$\left\|\mathscr{F}_q\{f\}\right\|_p = \left(\int_{\mathbb{R}^2} \left|\mathscr{F}_q\{f\}(\boldsymbol{\omega})\right|^p d\mathbf{x}\right)^{1/p}. \quad (55)$$

By Riesz's interpolation theorem, one can get the Hausdorff-Young inequality related to the QFT (see [10]):

$$\left\|\mathscr{F}_q\{f\}\right\|_{q,p'} \leq \|f\|_p \quad (56)$$

holds for $1 \leq p \leq 2$ with $1/p + 1/p'$. This gives the following important theorem.

Theorem 11 (Hausdorff-Young inequality). *If $1 \leq p \leq 2$ and letting p' be such that $1/p + 1/p' = 1$, then, for all $f \in L^p(\mathbb{R}^2;\mathbb{H})$, it holds that*

$$\left\|L_{A_1,A_2}^{\mathbb{H}}\{f\}\right\|_{q,p'} \leq \frac{\left|b_1 b_2\right|^{-1/2+1/p'}}{2\pi} \|f\|_p, \quad (57)$$

where

$$\left\|L_{A_1,A_2}^{\mathbb{H}}\{f\}\right\|_{q,p'} = \left(\int_{\mathbb{R}^2} \left|L_{A_1,A_2}^{\mathbb{H}}\{f\}(\boldsymbol{\omega})\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'}. \quad (58)$$

Proof. From the Hausdorff-Young inequality for the QFT, we have

$$\left(\int_{\mathbb{R}^2} \left|\mathscr{F}_q\{f\}(\boldsymbol{\omega})\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'} \leq \left(\int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^p d\mathbf{x}\right)^{1/p}. \quad (59)$$

Based on the arguments used in the proof of Theorem 9, we immediately get

$$\begin{aligned}\left(\int_{\mathbb{R}^2} \left|\mathscr{F}_q\{h\}(\boldsymbol{\omega})\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'} &\leq \left(\int_{\mathbb{R}^2} \left|h(\mathbf{x})\right|^p d\mathbf{x}\right)^{1/p} \cdot \left(\int_{\mathbb{R}^2} \frac{1}{\left|b_1 b_2\right|} \left|\mathscr{F}_q\{h\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right)\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'} \\ &\leq \left(\int_{\mathbb{R}^2} \left|\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}} e^{(\mathbf{i}a_1/2b_1)x_1^2} f(\mathbf{x}) \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi b_2}} \cdot e^{(\mathbf{j}a_2/2b_2)x_2^2}\right|^p d\mathbf{x}\right)^{1/p} \frac{1}{\left|b_1 b_2\right|^{1/p'}} \left(\int_{\mathbb{R}^2} \left|\mathscr{F}_q\{h\} \cdot \left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right)\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'} \\ &\leq \frac{1}{2\pi \left|b_1 b_2\right|^{1/2}} \left(\int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^p d\mathbf{x}\right)^{1/p}.\end{aligned} \quad (60)$$

Hence,

$$\left(\int_{\mathbb{R}^2} \left|e^{-(\mathbf{i}d_1/2b_1)\omega_1^2} L_{A_1,A_2}^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) e^{-(\mathbf{j}d_2/2b_2)\omega_2^2}\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'} \leq \frac{\left|b_1 b_2\right|^{-1/2+1/p'}}{2\pi} \left(\int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^p d\mathbf{x}\right)^{1/p}. \quad (61)$$

Or, equivalently,

$$\left(\int_{\mathbb{R}^2} \left|L_{A_1,A_2}^{\mathbb{H}}\{f\}(\boldsymbol{\omega})\right|_q^{p'} d\boldsymbol{\omega}\right)^{1/p'} \leq \frac{\left|b_1 b_2\right|^{-1/2+1/p'}}{2\pi} \left(\int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^p d\mathbf{x}\right)^{1/p}. \quad (62)$$

Thus, the theorem is completely proved. □

5. Conclusion

In this paper, we derived a version of the uncertainty principle for the QLCT using a relation between the QFT and the QLCT. We presented Hausdorff-Young inequality associated with the QLCT. This inequality is very useful for establishing a variation on Heisenberg's uncertainty principle related to the QLCT.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by Research Collaboration Between Universities (PEKERTI) (no. 622.48/UN28.2/PL/2017) of Indonesia.

References

[1] L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, USA, 2010.

[2] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, San Diego, Calif, USA, 2nd edition, 1998.

[3] T. G. Feeman, *The Mathematics of Medical Imaging*, Springer Science+Business Media, New York, NY, USA, 2010.

[4] K. Grochenig, *Foundations of Time–Frequency Analysis*, Birkhauser, Boston, Mass, USA, 2000.

[5] M. Bahri, R. Ashino, and R. Vaillancourt, "Continuous quaternion Fourier and wavelet transforms," *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 12, no. 4, Article ID 1460003, 21 pages, 2014.

[6] M. Bahri, A. Lawi, N. Aris, A. F. Saleh, and M. Nur, "Relationships between convolution and correlation for fourier transform and quaternion fourier transform," *International Journal of Mathematical Analysis*, vol. 7, no. 41-44, pp. 2101–2109, 2013.

