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EXISTENCE THEOREMS FOR NONLINEAR DIFFERENTIAL EQUATIONS HAVING TRICHOTOMY IN BANACH SPACES

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Abstract. We give existence theorems for weak and strong solutions with trichotomy of the nonlinear differential equation

$$\dot{x}(t) = \mathcal{L}(t)x(t) + f(t, x(t)), \quad t \in \mathbb{R} \tag{P}$$

where $\{\mathcal{L}(t)\colon t \in \mathbb{R}\}$ is a family of linear operators from a Banach space E into itself and $f\colon \mathbb{R} \times E \to E$. By $L(E)$ we denote the space of linear operators from E into itself. Furthermore, for $a < b$ and $d > 0$, we let $C([-d, 0], E)$ be the Banach space of continuous functions from $[-d, 0]$ into E and $f^d\colon [a, b] \times C([-d, 0], E) \to E$. Let $\widehat{\mathcal{L}}\colon [a, b] \to L(E)$ be a strongly measurable and Bochner integrable operator on $[a, b]$ and for $t \in [a, b]$ define $\tau_t x(s) = x(t + s)$ for each $s \in [-d, 0]$. We prove that, under certain conditions, the differential equation with delay

$$\dot{x}(t) = \widehat{\mathcal{L}}(t)x(t) + f^d(t, \tau_t x) \quad \text{if } t \in [a, b], \tag{Q}$$

has at least one weak solution and, under suitable assumptions, the differential equation (Q) has a solution. Next, under a generalization of the compactness assumptions, we show that the problem (Q) has a solution too.

Keywords: nonlinear differential equation; trichotomy; existence theorem

MSC 2010: 35F31, 34D09

1. Introduction

In Section 2, we investigate the weak and strong solutions of the problem having trichotomy

$$\dot{x}(t) = \mathcal{L}(t)x(t) + f(t, x(t)), \quad t \in \mathbb{R}. \tag{P}$$

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Main results of this section generalize many previous theorems. In fact, in the case $\mathcal{L}(t) = 0$ we have, as a special case, some improvement to the existence theorem of Cramer-Lakshmikantham-Mitchell in [9], Boudourides in [2], Ibrahim-Gomaa in [21], Szep in [36] and Papageorgiou in [30]. Cramer-Lakshmikantham-Mitchell in [9] studied the special case of Problem (P) in a nonreflexive Banach space, Boudourides in [2] and Papageorgiou in [30] found weak solutions for the special case of Problem (P) on a finite interval $[0, T]$ with $0 < T < \infty$. Szep in [36] studied the special case of Problem (P) in a reflexive Banach space, while we use in this section more general compactness assumptions. Ibrahim-Gomaa [21] proved the existence of weak solutions for the special case of Problem (P) on a finite interval $[0, T]$. Also in [14] we consider the Cauchy problem by using weak and strong measures of noncompactness while in [17] we consider some differential inclusions and its topological properties with delay. In [35] the authors present necessary and sufficient conditions for uniform exponential trichotomy of evolution families on the real line, but in [27] Megan-Stoica deal with necessary and sufficient conditions for uniform exponential trichotomy of nonlinear evolution operators in Banach spaces. Moreover, the nonlinear differential equations were studied by many authors ([6], [7], [15], [19], [22], [25], [26] for instance). Further, the paper [3] contains also a suggestion how to apply the results presented in that paper.

In fact, if $\mathcal{L}(t) \neq 0$ our main results generalize those of Cichoń in [4], [6] because we are able to reduce the compactness assumptions.

Finally, in Section 4 we examine the equation

$$\text{(Q)} \qquad \dot{x}(t) = \widehat{\mathcal{L}}(t)x(t) + f^d(t, \tau_t x) \quad \text{if } t \in [a, b],$$

and obtain results similar to that for problem (P). Recently the difference equations (even in the context of Banach spaces) have been investigated (cf. [31], [34]).

2. Preliminaries

Let E be a Banach space, E^* its dual space and E_w the Banach space E endowed with the weak topology. Let λ be the Lebesgue measure on $\mathbb{R}^+$, B_E the family of all nonempty bounded subsets of E and R_E the family of all nonempty and relatively weakly compact subsets of E. Assume that $\langle , \rangle$ is the pairing between E and E^* and $C_{(w)}(\mathbb{R}, E)$ is the space of all (weakly) continuous functions from $\mathbb{R}^+$ to E endowed with the topology of almost uniform weak convergence. Further, let $C([-d, 0], E)$ be the Banach space of continuous functions from the closed interval $[-d, 0]$, $d \geqslant 0$ into E. By $L(E)$ we will denote the space of linear operators from E into itself. A function $u\colon [a, b] \to E$, $(a, b) \in \mathbb{R}^2$ is called Pettis integrable if for any measurable subset D of $[a,b]$ there is an element v_D in E such that $\langle v_D, f\rangle = \int_D \langle u(s), f\rangle \,\mathrm{d}s$ for all $f \in E^*$; in this case we write $v_D = \int_D u(s)\,\mathrm{d}s$. A function $u\colon [a,b] \to E$ is called Bochner integrable if there exists a sequence of countable-valued functions $\{u_n\}$ converging almost everywhere on $[a,b]$ such that $\lim\limits_{n\to\infty} \int_a^b \|u_n(s) - u(s)\|\,\mathrm{d}s = 0$. We note that every Bochner integrable function is Pettis integrable (see [20]).

For any nonempty bounded subset Z of E we recall the definition of De Blasi's measure of weak noncompactness:

$$\beta(Z) = \inf\{\varepsilon > 0\colon \exists\, K = \text{weakly compact subset of } E,\ Z \subseteq K + \varepsilon B_1\}.$$

For the properties of β see [1], [13].

If we put $\mathbb{R}^a = \{x\colon\ z \leqslant x < \infty,\ z = \min\{a, 0\}\}$, then by a Kamke function we mean a function $w\colon\ [a,b] \times \mathbb{R}^a \to \mathbb{R}^+$ such that

(i) w satisfies the Carathéodory conditions,
(ii) for all $t \in [a,b]$; $w(t,a) = 0$,
(iii) for any $c \in (a,b]$, $u \equiv 0$ is the only absolutely continuous function on $[a,c]$ which satisfies $\dot{u}(t) \leqslant w(t, u(t))$ a.e. on $[a,c]$ and such that $u(a) = 0$.

A nonempty family $K \subset R_E$ is a kernel if it satisfies the following conditions:

(i) $A \in K \Rightarrow \operatorname{conv} A \in K$,
(ii) $B \neq \emptyset$, $B \subset A$, $A \in K \Rightarrow B \in K$,
(iii) a subfamily of all weakly compact sets in K is closed in the family of all bounded and closed subsets of E with the topology generated by the Hausdorff distance.

A function $\gamma\colon\ B_E \to [0,\infty)$ is a measure of noncompactness with the kernel K if it is subject to the following conditions:

(i) $\gamma(A) = 0 \Rightarrow A \in K$,
(ii) $\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ is the weak closure of the set A,
(iii) $\gamma(\operatorname{conv} A) = \gamma(A)$,
(iv) $A, B \in B_E$, $B \subset A \Rightarrow \gamma(B) \leqslant \gamma(A)$, see [1], [23].

Denote by N a basis of neighbourhoods of zero in a locally convex space composed of closed convex sets. Let $N' = \{rV\colon\ V \in N,\ r > 0\}$. The following two definitions can be found in [5], [6].

A function $p\colon\ N' \to [0,\infty)$ is a p-function if it satisfies the following conditions:

(i) $X, Y \in N'$, $X \subset Y \Rightarrow p(X) \leqslant p(Y)$,
(ii) for each $\varepsilon > 0$ there exists $X \in N'$ such that $p(X) < \varepsilon$,
(iii) $p(X) > 0$ whenever $X \notin K$.

A function $\gamma\colon B_E \to [0,\infty)$ is a (K,N,p)-measure of noncompactness if and only if

$$\gamma(U) = \inf\{\varepsilon > 0\colon \exists A \in K,\ X \in N',\ U \subset A + X,\ p(X) \leqslant \varepsilon\},$$

for each $U \in B_E$.

Each (K,N,p)-measure of noncompactness is a measure of weak noncompactness. De Blasi's measure is (K,N,p)-measure of noncompactness [1], [5].

For each $t \in \mathbb{R}$ and $\mathcal{L}(t) \in L(E)$, we consider the differential equation

$$\dot{x}(t) = \mathcal{L}(t)x(t). \tag{1}$$

Following Elaydi and Hájek in [11] we introduce:

Let $X(t)$ be the fundamental solution of the differential equation $\dot{X}(t) = \mathcal{L}(t)X(t)$ with the condition $X(0) = \mathrm{Id}$. A linear equation (1) is said to have a trichotomy on $\mathbb{R}$ if there exist linear projections P, Q such that

$$PQ = QP, \quad P + Q = PQ$$

and constants $\alpha \geqslant 1$, $\sigma > 0$ with

$$\begin{aligned}
|X(t)PX^{-1}(s)| &\leqslant \alpha \mathrm{e}^{-\sigma(t-s)} && \text{if } 0 \leqslant s \leqslant t,\\
|X(t)(\mathrm{Id}-P)X^{-1}(s)| &\leqslant \alpha \mathrm{e}^{-\sigma(s-t)} && \text{if } t \leqslant s,\ s \geqslant 0,\\
|X(t)QX^{-1}(s)| &\leqslant \alpha \mathrm{e}^{-\sigma(s-t)} && \text{if } 0 \leqslant s \leqslant 0,\\
|X(t)(\mathrm{Id}-Q)X^{-1}(s)| &\leqslant \alpha \mathrm{e}^{-\sigma(t-s)} && \text{if } s \leqslant t,\ s \leqslant 0.
\end{aligned}$$

Define the integral kernel $K(t,s) = X(t)L(t,s)X^{-1}(s)$, where

$$L(t,s) = \begin{cases} \mathrm{Id} - Q & \text{if } 0 \leqslant s \leqslant \max(t,0),\\ -Q & \text{if } \max(t,0) < s,\\ P & \text{if } s \leqslant \min(t,0),\\ P - \mathrm{Id} & \text{if } \min(t,0) < s \leqslant 0. \end{cases}$$

Moreover, in [24] the authors consider two trichotomy concepts in the sense of Elaydi-Hájek in the general case of abstract evolution operators. Now for each $t, s \in \mathbb{R}$ we have $|K(t,s)| \leqslant \alpha \mathrm{e}^{-\sigma(t-s)}$ ([11], Lemma 7).

