

LINEAR NATURAL OPERATORS LIFTING  $p$ -VECTORS  
TO TENSORS OF TYPE  $(q, 0)$  ON WEIL BUNDLES

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(Received June 10, 2015)

*Abstract.* We give a classification of all linear natural operators transforming  $p$ -vectors (i.e., skew-symmetric tensor fields of type  $(p, 0)$ ) on  $n$ -dimensional manifolds  $M$  to tensor fields of type  $(q, 0)$  on  $T^A M$ , where  $T^A$  is a Weil bundle, under the condition that  $p \geq 1$ ,  $n \geq p$  and  $n \geq q$ . The main result of the paper states that, roughly speaking, each linear natural operator lifting  $p$ -vectors to tensor fields of type  $(q, 0)$  on  $T^A$  is a sum of operators obtained by permuting the indices of the tensor products of linear natural operators lifting  $p$ -vectors to tensor fields of type  $(p, 0)$  on  $T^A$  and canonical tensor fields of type  $(q - p, 0)$  on  $T^A$ .

*Keywords:* natural operator; Weil bundle

*MSC 2010:* 58A32

## 1. INTRODUCTION

In this paper we give a classification of all linear natural operators transforming skew-symmetric tensor fields of type  $(p, 0)$  (which we call  $p$ -vectors) on  $n$ -dimensional manifolds  $M$  to tensor fields of type  $(q, 0)$  on  $T^A M$ , where  $T^A$  is a Weil bundle, under the condition that  $p \geq 1$ ,  $n \geq p$  and  $n \geq q$ . Similar problems in some special cases were studied earlier by Kolář [7], Grabowski and Urbański [4], and Mikulski [10]. The theorem we prove here generalizes the results of [2] and [1]. The former of the two papers was devoted to the case  $q = p$ , whereas in the latter canonical tensor fields of type  $(p, 0)$  on Weil bundles were studied. We now prove that in the general case each linear natural operator lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$  is a sum of operators obtained by permuting the indices of the tensor products of linear natural operators lifting  $p$ -vectors to tensors of type  $(p, 0)$  on  $T^A$  and canonical tensor fields of type  $(q - p, 0)$  on  $T^A$ . Therefore in the general case each natural operator under

consideration can be constructed from those described in [2] and [1] by using well known operations on tensors. However, the proof of this fact is much more difficult than the proofs in both the special cases and needs some new ideas.

## 2. BACKGROUND ON THE WEIL BUNDLES

For the convenience of the reader we first summarize without proofs some basic information on Weil bundles. As was proved by Eck [3], Kainz and Michor [5] and Luciano [9], every product preserving bundle functor is equivalent to a Weil bundle. A new approach to this matter was presented by Kolář in [6]. We give a brief sketch of this result following the last paper. For a general theory of natural bundles and natural operators we refer the reader to [8].

Let  $F$  be a functor which transforms each manifold  $M$  into a locally trivial bundle  $\pi_M: FM \rightarrow M$  and each smooth map  $f: M \rightarrow N$  into a smooth map  $Ff: FM \rightarrow FN$  such that  $\pi_N \circ Ff = f \circ \pi_M$ . We call  $F$  a *bundle functor* if for every integer  $n \geq 0$  and every embedding  $f: M \rightarrow N$  between  $n$ -dimensional manifolds  $Ff$  is an embedding and  $Ff(FM) = \pi_N^{-1}(f(M))$ . Hence we can identify  $FU$  with  $\pi_M^{-1}(U)$  for each open subset  $U$  of a manifold  $M$ . Such  $F$  is said to be *product preserving* if for all manifolds  $M$  and  $N$  the map  $(Fp_M, Fp_N): F(M \times N) \rightarrow FM \times FN$ , where  $p_M: M \times N \rightarrow M$  and  $p_N: M \times N \rightarrow N$  are the projections, is a diffeomorphism. Hence we can identify  $F(M \times N)$  with  $FM \times FN$ .

A *Weil algebra* is, by definition, a finite-dimensional associative and commutative  $\mathbb{R}$ -algebra  $A$  with unit which has an ideal  $N$  such that  $A/N$  is one-dimensional and  $N^{r+1} = 0$  for an integer  $r \geq 0$ . The basic examples are the algebras  $\mathbb{D}_k^r$  of  $r$ -jets at 0 of smooth functions  $\mathbb{R}^k \rightarrow \mathbb{R}$ . For an arbitrary Weil algebra  $A$  there is a surjective algebra homomorphism  $\mathbb{D}_k^r \rightarrow A$  for some integers  $r \geq 0$  and  $k \geq 0$ .