[7] M. Bahri, "Quaternion algebra-valued wavelet transform," *Applied Mathematical Sciences*, vol. 5, no. 69-72, pp. 3531–3540, 2011.

[8] M. Bahri, "Construction of quaternion-valued wavelets," *Matematika (Johor Bahru)*, vol. 26, no. 1, pp. 107–114, 2010.

[9] H. De Bie, N. De Schepper, T. A. Ell, K. Rubrecht, and S. J. Sangwine, "Connecting spatial and frequency domains for the quaternion Fourier transform," *Applied Mathematics and Computation*, vol. 271, pp. 581–593, 2015.

[10] L.-P. Chen, K. I. Kou, and M.-S. Liu, "Pitt's inequality and the uncertainty principle associated with the quaternion fourier transform," *Journal of Mathematical Analysis and Applications*, vol. 423, no. 1, pp. 681–700, 2015.

[11] E. M. Hitzer, "Directional uncertainty principle for quaternion fourier transform," *Advances in Applied Clifford Algebras (AACA)*, vol. 20, no. 2, pp. 271–284, 2010.

[12] E. M. Hitzer, "Quaternion Fourier transform on quaternion fields and generalizations," *Advances in Applied Clifford Algebras (AACA)*, vol. 20, no. 3, pp. 497–517, 2007.

[13] E. Hitzer, "General two-sided quaternion fourier transform, convolution and mustard convolution," *Advances in Applied Clifford Algebras (AACA)*, vol. 27, no. 1, pp. 381–395, 2017.

[14] M. Fei, Y. Xu, and J. Yan, "Real Paley-Wiener theorem for the quaternion Fourier transform," *Complex Variables and Elliptic Equations. An International Journal*, vol. 62, no. 8, pp. 1072–1080, 2017.

[15] Y. Yang, P. Dang, and T. Qian, "Tighter uncertainty principles based on quaternion Fourier transform," *Advances in Applied Clifford Algebras (AACA)*, vol. 26, no. 1, pp. 479–497, 2016.

[16] T. A. Ell, "Quaternion-fourier transforms for analysis of two-dimensional linear time-invariant partial differential systems," in *Proceeding of the 32nd Conference on Decision and Control, IEEE Control Systems Society*, pp. 1830–1841, December 1993.

[17] A. M. Grigoryan and S. S. Agaian, "Tensor transform-based quaternion fourier transform algorithm," *Information Sciences*, vol. 320, pp. 62–74, 2015.

[18] V. R. Dubey, "Quaternion Fourier transform for colour images," *International Journal of Computer Science and Information Technologies*, vol. 5, no. 3, pp. 4411–4416, 2014.

[19] S. Sangwine and T. Ell, "Hypercomplex Fourier transforms of color images," *IEEE Transactions on Image Processing*, vol. 16, no. 1, pp. 22–35, 2007.

[20] M. Bahri, Zulfajar, and R. Ashino, "Convolution and correlation theorem for linear canonical transform and properties," *Information*, vol. 17, no. 6B, pp. 2509–2521, 2014.

[21] D. Wei, Q. Ran, and Y. Li, "A convolution and correlation theorem for the linear canonical transform and its application," *Circuits, Systems and Signal Processing*, vol. 31, no. 1, pp. 301–312, 2012.

[22] S. Xu, L. Feng, Y. Chai, Y. Hu, and L. Huang, "The properties of generalized offset linear canonical Hilbert transform and its applications," *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 15, no. 4, Article ID 1750031, 21 pages, 2017.

[23] M. Bahri and R. Ashino, "Logarithmic uncertainty principle for quaternion linear canonical transform," in *Proceedings of the 2016 International Conference on Wavelet Analysis and Pattern Recognition, ICWAPR 2016*, pp. 140–145, South Korea, July 2016.

[24] K. I. Kou, J.-Y. Ou, and J. Morais, "On uncertainty principle for quaternionic linear canonical transform," *Abstract and Applied Analysis*, vol. 2013, Article ID 725952, 14 pages, 2013.

[25] X.-X. Hu and K. I. Kou, "Quaternion Fourier and linear canonical inversion theorems," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 7, pp. 2421–2440, 2017.

[26] M. Bahri and R. Ashino, "A simplified proof of uncertainty principle for quaternion linear canonical transform," *Abstract and Applied Analysis*, Article ID 5874930, Art. ID 5874930, 11 pages, 2016.

[27] K. I. Kou, J. Ou, and J. Morais, "Uncertainty principles associated with quaternionic linear canonical transforms," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 10, pp. 2722–2736, 2016.

[28] T. Bülow, *Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images [PhD Thesis]*, University of Kiel, Germany, 1999.

[29] M. Bahri, "A modified uncertainty principle for two-sided quaternion Fourier transform," *Advances in Applied Clifford Algebras (AACA)*, vol. 26, no. 2, pp. 513–527, 2016.

[30] Y.-N. Zhang and B.-Z. Li, "Novel uncertainty principles for two-sided quaternion linear canonical transform," *Advances in Applied Clifford Algebras (AACA)*, vol. 28, no. 15, 2018.