We will need the following lemmas in the proof of the main results.

Lemma 2.1 ([5]). *If γ is an (R_E,N,p)-measure of noncompactness such that $p(\alpha X) = \alpha p(X)$ with $X \in N'$, $\alpha \in \mathbb{R}^+$ and for each $X, Y \in N'$ we have $X + Y \in N'$, then*

(M_1) $\gamma(U+V) \leqslant \gamma(U)+\gamma(V)$,
(M_2) $\gamma(\alpha U) = \alpha\gamma(U)$,
(M_3) $\gamma(U \cup \{x\}) = \gamma(U)$, $x \in E$,
(M_4) $U \subseteq V \Rightarrow \gamma(U) \leqslant \gamma(V)$,
(M_5) $\gamma(\overline{\operatorname{conv}} U) = \gamma(U)$,
(M_6) $\gamma(U) = 0 \Rightarrow U$ *is relatively compact in* E.

Under the assumptions in Lemma 2.1 on the measure γ we state the following lemma.

Lemma 2.2 ([16]). *Let* $V \subseteq C(I,E)$ *be bounded equicontinuous in the strong topology and* $V(J) = \{x(t) \colon x \in V,\ t \in J\}$, *where* J *is a subinterval of* I. *Then, under the assumptions in Lemma 2.1,* $\gamma(V(J)) = \sup\limits_{t\in J} \gamma(V(\{t\})) = \gamma((J(s))$ *for some* $s \in J$.

Lemma 2.3 ([6]). *Let* γ *be an* (R_E, N, p)*-measure of noncompactness such that* $p(\alpha X) = \alpha p(X)$ *with* $X \in N'$, $\alpha \in \mathbb{R}$ *and* N *is composed of balanced sets. Then for each bounded subset* U *of* E *and for each* $A \in L(E)$, *we have* $\gamma(AU) \leqslant |A|\gamma(U)$.

Lemma 2.4 ([11]). *Let* $\xi(t)$ *be a nonnegative locally integrable function such that*

$$\int_t^{t+1} \xi(s)\,\mathrm{d}s \leqslant b, \quad t \in \mathbb{R}.$$

If $\alpha > 0$, *then for all* $t \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|t-s|}\xi(s)\,\mathrm{d}s \leqslant \frac{2b}{1-\mathrm{e}^{-\alpha}}.$$

Lemma 2.5 ([4]). *If* $D \colon [a,b] \to L(E)$ *is a continuous mapping and* U *is a bounded subset of* E, *then*

$$\gamma\Bigl(\bigcup_{t\in[a,b]} D(t)U\Bigr) \leqslant \sup_{t\in[a,b]} |D(t)|\gamma(U).$$

Lemma 2.6 ([10]). *Let* W *be a bounded, almost equicontinuous subset of* $C(\mathbb{R},E)$. *For any subset* X *of* W *set* $\aleph(X) = \sup\limits_{t\in\mathbb{R}} \gamma(X(t))$. *Then* $\aleph$ *has the properties* (M_1)–(M_5) *in Lemma 2.1 and if* $\aleph(x) = 0$, *then* x *is relatively compact in* $C(\mathbb{R},E)$.

Lemma 2.7 ([8])**.** *Let Y and E be two Banach spaces, $P_{fc}(Y)$ the set of all closed and convex subsets of Y and let $F\colon E \to P_{fc}(Y)$ be weakly sequentially upper hemicontinuous. Further let $(x_n)_{n\in\mathbb{N}} \subset C(I,E)$, $x_n(t) \to x_0(t)$ weakly a.e. on I and $(y_n)_{n\in\mathbb{N}\cup\{0\}} \subset L^1(I,E)$, $y_n \to y_0$ weakly. Suppose that there exists $a \in L^1(I,\mathbb{R})$ such that $\|F(x)\| \leqslant a(t)$ for all $x \in C(I,E)$ and $y_n(t) \in F(x_n(t))$ a.e. on I. Then $y_0(t) \in F(x_0(t))$ a.e. on I.*

Lemma 2.8 ([28])**.** *Let $V \subseteq C(I,E)$ be a family of strongly equicontinuous functions. Then*

$$\beta_c(V) = \sup_{t\in I} \beta(V(t)),$$

where $\beta_c(V)$ is the measure of weak noncompactness in $C(I,E)$ and $t \mapsto \beta(V(t))$ is a continuous function.

We need to state the well-known Darbo-Sadovskii's theorem [33].

Theorem 2.9. *Let μ be a measure of noncompactness defined on a normed space M such that $\mu(\overline{\operatorname{conv}} U) = \mu(U)$ for any nonempty and bounded subset U of M. Let D be a nonempty bounded closed and convex subset of M. If $T\colon D \to M$ is continuous and for each bounded $A \subseteq D$ with $\mu(A) > 0$, $\mu(T(A)) < \mu(A)$, then T has a fixed point.*

Now we consider the Cauchy problem

$$\text{(C)} \qquad \begin{cases} \dot{x}(t) = h(t, \tau_t x), \\ x(t) = \psi \in C([-d,0],E), \end{cases}$$

where $h\colon [0,\infty) \times C([-d,0],E) \to E$, $x \in C([-d,\infty),E)$ and $\tau_t x \in C([-d,0],E)$, $t \geqslant 0$ is defined by $\tau_t x(s) = x(t+s)$, $s \in [-d,0]$. Let $B_r = \{x \in C([-d,0),E)\colon \|x\| \leqslant r\}$.

Theorem 2.10 ([3], Theorem 5)**.** *Suppose that E is a separable Banach space. Let $h\colon [0,\infty) \times C([-d,0],E) \to E$ be sequentially weakly continuous in bounded sets. Further, let $h([0,T] \times B_r)$ be relatively compact in E_w for any $T, r > 0$. Then for each $r > 0$ there exists $\delta(r) > 0$ such that if $\psi \in C([-d,0],E)$ and $\|\psi\| \leqslant r$, problem* (C) *has a solution defined on $[0,\delta]$. Moreover, if h is continuous, then problem* (C) *has a solution in $C^1([0,\delta];E)$ and the separability of E is not needed.*

3. Existence results for problem (P)

In the following we study the problem (P) on $\mathbb{R}$ and use the (K, N, p)-measure of noncompactness so that we will generalize Theorem 8 with respect to the Cauchy problem in [14] and the references herein.

Theorem 3.1. *We introduce the following assumptions:*

(M_1) *f is a continuous function from $\mathbb{R} \times E_w$ to E_w.*

(M_2) *$\mathcal{L}\colon \mathbb{R} \to L(E)$ is strongly measurable and Bochner integrable on every finite subinterval of $\mathbb{R}$ and the linear equation*

$$\dot{x}(t) = \mathcal{L}(t)x(t)$$

has a trichotomy with constants $\alpha \geqslant 1$ and $\sigma > 0$.

(M_3) *There exist two real nonnegative functions c_1, c_2 which are locally integrable on $\mathbb{R}$ and, for each $t \in \mathbb{R}$, there exist two constants C_1 and C_2 such that*

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} c_1(s)\,\mathrm{d}s \leqslant C_1, \quad \sup_{t\in\mathbb{R}} \int_t^{t+1} c_2(s)\,\mathrm{d}s \leqslant C_2,$$

where $0 < C_2 < \frac{1}{2}(1 - \mathrm{e}^{-\sigma})/\alpha$ and $\|f(t,x)\| \leqslant c_1(t) + c_2(t)\|x\|$ for each $t \in \mathbb{R}$ and $x \in E$.

(M_4) *For each compact subset I of $\mathbb{R}$ and for each $\varepsilon > 0$ there exists a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ such that for any nonempty bounded subset U of E one has*

$$\gamma(f(J \times U)) \leqslant \sup_{t\in J} w(t, \gamma(U))$$

for any compact subset J of I_ε.