Let  $F$  be a product preserving bundle functor. Put  $A = F\mathbb{R}$ . Applying  $F$  to the addition and multiplication  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  in the field  $\mathbb{R}$  as well as to multiplying  $\mathbb{R} \rightarrow \mathbb{R}$  by any real number in  $\mathbb{R}$  we obtain an addition and multiplication  $A \times A \rightarrow A$  in  $A$  as well as multiplying  $A \rightarrow A$  by this real number in  $A$ , so  $A$  is an  $\mathbb{R}$ -algebra. In fact, it is a Weil algebra.

Conversely, let  $A$  be a Weil algebra and let  $p: \mathbb{D}_k^r \rightarrow A$  be a surjective algebra homomorphism. We say that two smooth maps  $\gamma, \delta: \mathbb{R}^k \rightarrow M$ , where  $M$  is a manifold, determine the same  $A$ -jet if  $p(j_0^r(\psi \circ \gamma)) = p(j_0^r(\psi \circ \delta))$  for every smooth function  $\psi: M \rightarrow \mathbb{R}$ . We will denote by  $j^A \gamma$  the  $A$ -jet of a smooth map  $\gamma: \mathbb{R}^k \rightarrow M$  and by  $T^A M$  the set of  $A$ -jets of all such maps. Since every chart  $\varphi: U \rightarrow \mathbb{R}^n$  on  $M$  induces the chart  $T^A U \ni j^A \gamma \mapsto (p(j_0^r(\varphi^1 \circ \gamma)), \dots, p(j_0^r(\varphi^n \circ \gamma))) \in A^n$  on  $T^A M$ ,  $T^A M$  is a manifold, and so a bundle over  $M$  with the projection  $T^A M \ni j^A \gamma \mapsto \gamma(0) \in M$ . If  $f: M \rightarrow N$  is a smooth map between manifolds then we define  $T^A f: T^A M \rightarrow T^A N$

by  $T^A f(j^A \gamma) = j^A(f \circ \gamma)$ . The functor  $T^A$  is called the *Weil bundle* induced by  $A$ . It is a product preserving bundle functor. Though the construction of  $T^A$  depends on the choice of  $p$ ,  $T^A$  is unique up to an equivalence.

Therefore we have a Weil algebra for every product preserving bundle functor and a product preserving bundle functor for every Weil algebra. These constructions are inverse to each other if isomorphic algebras and equivalent functors are identified. Thus we have a one-to-one correspondence between product preserving bundle functors and Weil algebras.

It is worth pointing out that the Weil bundle induced by the simplest nontrivial Weil algebra  $\mathbb{D}_1^1$  is nothing but the usual tangent bundle functor  $T$ .

### 3. CONSTRUCTION OF SOME NATURAL OPERATORS

We now turn to the main subject of the paper.

Fix a Weil algebra  $A$ , as well as integers  $n \geq 0$ ,  $p \geq 0$  and  $q \geq 0$ .

Let us denote by  $V^r(M)$ , where  $M$  is a smooth manifold and  $r \geq 0$  is an integer, the vector space of all tensor fields of type  $(r, 0)$  on  $M$ , and by  $SV^r(M)$  the subspace of  $V^r(M)$  consisting of all skew-symmetric tensor fields.

**Definition 3.1.** A natural operator lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$  is a system of maps  $L_M: SV^p(M) \rightarrow V^q(T^A M)$  indexed by  $n$ -dimensional manifolds and satisfying for all such manifolds  $M, N$ , every embedding  $f: M \rightarrow N$  and all  $t \in SV^p(M)$  and  $u \in SV^p(N)$  the implication

$$(3.1) \quad \bigwedge^p T f \circ t = u \circ f \implies \bigotimes^q T T^A f \circ L_M(t) = L_N(u) \circ T^A f.$$

Of course, such a natural operator  $L$  is called *linear* if the map  $L_M$  is linear for each  $n$ -dimensional manifold  $M$ .

For every integer  $r \geq 0$ , every  $k \in \{1, \dots, r\}$  and every  $a \in A$  we have the linear map  $Z_a^k: \bigotimes^r A \rightarrow \bigotimes^r A$  such that

$$Z_a^k(b_1 \otimes \dots \otimes b_r) = b_1 \otimes \dots \otimes b_{k-1} \otimes a b_k \otimes b_{k+1} \otimes \dots \otimes b_r$$

for all  $b_1, \dots, b_r \in A$ .