Then there exists a bounded weak solution of (P) *on $\mathbb{R}$.*

Proof. By virtue of assumption (M_2) there exist two constants α and σ such that for each $t, s \in \mathbb{R}$,

$$|K(t,s)| \leqslant \alpha \mathrm{e}^{-\sigma(t-s)}. \tag{2}$$

If $M = 2\alpha C_1/(1 - \mathrm{e}^{-\sigma} - 2\alpha C_2)$, then $M > 0$. Put

$$H = \Bigl\{x \in C_w(\mathbb{R}, E)\colon \|x(t)\| \leqslant M,\ \|x(t) - x(\tau)\| \leqslant M \int_\tau^t |\mathcal{L}(s)|\,\mathrm{d}s + \int_\tau^t c_1(s)\,\mathrm{d}s + M \int_\tau^t c_2(s)\,\mathrm{d}s,\ \tau \leqslant t\Bigr\}.$$

H is a nonempty, almost equicontinuous, bounded, closed and convex subset of $C_w(\mathbb{R}, E)$. For each $x \in H$ we can define a mapping Γ by

$$\Gamma(x)(t) = \int_{\mathbb{R}} K(t,s) f(s, x(s)) \,\mathrm{d}s \quad \text{for each } t \in \mathbb{R}.$$

By Lemma (2.4) and (2) we have $\|\Gamma(x)\| \leqslant 2\alpha(C_1 + MC_2)/(1 - \mathrm{e}^{-\sigma}) = M$, and so Γ is bounded on $\mathbb{R}$. Moreover, since $y = \Gamma(x)$ is a weak solution of the equation $\dot{y}(t) = \mathcal{L}(t)y(t) + f(t, x(t))$, we have

$$\begin{aligned} \|\Gamma(x)(t) - \Gamma(x)(\tau)\| &\leqslant \int_t^\tau \|\mathcal{L}(s)\Gamma(x)(s) + f(t, x(s))\| \,\mathrm{d}s \\ &\leqslant M \int_\tau^t |\mathcal{L}(s)| \,\mathrm{d}s + \int_\tau^t c_1(s) \,\mathrm{d}s + M \int_\tau^t c_2(s) \,\mathrm{d}s. \end{aligned}$$

Therefore $\Gamma(x) \in H$ and $\Gamma\colon H \to H$. Moreover, it can be shown as in [7] that Γ is continuous on H. Now we note that each nonempty subset X of H is equicontinuous. According to the definition of γ for each $\varepsilon > 0$ there exists $V \in N'$ with $p(V) < \varepsilon$. We can find two positive constants δ, q such that $M\mathrm{e}^{-\delta q} < 2\delta$ and $B_\delta \subset V$. In the sequel without loss of generality we will assume that $A = (t - q, t + q)$ and $0 \notin A$. Set $X_1 = \int_{-\infty}^{t-q} K(t,s) f(s, X(s)) \,\mathrm{d}s$, thus

$$\|X_1\| \leqslant \int_{-\infty}^{t-q} \alpha \mathrm{e}^{-\delta(t-s)} (c_1(s) + Mc_2(s)) \,\mathrm{d}s \leqslant \frac{M\mathrm{e}^{-\delta q}}{2} < \delta$$

and $\gamma(X_1) \leqslant p(V) \leqslant \varepsilon$, so $X_1 \subset B_\delta \subset V$. Moreover, from [32] we have

$$\gamma\Big(\int_{t+q}^{\infty} K(t,s) f(s, X(s)) \,\mathrm{d}s\Big) \leqslant \varepsilon.$$

By condition (M_4) there exists a closed subset J_ε of $[t - q, t + q]$ such that $\lambda([t - q, t + q] - J_\varepsilon) < \varepsilon$ and for any compact subset K of J_ε and any bounded subset Z of E,

$$\gamma(f(K \times Z)) \leqslant \sup_{s \in K} w(s, \gamma(Z)). \tag{3}$$

By Scorza-Dragoni theorem there exists a closed subset I_ε of the interval $[t - q, v]$ such that $\lambda(I - I_\varepsilon) < \delta$ and there exist $\delta(\varepsilon)$, $\eta > 0$ $(\eta < \delta)$ such that

$$s_1, s_2 \in I_\varepsilon;\ r_1, r_2 \in [a,b] \text{ with } |s_1 - s_2| < \delta,\ |r_1 - r_2| < \delta \Rightarrow |w(s_1, r_1) - w(s_2, r_2)| < \varepsilon.$$

Put $D = \{x \in C([t - q, v], E)\colon\ x \in X\}$, so

$$\gamma(D) = \sup\{\gamma(X(s))\colon\ t - q \leqslant s \leqslant v\} \leqslant \gamma(X)$$

and

$$|s_1 - s_2| < \eta \Rightarrow |\gamma(D(s_1)) - \gamma(D(s_2))| < \delta.$$

Let us fix u, v, $t-q \leqslant u < v < t+q$ and let $u = t_0 < t_1 < \ldots < t_m = v$ be a partition of $[u,v]$ with $t_i - t_{i-1} < \eta$ for $i = 1,\ldots,m$. Let $T_i = J_\varepsilon \cap [t_{i-1}, t_i] \cap I_\varepsilon$, $P = \bigcup_{i=1}^{m} T_i = [u,v] \cap J_\varepsilon \cap I_\varepsilon$ and $Q = [u,v] - P$. We can find $\eta' > 0$, $\eta' < \delta$, such that if $r_1, r_2 \in P$ and $|r_1 - r_2| < \eta'$, then

$$|K(t,r_1) - K(t,r_2)| < \varepsilon$$

and we can find s_i in T_i with

$$\sup_{s\in T_i} |K(t,s)| = |K(t,s_i)|. \tag{4}$$

Further, we have

$$\int_s^v K(t,s)f(s,D(s))\,\mathrm{d}s \subset \int_P K(t,s)f(s,D(s))\,\mathrm{d}s + \int_Q K(t,s)f(s,D(s))\,\mathrm{d}s. \tag{5}$$

By the mean value theorem for the Pettis-integral we obtain

$$\int_P K(t,s)f(s,D(s))\,\mathrm{d}s \subset \sum_{i=1}^{n} \lambda(T_i)\overline{\mathrm{conv}}\,\{K(t,s)f(s,w)\colon\ s\in T_i,\ w\in D(s)\}.$$

Let $D_i = \{x(t)\colon\ x\in D,\ t\in T_i\}$. Hence, by Lemma 2.8,

$$\gamma(D_i) = \sup\{\gamma(D(t))\colon\ t\in T_i\} = \gamma(D(s_i')) \quad \text{for some } s_i' \in T_i. \tag{6}$$

In view of (4), (6) and (3) we have

$$\gamma\Bigl(\int_P K(t,s)f(s,D(s))\,\mathrm{d}s\Bigr) \leqslant \sum_{i=1}^{m} \lambda(T_i)|K(t,s_i)|w(q_i,\gamma(D(s)), \quad q_i \in T_i.$$

Moreover, $|w(s,\gamma(D(s))) - w(q_i,\gamma(D(s_i^*)))| \leqslant \varepsilon'/\lambda(P)$ for all $s^* \in T_i$. So

$$\lambda(T_i)|K(t,s_i)|w(q_i,\gamma(D(s_i^*))) \leqslant \int_{T_i} |K(t,s)|w(s,\gamma(D(s)))\,\mathrm{d}s + \frac{\varepsilon'\lambda(T_i)}{\lambda(P)}$$

and

$$\begin{aligned}\gamma\Bigl(\int_P K(t,s)f(s,D(s))\,\mathrm{d}s\Bigr) &\leqslant \sum_{i=1}^{m}\Bigl(\int_{T_i} |K(t,s)|w(s,\gamma(D(s)))\,\mathrm{d}s + \frac{\varepsilon'\lambda(T_i)}{\lambda(P)}\Bigr)\\ &= \int_P |K(t,s)|w(s,\gamma(D(s)))\,\mathrm{d}s + \varepsilon'.\end{aligned} \tag{7}$$

Furthermore, we have

$$\gamma\Big(\int_Q K(t,s)f(s,D(s))\,\mathrm{d}s\Big) \leqslant \int_Q |K(t,s)|(c_1(s)+Mc_2(s))\,\mathrm{d}s. \tag{8}$$

From (5) we have

$$\begin{aligned}\gamma\Big(\int_u^v K(t,s)f(s,D(s))\,\mathrm{d}s\Big) \leqslant \gamma\Big(\int_P K(t,s)f(s,D(s))\,\mathrm{d}s\Big)\\ + \gamma\Big(\int_Q K(t,s)f(s,D(s))\,\mathrm{d}s\Big).\end{aligned}$$

If $\lambda(Q) < \varepsilon$, then from (7) and (8) we deduce that

$$\begin{aligned}\gamma\Big(\int_u^v K(t,s)f(s,D(s))\,\mathrm{d}s\Big) &\leqslant \int_P \|K(t,s)\| w(s,\gamma(D(s)))\,\mathrm{d}s\\ &\leqslant \int_u^v |K(t,s)| w(s,\gamma(D(s)))\,\mathrm{d}s.\end{aligned}$$

Moreover,

$$\gamma(\varphi(D)(v)) \leqslant \gamma(\varphi(D)(u)) + \gamma\Big(\int_u^v K(t,s)f(s,D(s))\,\mathrm{d}s\Big).$$

Defining $\varrho(t) := \gamma(D(t))$ we get

$$\varrho(v) - \varrho(u) \leqslant \gamma\Big(\int_u^v K(t,s)f(s,D(s))\,\mathrm{d}s\Big) \leqslant \gamma(B_1)\int_u^v |K(t,s)| w(s,\varrho(s))\,\mathrm{d}s.$$

Therefore $\dot{\varrho}(t) \leqslant \alpha\gamma(B_1)\mathrm{e}^{-\sigma(t-s)}w(t,\varrho(t))$ a.e. on $[u,v]$ and since $\varrho(u) = 0$, hence $\varrho \equiv 0$ and so $\overline{D}^w$ is weakly compact in $C_w(\mathbb{R},E)$. But D is closed, hence it is a convex and compact subset in $C_w(\mathbb{R},E)$. By the Schauder-Tichonov theorem, since φ is a continuous mapping from D to D, there is a fixed point y of φ such that y is the desired weak solution of (P). □

Theorem 3.2. *Let the following assumptions be fulfilled:*

(A_1) $\mathcal{L}\colon \mathbb{R} \to L(E)$ *is strongly measurable and Bochner integrable on every finite subinterval of* $\mathbb{R}$ *and the linear equation*

$$\dot{x}(t) = \mathcal{L}(t)x(t)$$

has a trichotomy with constants $\alpha \geqslant 1$ *and* $\sigma > 0$.

(A_2) $f\colon\ \mathbb{R}\times E\to E$ *is a function such that*

(i) *for each* $t\in\mathbb{R}$ *the function* $f(t,.)$ *is continuous,*

(ii) *for each* $x\in E$ *the function* $f(\cdot,x)$ *is measurable,*

(iii) *there exist two real nonnegative functions* c_1,c_2 *locally integrable on* $\mathbb{R}$ *and, for each* $t\in\mathbb{R}$*, two constants* C_1 *and* C_2 *with*

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}c_1(s)\,\mathrm{d}s\leqslant C_1,\qquad \sup_{t\in\mathbb{R}}\int_t^{t+1}c_2(s)\,\mathrm{d}s\leqslant C_2,$$

where $0<C_2<(1-\mathrm{e}^{-\sigma})/2\alpha$ *and* $\|f(t,x)\|\leqslant c_1(t)+c_2(t)\|x\|$ *for each* $t\in\mathbb{R}$ *and* $x\in E$.

(A_3) $h\colon\ \mathbb{R}\times[0,\infty)\to\mathbb{R}^+$ *satisfies the Carathéodory conditions.*

(A_4) $L=\sup\{\int_A\|K(t,s)\|h(t,\gamma(B(s))\,\mathrm{d}s\colon\ t\in\mathbb{R}\}\leqslant\sup\{\gamma(B(s))\colon\ s\in A\}$, *where* B *is a bounded subset of* $C(\mathbb{R},E)$, *for each compact subset* A *of* $\mathbb{R}$.

(A_5) *For each compact subset* I *of* $\mathbb{R}$ *and for each* $\varepsilon>0$, *there exists a closed subset* I_ε *of* I *with* $\lambda(I-I_\varepsilon)<\varepsilon$ *such that for any nonempty bounded subset* U *of* E *one has*

$$\gamma(f(J\times U))\leqslant\sup_{t\in J}h(t,\gamma(U))$$

for any compact subset J *of* I_ε.

Then there is at least one bounded solution of (P) *on* $\mathbb{R}$.

P r o o f. By the assumption (A_1) there exist two constants α and σ such that for each $t,s\in\mathbb{R}$, [11] Lemma 7 yields

$$|K(t,s)|\leqslant\alpha\mathrm{e}^{-\sigma(t-s)}. \tag{9}$$

Now if $M=2\alpha C_1/(1-\mathrm{e}^{-\sigma}-2\alpha C_2)$, then $M>0$. Put

$$H=\Big\{x\in C(\mathbb{R},E)\colon\ \|x(t)\|\leqslant M,\ \|x(t)-x(\tau)\|\leqslant M\int_\tau^t|A(s)|\,\mathrm{d}s+\int_\tau^t c_1(s)\,\mathrm{d}s+M\int_\tau^t c_2(s)\,\mathrm{d}s,\ \tau\leqslant t\Big\}.$$

H is a nonempty, almost equicontinuous, bounded, closed and convex subset of $C(\mathbb{R},E)$. For each $x\in H$ we can define a mapping ψ by

$$\psi(x)(t)=\int_{\mathbb{R}}K(t,s)f(s,x(s))\,\mathrm{d}s\quad\text{for each }t\in\mathbb{R},$$

and this mapping is bounded on $\mathbb{R}$. Since $y = \psi(x)$ is a solution of the equation $\dot{y} = A(t)y + f(t, x(t))$, we have

$$\begin{aligned}\|\psi(x)(t) - \psi(x)(\tau)\| &\leqslant \int_t^{\tau} \|A(s)\psi(x)(s) + f(t, x(s))\| \,\mathrm{d}s \\ &\leqslant M \int_{\tau}^{t} |A(s)| \,\mathrm{d}s + \int_{\tau}^{t} c_1(s) \,\mathrm{d}s + M \int_{\tau}^{t} c_2(s) \,\mathrm{d}s.\end{aligned}$$

By Lemma (2.4) and (9)

$$\|\psi(x)\| \leqslant \frac{2\alpha(C_1 + MC_2)}{1 - \mathrm{e}^{-\sigma}} = M.$$

Therefore $\psi(x) \in H$ and $\psi\colon H \to H$. Moreover, it can be shown as in [7] that ψ is a continuous function on H. Now we note that each subset X of H is equicontinuous. By the definition of γ for each $\varepsilon > 0$ there exists $V \in N'$ with $p(V) < \varepsilon$. We can find two positive constants δ, q such that $M\mathrm{e}^{-\delta q} < 2\delta$ and $B_\delta \subset V$. In the sequel without loss of generality we will assume that $A = (t-q, t+q)$ and $0 \notin A$. Set $X_1 = \int_{-\infty}^{t-q} K(t,s) f(s, X(s)) \,\mathrm{d}s$, $\|X_1\| \leqslant \int_{-\infty}^{t-q} \alpha \mathrm{e}^{-\delta(t-s)}(c_1(s) + Mc_2(s)) \,\mathrm{d}s \leqslant M\mathrm{e}^{-\delta q}/2 < \delta$ and

$$\gamma(X_1) \leqslant p(V) \leqslant \varepsilon.$$

Thus $X_1 \subset B_\delta \subset V$. Moreover [32],

$$\gamma\left(\int_{t+q}^{\infty} K(t,s) f(s, X(s)) \,\mathrm{d}s\right) \leqslant \varepsilon.$$

Condition (M_5) yields that there exists a closed subset J_ε of $[t-q, t+q]$ such that $\lambda([t-q, t+q] - J_\varepsilon) < \varepsilon$ and for any compact subset K of J_ε and any bounded subset Z of E,

$$\gamma(f(K \times Z)) \leqslant \sup_{s \in K} h(s, \gamma(Z)). \tag{10}$$

From the Scorza-Dragoni theorem there exists a closed subset I_ε of the interval $[t-q, t+q]$ such that $\lambda(I - I_\varepsilon) < \delta$ and there exist $\delta(\varepsilon)$, $\eta > 0$, $\eta < \delta$, such that

$$s_1, s_2 \in I_\varepsilon;\ r_1, r_2 \in [a, b] \text{ with } |s_1 - s_2| < \delta,\ |r_1 - r_2| < \delta \Rightarrow |h(s_1, r_1) - h(s_2, r_2)| < \varepsilon.$$

Put $D = \{X(s)\colon\ t - q \leqslant s \leqslant t + q\}$, so

$$\gamma(D) = \sup\{\gamma(X(s))\colon\ t - q \leqslant s \leqslant t + s\} \leqslant \gamma(X)$$

and

$$|s_1 - s_2| < \eta \Rightarrow |\gamma(D(s_1)) - \gamma(D(s_2))| < \delta.$$

Let $t-q = t_0 < t_1 < \ldots < t_m = t+q$ be a partition of $[t-q, t+q]$ with $t_i - t_{i-1} < \eta$ for $i = 1, \ldots, m$. Let $T_i = J_\varepsilon \cap [t_{i-1}, t_i] \cap I_\varepsilon$, $P = \bigcup\limits_{i=1}^{m} T_i = [t-q, t+q] \cap J_\varepsilon \cap I_\varepsilon$ and $Q = [t-q, t+q] - P$. We can find $\eta' > 0$ ($\eta' < \delta$) such that if $r_1, r_2 \in P$ and $|r_1 - r_2| < \eta'$, then

$$|K(t, r_1) - K(t, r_2)| < \varepsilon,$$

and we can find s_i in T_i with

$$\sup_{s \in T_i} |K(t,s)| = |K(t, s_i)|. \tag{11}$$

Further, we have

$$\int_{t-q}^{t+q} K(t,s) f(s, D(s))\,\mathrm{d}s \subset \int_P K(t,s) f(s, D(s))\,\mathrm{d}s + \int_Q K(t,s) f(s, D(s))\,\mathrm{d}s. \tag{12}$$

By the mean value theorem for the Pettis-integral we obtain

$$\int_P K(t,s) f(s, D(s))\,\mathrm{d}s \subset \sum_{i=1}^{n} \lambda(T_i)\, \overline{\operatorname{conv}}\, \{K(t,s) f(s,w)\colon\ s \in T_i,\ w \in D(s)\}.$$

Let $D_i = \{x(t)\colon\ x \in D,\ t \in T_i\}$. Hence, by Lemma 2.8,

$$\gamma(D_i) = \sup\{\gamma(D(t))\colon\ t \in T_i\} = \gamma(D(s_i')) \quad \text{for some } s_i' \in T_i. \tag{13}$$

In view of (11), (13) and (10) we have

$$\gamma\Bigl(\int_P K(t,s) f(s, D(s))\,\mathrm{d}s\Bigr) \leqslant \sum_{i=1}^{m} \lambda(T_i) |K(t, s_i)| h(q_i, \gamma(D(s)), \quad q_i \in T_i.$$

Moreover, $|h(s, \gamma(D(s))) - h(q_i, \gamma(D(s_i^*)))| \leqslant \varepsilon'/\lambda(P)$ for all $s^* \in T_i$. So

$$\lambda(T_i) |K(t, s_i)| h(q_i, \gamma(D(s_i^*))) \leqslant \int_{T_i} |K(t,s)| h(s, \gamma(D(s)))\,\mathrm{d}s + \frac{\varepsilon' \lambda(T_i)}{\lambda(P)}$$

and

$$\begin{aligned}\gamma\Bigl(\int_P K(t,s) f(s, D(s))\,\mathrm{d}s\Bigr) &\leqslant \sum_{i=1}^{m} \Bigl(\int_{T_i} |K(t,s)| h(s, \gamma(D(s)))\,\mathrm{d}s + \frac{\varepsilon' \lambda(T_i)}{\lambda(P)}\Bigr) \\ &= \int_P |K(t,s)| h(s, \gamma(D(s)))\,\mathrm{d}s + \varepsilon'.\end{aligned} \tag{14}$$