Suppose that  $q \geq p$ . Let  $\text{ED}_p^q(A)$  denote the vector space of all  $(q-p)$ -linear maps  $D: A \times \dots \times A \rightarrow \bigotimes^q A$  such that

$$(3.2) \quad Z_a^i \circ D = Z_a^j \circ D$$

for all  $i, j \in \{1, \dots, p\}$  and every  $a \in A$ , and

$$(3.3) \quad D(c_{p+1}, \dots, c_{k-1}, ab, c_{k+1}, \dots, c_q) = Z_a^k(D(c_{p+1}, \dots, c_{k-1}, b, c_{k+1}, \dots, c_q)) \\ + Z_b^k(D(c_{p+1}, \dots, c_{k-1}, a, c_{k+1}, \dots, c_q))$$

for every  $k \in \{p+1, \dots, q\}$  and all  $a, b, c_{p+1}, \dots, c_{k-1}, c_{k+1}, \dots, c_q \in A$ .

If  $p \geq 1$ , then elements of the vector space  $\text{ED}_p^q(A)$  may be multiplied by elements of the algebra  $A$ . Indeed, it suffices to take any  $k \in \{1, \dots, p\}$  and put

$$aD = Z_a^k \circ D$$

for every  $a \in A$  and every  $D \in \text{ED}_p^q(A)$ . By (3.2), it is immaterial which  $k \in \{1, \dots, p\}$  we choose. In addition, we see that  $\text{ED}_p^q(A)$  is an  $A$ -module.

Let  $e_1, \dots, e_n$  denote the standard basis of the vector space  $\mathbb{R}^n$ .

**Proposition 3.1.** *If  $p \geq 1$  and  $q \geq p$ , then for every  $D \in \text{ED}_p^q(A)$  there is a unique natural operator  $\overline{D}$  lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$  such that*

$$(3.4) \quad \overline{D}_U(t)(X) = \sum_{i_1=1}^n \dots \sum_{i_q=1}^n (T^A t^{i_1 \dots i_p}(X) \cdot D)(X^{i_{p+1}}, \dots, X^{i_q}) \otimes e_{i_1} \otimes \dots \otimes e_{i_q}$$

for every open subset  $U$  of  $\mathbb{R}^n$ , every  $t \in \text{SV}^p(U)$  and every  $X \in T^A U$ .

The right hand side of the above equality needs some explanation. Since  $T^A \mathbb{R} = A$  and  $t^{i_1 \dots i_p} : U \rightarrow \mathbb{R}$ , we have  $T^A t^{i_1 \dots i_p}(X) \in A$  for all  $i_1, \dots, i_p \in \{1, \dots, n\}$ . Moreover, since  $T^A U$  is an open subset of  $A^n$ , the tangent bundle  $TT^A U$  can be identified with  $T^A U \times A^n$ . But the isomorphism  $A^n \ni X \mapsto \sum_{i=1}^n X^i \otimes e_i \in A \otimes \mathbb{R}^n$  enables us to identify  $A^n$  with  $A \otimes \mathbb{R}^n$ , and consequently  $\bigotimes^q A^n$  with  $\bigotimes^q A \otimes \bigotimes^q \mathbb{R}^n$ .

In order to prove the proposition, we first show a lemma.

Suppose now that  $q \geq p$ . Let  $E_p(A)$  denote the subspace of the vector space  $\bigotimes^p A$  consisting of the tensors  $V$  which for all  $i, j \in \{1, \dots, p\}$  and every  $a \in A$  satisfy the condition  $Z_a^i(V) = Z_a^j(V)$ , and let  $D_{q-p}(A)$  denote the vector space of all  $(q-p)$ -linear maps  $F : A \times \dots \times A \rightarrow \bigotimes^{q-p} A$  such that  $F(c_{p+1}, \dots, c_{k-1}, ab, c_{k+1}, \dots, c_q) = Z_a^{k-p}(F(c_{p+1}, \dots, c_{k-1}, b, c_{k+1}, \dots, c_q)) + Z_b^{k-p}(F(c_{p+1}, \dots, c_{k-1}, a, c_{k+1}, \dots, c_q))$  for all  $a, b, c_{p+1}, \dots, c_{k-1}, c_{k+1}, \dots, c_q \in A$  and every  $k \in \{p+1, \dots, q\}$ .

**Lemma 3.1.** *Let  $I: E_p(A) \otimes D_{q-p}(A) \rightarrow \text{ED}_p^q(A)$  be the unique linear map such that for every  $V \in E_p(A)$ , every  $F \in D_{q-p}(A)$  and all  $a_{p+1}, \dots, a_q \in A$*

$$I(V \otimes F)(a_{p+1}, \dots, a_q) = V \otimes F(a_{p+1}, \dots, a_q).$$

*Then  $I$  is an isomorphism of vector spaces.*

**Proof.** Fix a  $D \in \text{ED}_p^q(A)$ . Let  $v_1, \dots, v_m$  be a basis of the vector space  $A$ . There are uniquely determined  $F_{i_1 \dots i_p}: A \times \dots \times A \rightarrow \bigotimes^{q-p} A$ , where  $i_1, \dots, i_p \in \{1, \dots, m\}$ , such that for all  $a_{p+1}, \dots, a_q \in A$

$$D(a_{p+1}, \dots, a_q) = \sum_{i_1=1}^m \dots \sum_{i_p=1}^m v_{i_1} \otimes \dots \otimes v_{i_p} \otimes F_{i_1 \dots i_p}(a_{p+1}, \dots, a_q).$$

From the uniqueness it follows that  $F_{i_1 \dots i_p} \in D_{q-p}(A)$  for all  $i_1, \dots, i_p \in \{1, \dots, m\}$ . Let  $F_1, \dots, F_d$  be a basis of the vector space  $D_{q-p}(A)$ . By the above, there are uniquely determined  $V_1, \dots, V_d \in \bigotimes^p A$  such that for all  $a_{p+1}, \dots, a_q \in A$

$$D(a_{p+1}, \dots, a_q) = \sum_{j=1}^d V_j \otimes F_j(a_{p+1}, \dots, a_q).$$

From the uniqueness it follows that  $V_1, \dots, V_d \in E_p(A)$ . Therefore

$$D = I\left(\sum_{j=1}^d V_j \otimes F_j\right)$$

and  $I$  is a surjection. It is also an injection, because of the uniqueness of  $V_1, \dots, V_d$ , and the lemma follows.  $\square$

**Proof of Proposition 3.1.** Fix a  $D \in \text{ED}_p^q(A)$ . From what has already been proved, we have  $D = I\left(\sum_{j=1}^d V_j \otimes F_j\right)$ , where  $F_1, \dots, F_d$  is a basis of the vector space  $D_{q-p}(A)$  and  $V_1, \dots, V_d \in E_p(A)$  are uniquely determined. For every  $n$ -dimensional manifold  $M$  and every  $t \in \text{SV}^p(M)$  we put

$$\overline{D}_M(t) = \sum_{j=1}^d \overline{V}_j(t) \otimes \overline{F}_j,$$

where  $\overline{V}_j$  with  $j \in \{1, \dots, d\}$  is the linear natural operator lifting  $p$ -vectors to tensors of type  $(p, 0)$  on  $T^A$  induced by  $V_j$  in the manner described in [2], and where  $\overline{F}_j$  with  $j \in \{1, \dots, d\}$  is the canonical tensor of type  $(q-p, 0)$  on  $T^A$  induced by  $F_j$  in

the manner described in [1]. It is known that for every  $j \in \{1, \dots, d\}$ , every open subset  $U$  of  $\mathbb{R}^n$ , every  $t \in \text{SV}^p(U)$  and every  $X \in T^A U$

$$\begin{aligned}\overline{V}_{j,U}(t)(X) &= \sum_{i_1=1}^n \dots \sum_{i_p=1}^n (T^A t^{i_1 \dots i_p}(X) \cdot V_j) \otimes e_{i_1} \otimes \dots \otimes e_{i_p}, \\ \overline{F}_{j,U}(X) &= \sum_{i_{p+1}=1}^n \dots \sum_{i_q=1}^n F_j(X^{i_{p+1}}, \dots, X^{i_q}) \otimes e_{i_{p+1}} \otimes \dots \otimes e_{i_q}.\end{aligned}$$

Using these formulas it is easily seen that  $\overline{D}$  satisfies (3.4). Since we may take as  $f$  in (3.1) the inverse of any chart on an  $n$ -dimensional manifold, it is obvious that  $\overline{D}$  satisfying (3.4) is unique. This proves the proposition.  $\square$

#### 4. THE MAIN RESULT

Let  $\Omega$  denote the group of all permutations of the set  $\{1, \dots, q\}$  and let  $\omega \in \Omega$ . For every manifold  $M$  we define  $\omega_M$  to be the linear map  $V^q(M) \rightarrow V^q(M)$  such that

$$\omega_M(V_1 \otimes \dots \otimes V_q) = V_{\omega(1)} \otimes \dots \otimes V_{\omega(q)}$$

for all  $V_1, \dots, V_q \in V^1(M)$ . Of course, if  $L$  is a linear natural operator lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$ , then so is the system of maps  $\omega_{T^A M} \circ L_M$  indexed by  $n$ -dimensional manifolds. We will denote it by  $\omega \circ L$ .

For all  $k_1, \dots, k_p \in \{1, \dots, q\}$  such that  $k_1 < \dots < k_p$  we define  $\omega_{k_1 \dots k_p}$  to be the permutation of the set  $\{1, \dots, q\}$  satisfying  $\omega_{k_1 \dots k_p}(1) = k_1, \dots, \omega_{k_1 \dots k_p}(p) = k_p$  and  $\omega_{k_1 \dots k_p}(i) < \omega_{k_1 \dots k_p}(j)$  for all  $i, j \in \{p+1, \dots, q\}$  such that  $i < j$ .