Furthermore, we have

$$\gamma\Big(\int_Q K(t,s)f(s,D(s))\,\mathrm{d}s\Big) \leqslant \int_Q |K(t,s)|(c_1(s)+Mc_2(s))\,\mathrm{d}s. \tag{15}$$

From (12) we have

$$\begin{aligned}\gamma\Big(\int_{t-q}^{t+q} K(t,s)f(s,D(s))\,\mathrm{d}s\Big) &\leqslant \gamma\Big(\int_P K(t,s)f(s,D(s))\,\mathrm{d}s\Big)\\ &\quad+\gamma\Big(\int_Q K(t,s)f(s,D(s))\,\mathrm{d}s\Big).\end{aligned}$$

If $\lambda(Q)<\varepsilon$, then from (14) and (15) we deduce that

$$\begin{aligned}\gamma\Big(\int_{t-q}^{t+q} K(t,s)f(s,D(s))\,\mathrm{d}s\Big) &\leqslant \int_P |K(t,s)|h(s,\gamma(D(s)))\,\mathrm{d}s\\ &\leqslant \int_{t-q}^{t+q} |K(t,s)|h(s,\gamma(D(s)))\,\mathrm{d}s\\ &\leqslant \sup\{\gamma(D(s))\colon\ t-q<s<t+q\}=\gamma(D).\end{aligned}$$

Thus

$$\gamma(\psi(X(t))) \leqslant 2\varepsilon+\gamma(D) \leqslant 2\varepsilon+\gamma(X).$$

If we put $\aleph(X)=\sup\{\gamma(X(t))\colon\ t\in\mathbb{R}\}$ then, by Lemma 2.6, $\aleph$ satisfies the condition (M_5) in Lemma 2.1 and moreover $\aleph(\psi(X))\leqslant\aleph(X)$. By Theorem 2.9 ψ has a fixed point in H which, due to Lemma 7 of [12], is a bounded solution of (P). □

In the next theorem we will deal with the differential equation

$$\dot{x}(t)=\mathcal{L}(t)x(t)+f'(t,x(t)),\quad t\in\mathbb{R} \tag{P$'$}$$

where $f'\colon\ \mathbb{R}\times E\to E$ is a Carathéodory function, $\mathcal{L}\colon\ \mathbb{R}\to L(E)$ is a strongly measurable and Bochner integrable operator on every closed finite interval I of $\mathbb{R}$ and γ is a (K,N,p)-measure of weak noncompactness. The Kuratowski measure of noncompactness is a (K,N,p)-measure of noncompactness [5], [1], hence we get generalizations of results such as Theorem 2 in [37] and Theorem 9 in [14].

Theorem 3.3. *Assume that $f'\colon\ \mathbb{R}\times E\to E$ satisfies* (M_3) *and* (M_4) *of Theorem* 3.1 *while $\mathcal{L}\colon\ \mathbb{R}\to L(E)$ is a strongly measurable and Bochner integrable operator on every closed finite interval I of $\mathbb{R}$. Moreover, assume*

(i) *for each $t\in\mathbb{R}$, $f'(t,\cdot)$ is continuous,*
(ii) *for each $x\in E$, $f'(\cdot,x)$ is measurable,*
(iii) *for each $x\in E$ and each closed finite interval I of $\mathbb{R}$, $f'(I\times\{x\})$ is separable.*

Then problem (P') *has at least one bounded solution.*

P r o o f. Let

$$W=\Big\{x\in C(\mathbb{R},E)\colon \ \|x(t)\|\leqslant M,\ \|x(t)-x(\tau)\|\leqslant M\int_\tau^t|\mathcal{L}(s)|\,\mathrm{d}s \\ +\int_\tau^t c_1(s)\,\mathrm{d}s+M\int_\tau^t c_2(s)\,\mathrm{d}s,\ \tau\leqslant t\Big\}.$$

We can define a mapping $\psi\colon\ W\to W$ by

$$\psi(x)(t)=\int_{\mathbb{R}}K(t,s)f(s,x(s))\,\mathrm{d}s\quad\text{for each } t\in\mathbb{R}.$$

Let x_0 be an arbitrary element in W, $\psi(x_n)=x_{n+1}$ and $Y=\{x_n\colon\ n=0,1,2,3,\ldots\}$. As in the proof of Theorem 3.1, there exist two constant u,v such that if $V=\{x_n\in C([t-q,v],E)\colon\ x_n\in Y\}$ and we define $\varrho(t):=\gamma(V(t))$, then

$$\varrho(v)-\varrho(u)\leqslant\gamma\Big(\int_u^v K(t,s)f(s,D(s))\,\mathrm{d}s\Big)\leqslant\gamma(B_1)\int_u^v|K(t,s)|w(s,\varrho(s))\,\mathrm{d}s.$$

Therefore $\dot\varrho(t)\leqslant\alpha\gamma(B_1)\mathrm{e}^{-\sigma(t-s)}w(t,\varrho(t))$ a.e. on $[u,v]$ and since $\varrho(u)=0$, we have $\varrho\equiv0$. Thus the closure of V is compact and so we can find a subsequence (x_{n_k}) of (x_n) which converges to a limit x. Since $\|x_n-\varphi(x_n)\|\to0$ as $n\to\infty$ and φ is continuous, hence $x=\varphi(x)$ so that x is the desired solution of (P$'$). $\square$

We are in a position to prove the following result.

Theorem 3.4. *Let $\mathfrak{h}\colon\ [a,b]\times\mathbb{R}^a\to\mathbb{R}^+$ be a Carathéodry function and, for each bounded subset Z of $[a,b]\times\mathbb{R}^a$, let there exist a measurable function m_Z such that $\mathfrak{h}(t,s)\leqslant m_Z(t)$ for each $(t,s)\in Z$ and m is integrable on $[c,T]$ for each c; $a<c\leqslant b$. Moreover, let for each c; $a<c\leqslant b$, the identically zero function be the only absolutely continuous function on $[a,c]$ which satisfies $\dot u(t)=\mathfrak{h}(t,u(t))$ a.e. on $[a,c]$, such that the right hand derivative of $u(t)$ at $t=a$, $D_+u(a)$, exists and $D_+u(a)=u(a)=0$. If we replace in the setting of Theorem* 3.3 *a Kamke function w by a function $\mathfrak{h}$ and suppose that f' is bounded and continuous, then the problem* (P) *has at least one solution.*

P r o o f. Due to the assumption that f' is bounded we can find a constant C such that $\|f'(t,x)\|\leqslant C$. Let $\mathcal{L}\colon\ [a,b]\to\mathbb{R}$ be defined by

$$\mathcal{L}(t)=\sup_{\|x\|,\|y\|\leqslant Ct}\|f'(t,x)-f'(t,y)\|.$$

It can be shown as in [14], [29], that $\mathcal{L}$ is continuous at a and lower semicontinuous on $]a,b]$. Consequently, we can say that $\big\|\int_t^\tau f'(s,x(s))-\int_t^\tau f'(s,y(s))\,\mathrm{d}s\big\|\leqslant\int_t^\tau\mathcal{L}(s)\,\mathrm{d}s$

for each $x, y \in Y$. Now by the same argument as in the proof of Theorem 3.3 if we put $Y = \{x_n\colon n = 0, 1, 2, 3, \ldots\}$ and $V = \{x_n \in C([t-q, v], E)\colon x_n \in Y\}$ while $\varrho(t) = \gamma(V(t))$ we get

$$\varrho(v) - \varrho(u) \leqslant \gamma\biggl(\int_u^v K(t,s) f(s, D(s))\,\mathrm{d}s\biggr) \leqslant \gamma(B_1)\int_u^v |K(t,s)| w(s, \varrho(s))\,\mathrm{d}s.$$

Now we can conclude that

$$\varrho(\tau) - \varrho(t) \leqslant \min\biggl(\int_t^\tau \mathcal{L}(s)\,\mathrm{d}s, \gamma(B_1)\int_t^\tau K(t,s) f(s, D(s))\,\mathrm{d}s\biggr), \quad t - q < t \leqslant \tau \leqslant v.$$

Since ϱ is an absolutely continuous function on $[t-q, v]$ so

$$\dot{\varrho}(t) \leqslant \min(\mathcal{L}(t), \gamma(B_1)\alpha f(t, D(t)), \quad \text{a.e. on } [t-q, v]. \tag{16}$$

By Lemma 1 in [29] $\varrho \equiv 0$ on $[t-q, v]$ and thus we obtain the result. $\square$

4. Existence results for problem (Q)

For $t \in [a, b]$ we let $\widehat{\mathcal{L}}(t) \in L(E)$ and $\tau_t x(s) = x(t+s)$ for all $s \in [-d, 0]$. Assume that $C([-d, 0], E)$ is the Banach space of continuous functions from $[-d, 0]$ into E and $f^d\colon [a, b] \times C([-d, 0], E) \to E$. In the next theorem we deal with the problem

$$\dot{x}(t) = \widehat{\mathcal{L}}(t) x(t) + f^d(t, \tau_t x), \quad t \in [a, b] \tag{Q}$$

and obtain a generalization of Theorem 3.1.

Theorem 4.1. *We assume:*

(H_1) $f^d\colon [a, b] \times C_{(w)}([-d, 0], E) \to E$ *is continuous, where* $C_{(w)}([-d, 0], E)$ *is the space of all weakly continuous functions from* $[-d, 0]$ *to* E.

(H_2) $\widehat{\mathcal{L}}\colon [a, b] \to L(E)$ *is a strongly measurable and Bochner integrable operator on* $[a, b]$ *and the linear equation*

$$\dot{x}(t) = \widehat{\mathcal{L}}(t) x(t)$$

has a trichotomy with constants $\alpha \geqslant 1$ *and* $\sigma > 0$.