We can now formulate our main result.

**Theorem 4.1.** *Suppose that  $p \geq 1$ ,  $n \geq p$  and  $n \geq q$ . Then for every linear natural operator  $L$  lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$  there are uniquely determined  $D_{k_1 \dots k_p} \in \text{ED}_p^q(A)$ , where  $k_1, \dots, k_p \in \{1, \dots, q\}$  and  $k_1 < \dots < k_p$ , such that*

$$L = \sum_{1 \leq k_1 < \dots < k_p \leq q} \omega_{k_1 \dots k_p}^{-1} \circ \overline{D}_{k_1 \dots k_p}.$$

## 5. PROOF OF THE MAIN RESULT

The remainder of the paper will be devoted to the proof of this theorem. Throughout the proof,  $L$  denotes a linear natural operator lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$ .

Our proof starts with several lemmas.

Let  $n \geq p$  and let  $e$  be the  $p$ -vector on  $\mathbb{R}^n$  given by the formula

$$e: \mathbb{R}^n \ni x \mapsto e_1 \wedge \dots \wedge e_p \in \bigwedge^p \mathbb{R}^n,$$

where, as usual,  $e_1, \dots, e_n$  stands for the standard basis of the vector space  $\mathbb{R}^n$ .

**Lemma 5.1.** *Suppose that  $p \geq 1$  and  $n \geq p$ . If  $L_{\mathbb{R}^n}(e) = 0$ , then  $L = 0$ .*

*Proof.* The proof of this lemma is similar to that of the analogous lemma in [2].

Let  $\alpha_1 \geq 0, \dots, \alpha_n \geq 0$  be integers. We first prove that for every  $i \in \{0, \dots, p-1\}$  we have  $L_{\mathbb{R}^n}(e_{\alpha, i})|_{T_0^A \mathbb{R}^n} = 0$ , where  $e_{\alpha, i}: \mathbb{R}^n \ni x \mapsto (x^1)^{\alpha_1} \dots (x^i)^{\alpha_i} e_1 \wedge \dots \wedge e_p \in \bigwedge^p \mathbb{R}^n$ . The proof is by induction on  $i$ . Let  $i \geq 1$  and let  $g: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ , where  $\varepsilon > 0$ , be an embedding such that  $g(0) = 0$  and  $g' = 1 + g^{\alpha_i}$ . If  $L_{\mathbb{R}^n}(e_{\alpha, i-1})|_{T_0^A \mathbb{R}^n} = 0$ , then (3.1) with

$$f: \mathbb{R}^{i-1} \times (-\varepsilon, \varepsilon) \times \mathbb{R}^{n-i} \ni x \mapsto (x^1, \dots, x^{i-1}, g(x^i), x^{i+1}, \dots, x^n) \in \mathbb{R}^n,$$

$t = e_{\alpha, i-1}$  and  $u = e_{\alpha, i-1} + e_{\alpha, i}$  yields  $L_{\mathbb{R}^n}(e_{\alpha, i})|_{T_0^A \mathbb{R}^n} = 0$ , as desired. Next, consider  $e_{\alpha}: \mathbb{R}^n \ni x \mapsto (x^1)^{\alpha_1} \dots (x^n)^{\alpha_n} e_1 \wedge \dots \wedge e_p \in \bigwedge^p \mathbb{R}^n$ . Let  $g: (-\varepsilon, \varepsilon)^{n-p+1} \rightarrow \mathbb{R}$  be such that  $g(0) = 0$ ,  $(\partial g / \partial x^p)(x^p, \dots, x^n) = 1 + g(x^p, \dots, x^n)^{\alpha_p} (x^{p+1})^{\alpha_{p+1}} \dots (x^n)^{\alpha_n}$  for every  $(x^p, \dots, x^n) \in (-\varepsilon, \varepsilon)^{n-p+1}$  and that

$$f: \mathbb{R}^{p-1} \times (-\varepsilon, \varepsilon)^{n-p+1} \ni x \mapsto (x^1, \dots, x^{p-1}, g(x^p, \dots, x^n), x^{p+1}, \dots, x^n) \in \mathbb{R}^n$$

is an embedding. Then (3.1) with the above  $f$ ,  $t = e_{\alpha, p-1}$  and  $u = e_{\alpha, p-1} + e_{\alpha}$  leads to the equality  $L_{\mathbb{R}^n}(e_{\alpha})|_{T_0^A \mathbb{R}^n} = 0$ . Finally, for all  $i_1, \dots, i_p \in \{1, \dots, n\}$  such that  $i_1 < \dots < i_p$  we consider  $e_{\alpha, i_1, \dots, i_p}: \mathbb{R}^n \ni x \mapsto (x^1)^{\alpha_1} \dots (x^n)^{\alpha_n} e_{i_1} \wedge \dots \wedge e_{i_p} \in \bigwedge^p \mathbb{R}^n$ . Let  $\tau$  be the permutation of the set  $\{1, \dots, n\}$  such that  $\tau(1) = i_1, \dots, \tau(p) = i_p$  and let us denote  $\beta_1 = \alpha_{\tau(1)}, \dots, \beta_n = \alpha_{\tau(n)}$ . Then (3.1) with

$$f: \mathbb{R}^n \ni x \mapsto (x^{\tau^{-1}(1)}, \dots, x^{\tau^{-1}(n)}) \in \mathbb{R}^n,$$

$t = e_{\beta}$  and  $u = e_{\alpha, i_1, \dots, i_p}$  leads to the equality  $L_{\mathbb{R}^n}(e_{\alpha, i_1, \dots, i_p})|_{T_0^A \mathbb{R}^n} = 0$ .

Obviously, for every smooth  $t: \mathbb{R}^n \rightarrow \bigwedge^p \mathbb{R}^n$  and every integer  $r \geq 0$  there are polynomials  $u_{i_1 \dots i_p} \in \mathbb{R}[x^1, \dots, x^n]$  for all  $i_1, \dots, i_p \in \{1, \dots, n\}$  such that  $i_1 < \dots < i_p$  with the property that  $j_0^r t = j_0^r u$ , where  $u = \sum_{1 \leq i_1 < \dots < i_p \leq n} u_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$ . But from what has already been proved, we have the equality  $L_{\mathbb{R}^n}(u)|_{T_0^A \mathbb{R}^n} = 0$ . Therefore the Peetre theorem applied to the operator which maps each smooth  $t: \mathbb{R}^n \rightarrow \bigwedge^p \mathbb{R}^n$  to  $\mathbb{R}^n \ni x \mapsto L_{\mathbb{R}^n}(t)(x, y) \in \bigotimes^q A^n$ , where  $y$  is any point of the standard fibre of the bundle  $T^A \mathbb{R}^n \rightarrow \mathbb{R}^n$ , implies  $L_{\mathbb{R}^n}(t)|_{T_0^A \mathbb{R}^n} = 0$  for every smooth  $t: \mathbb{R}^n \rightarrow \bigwedge^p \mathbb{R}^n$ .

Now (3.1) with  $f: \mathbb{R}^n \ni x \mapsto x - c \in \mathbb{R}^n$ , where  $c \in \mathbb{R}^n$ , any smooth  $t: \mathbb{R}^n \rightarrow \bigwedge^p \mathbb{R}^n$  and  $u = t \circ f^{-1}$  shows that  $L_{\mathbb{R}^n}(t)|_{T_c^A \mathbb{R}^n} = 0$  for every  $c \in \mathbb{R}^n$ , which proves the lemma.  $\square$

If  $L$  is a linear natural operator lifting  $p$ -vectors to tensors of type  $(q, 0)$  on  $T^A$ , then there are unique smooth functions  $B^{i_1 \dots i_q}: A^n \rightarrow \bigotimes^q A$ , where  $i_1, \dots, i_q \in \{1, \dots, n\}$ , such that

$$L_{\mathbb{R}^n}(e)(X) = \sum_{i_1=1}^n \dots \sum_{i_q=1}^n B^{i_1 \dots i_q}(X) \otimes e_{i_1} \otimes \dots \otimes e_{i_q}$$

for every  $X \in A^n$ . We will call them the *coordinates of  $L$* . On account of Lemma 5.1,  $L$  is fully determined by its coordinates, provided  $p \geq 1$  and  $n \geq p$ , which we assume from now on.

Note that using the coordinates of  $L$  we may rewrite the left hand side of the consequent of (3.1) in a more convenient form. Namely, if  $U$  is an open subset of  $\mathbb{R}^n$  and  $f: U \rightarrow \mathbb{R}^n$  is an embedding, then

$$\bigotimes^q TT^A f(L_{\mathbb{R}^n}(e)(X)) = \sum_{i_1=1}^n \dots \sum_{i_q=1}^n \sum_{j_1=1}^n \dots \sum_{j_q=1}^n \left( Z_{T^A \frac{\partial f^{i_1}}{\partial x^{j_1}}(X)}^1 \circ \dots \circ Z_{T^A \frac{\partial f^{i_q}}{\partial x^{j_q}}(X)}^q \right) (B^{j_1 \dots j_q}(X)) \otimes e_{i_1} \otimes \dots \otimes e_{i_q}$$

for every  $X \in T^A U$ .