(H_3) *There exist two real nonnegative functions* c_1, c_2 *integrable on* $[a, b]$ *and two constants* C_1 *and* C_2 *such that*

$$\int_a^b c_1(s)\,\mathrm{d}s \leqslant C_1, \quad \int_a^b c_2(s)\,\mathrm{d}s \leqslant C_2,$$

where $0 < C_2 < \frac{1}{2}(1-\mathrm{e}^{-\sigma})/\alpha$ *and* $\|f(t,\varphi)\| \leqslant c_1(t) + c_2(t)\|\varphi(0)\|$ *for each* $t \in [a,b]$ *and* $\varphi \in C([-d,0],E)$.

(H_4) *For each* $\varepsilon > 0$ *there exists a closed subset* I_ε *of* $[a,b]$ *with* $\lambda([a,b]-I_\varepsilon) < \varepsilon$ *such that for any nonempty bounded subset* A *of* $C([-d,0],E)$ *and for each closed subset* $J \subseteq I_\varepsilon$*, one has*

$$\gamma(F(J\times A)) \leqslant \sup_{t\in J} h(t,\gamma(A(0))).$$

Then, for each $\psi \in C_E([a-d,a])$ *such that* $\psi(a) = 0$*, the problem* (Q) *has a weak solution on the interval* $[a-d,b]$.

Proof. Along the same lines as in [17], [18], [16] we use some methods for functional equations. We partition the closed interval $[a,b]$ by the points $t_i^n = (ib+(n-i)a)/n$ where $i = 0,1,2,\dots,n$. Let $\xi_1^n\colon [a-d,t_1^n]\times E \to E$ be a function defined by

$$\xi_1^n(t,x) = \begin{cases} \psi(t) & \text{if } t\in[a-d,a],\\ n(t-a)x & \text{if } t\in[a,t_1^n],\end{cases}$$

where n is a positive integer. Let $f_1^n\colon [a,t_1^n]\times E \to E$ be a function defined by $f_1^n(t,x) = f^d(t,\tau_{t_1^n}(\xi_1^n(\cdot,x)))$. Due to Theorem 3.1 there is a function v_n such that $v_n = \psi$ on $[a-d,a]$ and for each $t\in[a,t_1^n]$

$$v_n(t) = \int_a^t K(t,s) f_1^n(s,v_n(s))\,\mathrm{d}s.$$

Moreover, there exists a function $u_n\colon [-d,t_k^n] \to E$ defined by $u_n = \psi$ on $[a-d,a]$ and

$$u_n(t) = \int_a^t K(t,s) f_k^n(s,u_n(s))\,\mathrm{d}s, \quad t\in[a,t_k^n]$$

where $f_k^n(t,x) = f^d(t,\tau_{t_k^n}\xi_n^k(\cdot,x))$ and $\xi_k^n\colon [a-d,t_k^n]\times E\to E$ is defined by

$$\xi_k^n(t,x) = \begin{cases} u_n(t) & \text{if } t\in[a-d,t_{k-1}^n],\\ u_n(t_{k-1}^n) + n(t-t_{k-1}^n)(x-u_n(t_{k-1}^n)) & \text{if } t\in[t_{k-1}^n,t_k^n].\end{cases}$$

Assume that $\xi_{k+1}^n\colon [a-d,t_{k+1}^n]\times E \to E$ is a function defined by

$$\xi_{k+1}^n(t,x) = \begin{cases} u_n(t) & \text{if } t\in[a-d,t_k^n],\\ u_n(t_k^n) + n(t-t_k^n)(x-u_n(t_k^n)) & \text{if } t\in[t_k^n,t_{k+1}^n].\end{cases}$$

Let $f^n_{k+1}\colon [a,t^n_{k+1}]\times E\to E$ be defined by $f^n_{k+1}(t,x)=f^d(t,\tau_{t^n_{k+1}}(\xi^n_{k+1}(\cdot,x)))$. According to Theorem 3.1 there exists a function $u^{k+1}_n\colon [a,t^n_{k+1}]\to E$ such that for each $t\in[a,t^n_{k+1}]$

$$u^{k+1}_n(t)=\int_a^t K(t,s)f^n_{k+1}(s,u^{k+1}_n(s))\,\mathrm{d}s.$$

Put $u_n=u^{k+1}_n$ on $[t^n_k,t^n_{k+1}]$. Then we can consider u_n is defined on $[a-d,t^n_{k+1}]$ so that $u_n=\psi$ on $[a-d,a]$ and for each $t\in[a,t^n_{k+1}]$

$$u_n(t)=\int_a^t K(t,s)f^n_{k+1}(s,u_n(s))\,\mathrm{d}s.$$

Therefore for each $n\in\mathbb{N}$, there exists a continuous function u_n such that $u_n=\psi$ on $[a-d,a]$ and for each $t\in[a,b]$

$$u_n(t)=\int_a^t K(t,s)f^d(s,\tau_{t^n_k}\xi^n_k(\cdot,u_n(s)))\,\mathrm{d}s,$$

where $k\in\{1,2,3,\dots,n\}$ and $t^n_{k-1}\leqslant t\leqslant t^n_k$. Set $H=\{u_n\colon\ n\in\mathbb{N}\}$. If $t_1,t_2\in[a,b]$ and $t_1<t_2$, then

$$\begin{aligned}\|u_n(t_1)-u_n(t_2)\| &\leqslant \int_a^{t_1}|K(t_1,s)-K(t_2,s)|\,\|f^d(s,\tau_{t^n_k}\xi^n_k(\cdot,u_n(s)))\|\,\mathrm{d}s\\ &\quad+\int_{t_1}^{t_2}|K(t_2,s)|\,\|f^d(s,\tau_{t^n_k}\xi^n_k(\cdot,u_n(s)))\|\,\mathrm{d}s\\ &\leqslant \int_a^{t_1}|K(t_1,s)-K(t_2,s)|(c_1(s)+c_2(s)\|u_n(s)\|)\,\mathrm{d}s\\ &\quad+\alpha\int_{t_1}^{t_2}\mathrm{e}^{-\sigma|t_2-s|}(c_1(s)+c_2(s)\|u_n(s)\|)\,\mathrm{d}s.\end{aligned}$$

Furthermore, $|K(t,s)|\leqslant\alpha\mathrm{e}^{-\sigma|t-s|}$ and $u_n=\psi$ on $[a-d,a]$; hence, H is equicontinuous in $C([a-d,b],E)$. Moreover, we can define a mapping ψ' by

$$\psi'(x)(t)=\int_a^t K(t,s)f^d(s,\tau_{t^n_k}\xi^n_k(\cdot,x(s)))\,\mathrm{d}s\quad\text{for each } t\in[a,b],$$

so $\psi'(H(t))=\psi'(\{u_n(t)\colon\ n\in\mathbb{N}\})$ and $\psi(H(a))=0$.

We can show that $\gamma(\psi'(H(t)))=0$ for all $t\in[a,b]$. Let $a\leqslant t<x\leqslant b$. In the same way as in the proof of Theorem 3.1 if we replace the interval $[t-q,t+q]$ by $[t,x]$ and the set D by H, then

$$\gamma(\psi'(H(t)))\leqslant\int_t^x|K(t,s)|w(s,\gamma(H(s)))\,\mathrm{d}s.$$

Define $\varrho(t) := \gamma(H(t))$; since $\gamma(H(t)) = \gamma(\psi'(H(t)))$, so $\varrho(a) = 0$ and

$$\varrho(x) - \varrho(t) \leqslant \int_t^x |K(t,s)| w(s, \varrho(s)) \, \mathrm{d}s.$$

Therefore $\dot{\varrho}(t) \leqslant \alpha \mathrm{e}^{-\sigma|t-s|} w(t, \varrho(t))$ a.e., thus $\varrho \equiv 0$. By Ascoli's theorem the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly uniformly to a function $u \in C_E([a-d,b],E)$ such that $u = \psi$ on $[a-d,a]$. For simplicity we will denote the function $f^d(s, \tau_{t_k^n} \xi_k^n(\cdot, u_n(s)))$ by $h_n^k(s)$ and we have $\xi(\{h_n^k(t)\colon\ n \in \mathbb{N}\}) = 0$, so $\{h_n^k(t)\colon\ n \in \mathbb{N}\}$ is relatively weakly compact. If we create a multivalued function $F(t) = \overline{\operatorname{conv}}\, \{h_n^k(t)\colon n \in \mathbb{N}\}$, then $F(t)$ is nonempty convex and weakly compact. The set

$$\delta_F^1 := \{l \in L^1(I,E)\colon\ l(t) \in F(t)\}$$

is nonempty convex and weakly compact, thus by the Eberlein-Śmulian theorem there exists a subsequence $(h_{n_j}^k)$ of (h_n^k) such that $h_{n_j}^k \to l$ weakly, $l \in \delta_F^1$. Thus u_n tends weakly to $\int_a^t K(t,s) l(s) \, \mathrm{d}s$. Moreover, $u_n \in C_E([a-d,b])$ and $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u on each compact subset of $[a-d,b]$ and u is uniformly continuous on $[a-d,a]$. But for each $t \in [a,b]$ we can find $n \in \mathbb{N}$ such that $d > (b-a)/n$ and $t \in [t_{k-1}^n, t_k^n]$ for some k in the set $\{1,2,\ldots,n\}$. Moreover,

$$\begin{aligned}
\|\tau_{t_k^n} \xi_k^n(\cdot, u_n(t)) - \tau_t u\| \leqslant{}& \sup_{s\in[-d,(a-b)/n]} [\|\xi_k^n(t_k^n + s, u_n(t)) - u(t_k^n + s)\| \\
&\qquad + \|u(t_k^n + s) - u(t+s)\|] \\
&+ \sup_{s\in[(a-b)/n,0]} [(\|u_n(t_{k-1}^n) + n(t_k^n + s - t_{k-1}^n) \\
&\qquad \times (u_n(t) - u_n(t_{k-1}^n)) - u(t_k^n + s)\|) \\
&\qquad + \|u(t_k^n + s) - u(t+s)\|] \\
\leqslant{}& \sup_{s\in[-d,(a-b)/n]} [\|u_n(t_k^n + s) - u(t_k^n + s)\| \\
&\qquad + \|u(t_k^n + s) - u(t+s)\|] \\
&+ \sup_{s\in[(a-b)/n,0]} [((b-a)\|(u_n(t) - u_n(t_{k-1}^n))\| \\
&\qquad + \|u_n(t_{k-1}^n) - u(t_k^n + s)\| \\
&\qquad + \|u(t_k^n + s) - u(t+s)\|)] \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus by Lemma 2.7 we conclude that $u(\cdot)$ is the desired solution of (Q). □

There are really only a few results dealing with weak solutions for delayed problems and the proposed one seems to be interesting in this subject. The results presented here are of a more general form (quasi-linear problem and much better compactness-type assumption). In the important case $\widehat{\mathcal{L}}(t) \equiv 0$ Theorem 4.1 generalizes Theorem 2.10. In [3] the authors formulated a suggestion how to apply the results presented in this paper to retarded lattice dynamical systems.