**Lemma 5.2.** *If  $\{i_1, \dots, i_q\}$  does not contain  $\{1, \dots, p\}$ , then  $B^{i_1 \dots i_q} = 0$ . Otherwise there is a unique  $(q - p)$ -linear map  $C^{i_1 \dots i_q}: A \times \dots \times A \rightarrow \bigotimes^q A$  such that for every  $X \in A^n$  we have*

$$(5.1) \quad B^{i_1 \dots i_q}(X) = C^{i_1 \dots i_q}(X^{j_1}, \dots, X^{j_{q-p}}),$$

where the sequence  $(j_1, \dots, j_{q-p})$  is determined by the conditions  $j_1 \leq \dots \leq j_{q-p}$  and  $(1, \dots, p, j_1, \dots, j_{q-p}) = (i_{\sigma(1)}, \dots, i_{\sigma(q)})$  for a permutation  $\sigma$  of the set  $\{1, \dots, q\}$ .



**P r o o f.** Since  $L$  is linear, from (3.1) with  $f: \mathbb{R}^n \ni x \mapsto (\lambda_1 x^1, \dots, \lambda_n x^n) \in \mathbb{R}^n$ , where  $\lambda_1, \dots, \lambda_n \in \mathbb{R} \setminus \{0\}$ ,  $t = e$  and  $u = \lambda_1 \dots \lambda_p e$  we have

$$(5.2) \quad \lambda_{i_1} \dots \lambda_{i_q} B^{i_1 \dots i_q}(X) = \lambda_1 \dots \lambda_p B^{i_1 \dots i_q}(\lambda_1 X^1, \dots, \lambda_n X^n)$$

for all  $i_1, \dots, i_q \in \{1, \dots, n\}$  and every  $X \in A^n$ . By continuity, (5.2) is also true for all  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ . The homogeneous function theorem now gives the assertion of the lemma, and the proof is complete.  $\square$

Note that if  $q < p$ , then Lemmas 5.1 and 5.2 yield  $L = 0$ , which completes the proof of the theorem in this case. Hence from now on we make the assumption  $q \geq p$ . We will also need the assumption  $n \geq q$  throughout the rest of the proof.

Let  $\omega \in \Omega$ . The coordinates of  $\omega \circ L$  will be denoted by  $B_\omega^{i_1 \dots i_q}$ , where  $i_1, \dots, i_q \in \{1, \dots, n\}$ . We also define  $\omega_A$  to be the linear map  $\bigotimes^q A \rightarrow \bigotimes^q A$  such that

$$\omega_A(a_1 \otimes \dots \otimes a_q) = a_{\omega(1)} \otimes \dots \otimes a_{\omega(q)}$$

for all  $a_1, \dots, a_q \in A$ . It is a simple matter to observe that

$$(5.3) \quad B_\omega^{i_1 \dots i_q} = \omega_A \circ B^{i_{\omega^{-1}(1)} \dots i_{\omega^{-1}(q)}}$$

for all  $i_1, \dots, i_q \in \{1, \dots, n\}$ .

**Lemma 5.3.** *Suppose  $B^{i_1 \dots i_q} = 0$  for all  $i_1, \dots, i_q \in \{1, \dots, n\}$  such that for every  $k \in \{1, \dots, p\}$  there is one and only one  $l \in \{1, \dots, q\}$  for which  $i_l = k$ . Then  $L = 0$ .*

**P r o o f.** Let  $g_1, \dots, g_q \in \{1, \dots, n\}$  be such that there exist integers  $r_1, \dots, r_p$  which satisfy the following conditions:

$$r_1, \dots, r_p \geq 1, \quad r_1 + \dots + r_p \leq q, \quad g_{r_1 + \dots + r_{s-1} + k} = s$$

for all  $s \in \{1, \dots, p\}$ ,  $k \in \{1, \dots, r_s\}$ , and  $p < g_{r_1 + \dots + r_{p+1}} \leq \dots \leq g_q$ . Since  $n \geq q$ , we can choose  $h_1, \dots, h_q \in \{1, \dots, n\}$  with the properties that  $h_{r_1 + \dots + r_s} = s$  for every  $s \in \{1, \dots, p\}$ ,  $h_k \neq h_l$  if either  $k, l \leq r_1 + \dots + r_p$ ,  $k \neq l$  or  $k \leq r_1 + \dots + r_p$ ,  $l > r_1 + \dots + r_p$ , and  $h_m = g_m$  if  $m > r_1 + \dots + r_p$ . We define the embedding  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by the formula

$$f^s(x) = \begin{cases} \sum_{k=1}^{r_s} x^{h_{r_1 + \dots + r_{s-1} + k}} & \text{if } s \in \{1, \dots, p\}, \\ x^s & \text{if } s \in \{p+1, \dots, n\}. \end{cases}$$