In the next theorem we use a (K, N, p)-measure of weak noncompactness. The Kuratowski measure of noncompactness is (K, N, p)-measure of weak noncompactness, see [5], [1]; hence, we get generalizations of results so we have a generalization for Theorem 3.3 and improvement for Theorem 2 in [37] and Theorem 9 in [14]. In the following theorem we have a finite delay and we obtain similar result to that for problem (P).

Theorem 4.2. *We assume:*

(H_1) $f^d\colon [a,b] \times C_E([-d,0]) \to E$ *is a function such that*
 (i) $t \mapsto f^d(t,\varphi)$ *is measurable,*
 (ii) $\varphi \mapsto f^d(t,\varphi)$ *is continuous,*
 (iii) *there exist two real nonnegative functions* c_1, c_2 *integrable on* $[a,b]$ *and two constants* C_1 *and* C_2 *with*

$$\int_a^b c_1(s)\,\mathrm{d}s \leqslant C_1, \quad \int_a^b c_2(s)\,\mathrm{d}s \leqslant C_2,$$

where $0 < C_2 < \frac{1}{2}(1-\mathrm{e}^{-\sigma})/\alpha$ *and* $\|f(t,\varphi)\| \leqslant c_1(t) + c_2(t)\|\varphi(0)\|$ *for each* $t \in [a,b]$ *and* $\varphi \in C_E([-d,0])$.

(H_2) $\widehat{\mathcal{L}}\colon [a,b] \to L(E)$ *is a strongly measurable and Bochner integrable operator on* $[a,b]$.

(H_3) *For each* $\varepsilon > 0$ *there exists a closed subset* I_ε *of* $[a,b]$ *with* $\lambda([a,b] - I_\varepsilon) < \varepsilon$ *such that for any nonempty bounded subset* A *of* $C_E([-d,0])$ *and for each closed subset* $J \subseteq I_\varepsilon$, *one has*

$$\gamma(F(J \times A)) \leqslant \sup_{t \in J} h(t, \beta(A(0))).$$

(H_4) *Let*

$$L = \sup\biggl\{\int_a^b |K(t,s)| h(t,\gamma(B(s))\,\mathrm{d}s\colon\ t \in [a,b]\biggr\} \\ \leqslant \sup\{\gamma(B(s))\colon\ s \in [a,b]\},$$

where B *is a bounded subset of* $C([a,b],E)$.

Then, for each $\psi \in C_E([a-d,a])$ *such that* $\psi(a) = 0$, *problem* (Q) *has at least one bounded solution on the interval* $[a-d,b]$.

P r o o f. We partition the closed interval $[a,b]$ by the points: $t_i^n=(ib+(n-i)a)/n$ where $i=0,1,2,\dots,n$ and u_n will be defined by mathematical induction. Along the same lines as in [17], [16] we use some methods for functional equations. For each $(t,x)\in[a-d,t_1^n]\times E$ put

$$\Phi_1^n(t,x)=\begin{cases}\psi(t) & \text{if } t\in[a-d,a],\\ n(t-a)x & \text{if } t\in[a,t_1^n],\end{cases}$$

where n is a positive integer. Let $f_1^n\colon [a,t_1^n]\times E\to E$ be a function defined by $f_1^n(t,x)=f^d(t,\tau_{t_1^n}(\Phi_1^n(\cdot,x)))$. By Theorem 3.2 there is a bounded function $u_n\colon [a-d,t_1^n]\to E$ with $u_n=\psi$ on $[a-d,a]$ and for each $t\in[a,t_1^n]$

$$u_n(t)=\int_a^t K(t,s)f_1^n(s,u_n(s))\,\mathrm{d}s.$$

Now we can assume that the function u_n such that $u_n=\psi$ on $[a-d,a]$ and

$$u_n(t)=\int_a^t K(t,s)f_k^n(s,u_n(s))\,\mathrm{d}s,\quad t\in[a,t_k^n]$$

with $f_k^n(t,x)=f^d(t,\tau_{t_k^n}\Phi_n^k(\cdot,x))$ where $\Phi_k^n\colon [a-d,t_k^n]\times E\to E$ is defined by

$$\Phi_k^n(t,x)=\begin{cases}u_n(t) & \text{if } t\in[a-d,t_{k-1}^n],\\ u_k^n(t_{k-1}^n)+n(t-t_{k-1}^n)(x-u_k^n(t_{k-1}^n)) & \text{if } t\in[t_{k-1}^n,t_k^n].\end{cases}$$

We define $\Phi_{k+1}^n\colon [a-d,t_{k+1}^n]\times E\to E$ by

$$\Phi_{k+1}^n(t,x)=\begin{cases}u_n(t) & \text{if } t\in[a-d,t_k^n],\\ u_n(t_k^n)+n(t-t_k^n)(x-u_n(t_k^n)) & \text{if } t\in[t_k^n,t_{k+1}^n].\end{cases}$$

Now if $f_{k+1}^n\colon [a,t_{k+1}^n]\times E\to E$ is defined by $f_{k+1}^n(t,x)=f^d(t,\tau_{t_{k+1}^n}(\Phi_{k+1}^n(\cdot,x)))$, then f_{k+1}^n satisfies the conditions of Theorem 3.1. Hence there is a bounded function $u_n^{k+1}\colon [a,t_{k+1}^n]\to E$ such that for each $t\in[a,t_{k+1}^n]$

$$u_n^{k+1}(t)=\int_a^t K(t,s)f_{k+1}^n(s,u_n^{k+1}(s))\,\mathrm{d}s.$$

Put $u_n=u_n^{k+1}$ on $[t_k^n,t_{k+1}^n]$. Then we can consider u_n is defined on $[a-d,t_{k+1}^n]$ with $u_n=\psi$ on $[a-d,a]$ and for each $t\in[a,t_{k+1}^n]$, u_n is defined by

$$u_n(t)=\int_a^t K(t,s)f_{k+1}^n(s,u_n(s))\,\mathrm{d}s.$$

Consequently, for all $n \in \mathbb{N}$ we have a continuous bounded function u_n such that $u_n = \psi$ on $[a-d, a]$ and for each $t \in [a, b]$, u_n is defined by

$$u_n(t) = \int_a^t K(t,s) f^d(s, \tau_{t_k^n} \Phi_k^n(\cdot, u_n(s)))\,\mathrm{d}s,$$

where $k \in \{1, 2, 3, \dots, n\}$ is such that $t_{k-1}^n \leqslant t \leqslant t_k^n$. Set $W = \{u_n \colon n \in \mathbb{N}\}$. Now if $t_1, t_2 \in [a, b]$ and $t_1 < t_2$, then

$$\begin{aligned}
\|u_n(t_1) - u_n(t_2)\| &\leqslant \int_a^{t_1} |K(t_1,s) - K(t_2,s)|\, \|f^d(s, \tau_{t_k^n} \Phi_k^n(\cdot, u_n(s)))\|\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} |K(t_2,s)|\, \|f^d(s, \tau_{t_k^n} \Phi_k^n(\cdot, u_n(s)))\|\,\mathrm{d}s \\
&\leqslant \int_a^{t_1} |K(t_1,s) - K(t_2,s)| (c_1(s) + c_2(s)\|u_n(s)\|)\,\mathrm{d}s \\
&\quad + \alpha \int_{t_1}^{t_2} \mathrm{e}^{-\sigma|t_2 - s|} (c_1(s) + c_2(s)\|u_n(s)\|)\,\mathrm{d}s.
\end{aligned}$$

Since u_n is bounded, $|K(t,s)| \leqslant \alpha \mathrm{e}^{-\sigma|t-s|}$ and $u_n = \psi$ on $[a-d, a]$ hence W is equicontinuous in $C_E[a-d, b]$. Moreover, we can define a mapping ψ' by

$$\psi'(x)(t) = \int_a^t K(t,s) f(s, x(s))\,\mathrm{d}s \quad \text{for each } t \in [a, b],$$

so $\psi'(H(t)) = \psi'(\{u_n(t) \colon n \in \mathbb{N}\})$ and $\psi(H(a)) = 0$. We can show that $\psi'(H(t)) = 0$ for all $t \in [a, b]$.

Consider $a \leqslant t < x \leqslant b$. Along the same lines as in the proof Theorem 3.1 if we replace the interval $[t-q, t+q]$ by $[t, x]$ and the set D by W, then we have

$$\gamma(\psi'(H(t))) \leqslant \int_P |K(t,s)| h(s, \gamma(H(s)))\,\mathrm{d}s \leqslant \int_t^x |K(t,s)| h(s, \gamma(H(s)))\,\mathrm{d}s$$

and

$$\gamma(\psi'(H(x)) \leqslant \gamma(\psi'(W)(t)) + \gamma\Big(\int_t^x K(t,s) f(s, H(s))\,\mathrm{d}s\Big).$$

Define $\varrho(t) := \gamma(H(t))$; since $\gamma(H(t)) = \gamma(\psi'(H(t)))$, so $\varrho(a) = 0$ and we get

$$\varrho(x) - \varrho(t) \leqslant \gamma\Big(\int_t^x K(t,s) f(s, H(s))\,\mathrm{d}s\Big) \leqslant \int_t^x |K(t,s)| h(s, \varrho(s))\,\mathrm{d}s.$$

Therefore $\dot{\varrho}(t) \leqslant \alpha \mathrm{e}^{-\sigma|t-s|} h(t, \varrho(t))$ a.e., thus $\varrho \equiv 0$.

By Ascoli's theorem the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly uniformly to a function $u \in C_E([a-d,b])$ with $u=\psi$ on $[a-d,a]$.

For simplicity we will denote the function $f^d(s,\tau_{t_k^n}\Phi_k^n(\cdot,u_n(s)))$ by $h_n^k(s)$ and we have $\Phi(\{h_n^k(t)\colon\ n\in\mathbb{N}\})=0$, so $\{h_n^k(t)\colon\ n\in\mathbb{N}\}$ is relatively weakly compact.

Now if we create a multivalued function

$$F(t)=\overline{\operatorname{conv}}\,\{h_n^k(t)\colon\ n\in\mathbb{N}\},$$

then $F(t)$ is nonempty convex and weakly compact. The set

$$\delta_F^1 := \{l \in L^1(I,E)\colon\ l(t)\in F(t)\}$$

is nonempty convex and weakly compact, thus by the Eberlein-Śmulian theorem there exists a subsequence $(h_{n_j}^k)$ of (h_n^k) such that $h_{n_j}^k \to l$ weakly, $l\in\delta_F^1$. Thus u_n tends weakly to $\int_a^t K(t,s)l(s)\,\mathrm{d}s$. Moreover, for each $n\in\mathbb{N}$, $u_n\in C_E([a-d,b])$, u_n converges uniformly to u on each compact subset of $[a-d,b]$ and u is uniformly continuous on $[a-d,a]$. But for each $t\in[a,b]$ we can find $n\in\mathbb{N}$ such that $d>(b-a)/n$ and $t\in[t_{k-1}^n,t_k^n]$ for some k in the set $\{1,2,\dots,n\}$. Now

$$\begin{aligned}
\|\tau_{t_k^n}\Phi_k^n(\cdot,u_n(t))-\tau_t u\| \leqslant{}& \sup_{s\in[-d,(a-b)/n]}[\|\Phi_k^n(t_k^n+s,u_n(t))-u(t_k^n+s)\| \\
&\qquad + \|u(t_k^n+s)-u(t+s)\|] \\
&+ \sup_{s\in[(a-b)/n,0]}[(\|u_n(t_{k-1}^n)+n(t_k^n+s-t_{k-1}^n) \\
&\qquad \times (u_n(t)-u_n(t_{k-1}^n))-u(t_k^n+s)\|) \\
&\qquad + \|u(t_k^n+s)-u(t+s)\|] \\
\leqslant{}& \sup_{s\in[-d,(a-b)/n]}[\|u_n(t_k^n+s)-u(t_k^n+s)\| \\
&\qquad + \|u(t_k^n+s)-u(t+s)\|] \\
&+ \sup_{s\in[(a-b)/n,0]}[((b-a)\|u_n(t)-u_n(t_{k-1}^n)\| \\
&\qquad + \|u_n(t_{k-1}^n)-u(t_k^n+s)\| \\
&\qquad + \|u(t_k^n+s)-u(t+s)\|)] \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Thus by Lemma 2.7 we conclude that $u(\cdot)$ is the desired solution of (Q). □

Theorem 4.3. *We assume:*

(H$_1'$) $f'^d\colon\ [a,b]\times C([-d,0],E)\to E$ *is a function such that*

(i) $t\mapsto f'^d(t,\varphi)$ *is measurable,*

(ii) $\varphi \mapsto f'^d(t,\varphi)$ *is continuous,*

(iii) *for all* $\varphi \in C([-d,0],E)$, $f'^d([a,b]\times\{\varphi\})$ *is separable.*

(H_2) $\widehat{\mathcal{L}}\colon [a,b] \to L(E)$ *is a strongly measurable and Bochner integrable operator on* $[a,b]$ *and the linear equation*

$$\dot{x}(t) = \widehat{\mathcal{L}}(t)x(t)$$

has a trichotomy with constants $\alpha \geqslant 1$ *and* $\sigma > 0$.

(H_3) *There exist two real nonnegative functions* c_1, c_2 *integrable on* $[a,b]$ *and two constants* C_1 *and* C_2 *with*

$$\int_a^b c_1(s)\,\mathrm{d}s \leqslant C_1, \quad \int_a^b c_2(s)\,\mathrm{d}s \leqslant C_2,$$

where $0 < C_2 < (1-\mathrm{e}^{-\sigma})/(2\alpha)$ *and* $\|f'^d(t,\varphi)\| \leqslant c_1(t) + c_2(t)\|\varphi(0)\|$ *for each* $t \in [a,b]$ *and* $\varphi \in C([-d,0],E)$.

(H_4) *For each* $\varepsilon > 0$ *there exists a closed subset* I_ε *of* $[a,b]$ *with* $\lambda([a,b]-I_\varepsilon) < \varepsilon$ *such that for any nonempty bounded subset* A *of* $C([-d,0],E)$ *and for each closed subset* $J \subseteq I_\varepsilon$, *one has*

$$\gamma(f'^d(J\times A)) \leqslant \sup_{t\in J} h(t,\gamma(A(0))).$$

Then, for each $\psi \in C([a-d,a],E)$ *such that* $\psi(a) = 0$, *problem* (Q) *has a weak solution on the interval* $[a-d,b]$.

P r o o f. We partition the closed interval $[a,b]$ by the points: $t_i^n = (ib+(n-i)a)/n$ where $i = 0,1,2,\ldots,n$. For each $n \in \mathbb{N}$, let $\xi_1^n\colon [a-d,t_1^n]\times E \to E$ be a function defined by

$$\xi_1^n(t,x) = \begin{cases} \psi(t) & \text{if } t \in [a-d,a],\\ n(t-a)x & \text{if } t \in [a,t_1^n]. \end{cases}$$

Assume that $f_1'^n\colon [a,t_1^n]\times E \to E$ is defined by $f_1'^n(t,x) = f'^d(t,\tau_{t_1^n}(\xi_1^n(\cdot,x)))$. By Theorem 3.3 there is a function $v_n'\colon [a-d,t_1^n] \to E$ such that $v_n' = \psi$ on $[a-d,a]$ and for each $t \in [a,t_1^n]$

$$v_n'(t) = \int_a^t K(t,s) f_1'^n(s,v_n'(s))\,\mathrm{d}s.$$

As in Theorem 4.1 there exists a function $u_n\colon [-d,t_k^n] \to E$ defined by $u_n = \psi$ on $[a-d,a]$ and

$$u_n(t) = \int_a^t K(t,s) f_k'^n(s,u_n(s))\,\mathrm{d}s, \quad t \in [a,t_k^n]$$

where $f_k'^n(t,x) = f'^d(t, \tau_{t_k^n}\xi_n^k(\cdot,x))$ and $\xi_k^n\colon [a-d,t_k^n]\times E\to E$ is defined by

$$\xi_k^n(t,x) = \begin{cases} u_n(t) & \text{if } t\in[a-d,t_{k-1}^n],\\ u_n(t_{k-1}^n)+n(t-t_{k-1}^n)(x-u_n(t_{k-1}^n)) & \text{if } t\in[t_{k-1}^n,t_k^n].\end{cases}$$

At this point we can complete the proof as that of Theorem 4.1. □

In the next theorem we let $\mathfrak{h}\colon [a,b]\times\mathbb{R}^a\to\mathbb{R}^+$ be a Carathéodory function and, for each bounded subset Z of $[a,b]\times\mathbb{R}^a$, let there exist a measurable function m_Z such that $\mathfrak{h}(t,s)\leqslant m_Z(t)$ for each $(t,s)\in Z$ and m is integrable on $[c,T]$ for each c; $a<c\leqslant b$. Moreover, let for each c; $a<c\leqslant b$, the identically zero function be the only absolutely continuous function on $[a,c]$ which satisfies $\dot u(t)=\mathfrak{h}(t,u(t))$ a.e. on $[a,c]$ such that the right hand derivative of $u(t)$ at $t=a$, $D_+u(a)$, exists and $D_+u(a)=u(a)=0$.

We note that the assumptions on $\mathfrak{h}$ are weaker than those on a Kamke function w.

Theorem 4.4. *In the setting of Theorem* 4.3 *we replace a Kamke function w by a function $\mathfrak{h}$ and suppose that f'^d is bounded and continuous instead of* (i) *and* (ii) *in condition* (H_1'). *Then, for each $\psi\in C([a-d,a],E)$ such that $\psi(a)=0$, problem* (Q) *has a weak solution on the interval $[a-d,b]$.*

We omit the proof since it runs as the proof of Theorem 4.3 except that we replace the use of Theorem 3.3 by that of Theorem 3.4 to find a continuous function v_n such that $v_n=\psi$ on $[a-d,a]$ and for each $t\in[a,t_1^n]$

$$v_n(t)=\int_a^t K(t,s)f_1^n(s,v_n(s))\,\mathrm{d}s.$$

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