Consider the consequent of (3.1) with the above  $f, t = e$  and  $u = e$ . Comparing the parts of both sides which contain  $e_{g_1} \otimes \dots \otimes e_{g_q}$  and are linear with respect to each variable  $X^{h_{r_1+\dots+r_{s-1}+k}}$  with  $s \in \{1, \dots, p\}$ ,  $k \in \{1, \dots, r_s - 1\}$  we obtain for every  $X \in A^n$

$$\begin{aligned} \sum_{\phi \in \Phi} B^{h_{\phi(1)} \dots h_{\phi(q)}}(X) = \sum_{\psi_1 \in \Psi_1} \dots \sum_{\psi_p \in \Psi_p} C^{g_1 \dots g_q} (X^{h_{\psi_1(1)}}, \dots, X^{h_{\psi_1(r_1-1)}}, \dots, \\ X^{h_{\psi_p(r_1+\dots+r_{p-1}+1)}}, \dots, \\ X^{h_{\psi_p(r_1+\dots+r_p-1)}}, X^{h_{r_1+\dots+r_p+1}}, \dots, X^{h_q}), \end{aligned}$$

where  $\Phi$  is the group of all permutations  $\phi$  of the set  $\{1, \dots, q\}$  satisfying the conditions  $\phi\{r_1 + \dots + r_{s-1} + 1, \dots, r_1 + \dots + r_s\} \subset \{r_1 + \dots + r_{s-1} + 1, \dots, r_1 + \dots + r_s\}$  for every  $s \in \{1, \dots, p\}$  and  $\phi\{r_1 + \dots + r_p + 1, \dots, q\} = \text{id}_{\{r_1+\dots+r_p+1, \dots, q\}}$ , whereas  $\Psi_s$  for every  $s \in \{1, \dots, p\}$  is the group of all permutations of  $\{r_1 + \dots + r_{s-1} + 1, \dots, r_1 + \dots + r_s - 1\}$ . Combining this formula with (5.1) yields for every  $Y \in A^n$

$$(5.4) \quad B^{g_1 \dots g_q}(Y) = \frac{1}{(r_1 - 1)! \dots (r_p - 1)!} \sum_{\phi \in \Phi} B^{h_{\phi(1)} \dots h_{\phi(q)}}(X)$$

where  $X$  is an element of the set  $A^n$  such that  $X^{h_{r_1+\dots+r_{s-1}+k}} = Y^s$  for all  $s \in \{1, \dots, p\}$ ,  $k \in \{1, \dots, r_s - 1\}$ , and  $X^{h_m} = Y^{h_m}$  for every  $m \in \{r_1 + \dots + r_p + 1, \dots, q\}$ .

Let now  $i_1, \dots, i_q \in \{1, \dots, n\}$  be such that  $\{1, \dots, p\} \subset \{i_1, \dots, i_q\}$ . Then there are  $g_1, \dots, g_q \in \{1, \dots, n\}$  such that there exist integers  $r_1, \dots, r_p$  which satisfy the following conditions:  $r_1, \dots, r_p \geq 1$ ,  $r_1 + \dots + r_p \leq q$ ,  $g_{r_1+\dots+r_{s-1}+k} = s$  for all  $s \in \{1, \dots, p\}$ ,  $k \in \{1, \dots, r_s\}$ , and  $p < g_{r_1+\dots+r_p+1} \leq \dots \leq g_q$ , as well as an  $\omega \in \Omega$  such that  $g_k = i_{\omega(k)}$  for every  $k \in \{1, \dots, q\}$ . Applying (5.4) to  $\omega \circ L$  instead of  $L$  and using (5.3) we obtain for every  $Y \in A^n$

$$\begin{aligned} B^{i_1 \dots i_q}(Y) &= \omega_A^{-1}(B_\omega^{g_1 \dots g_q}(Y)) \\ &= \frac{1}{(r_1 - 1)! \dots (r_p - 1)!} \sum_{\phi \in \Phi} \omega_A^{-1}(B_\omega^{h_{\phi(1)} \dots h_{\phi(q)}}(X)) \\ &= \frac{1}{(r_1 - 1)! \dots (r_p - 1)!} \sum_{\phi \in \Phi} B^{h_{\phi(\omega^{-1}(1))} \dots h_{\phi(\omega^{-1}(q))}}(X), \end{aligned}$$

where  $h_1, \dots, h_q$  and  $X$  are chosen for  $g_1, \dots, g_q$  and  $Y$  in the same manner as in (5.4). But for every  $\phi \in \Phi$  and every  $k \in \{1, \dots, p\}$  there is one and only one  $l \in \{1, \dots, q\}$  for which  $h_{\phi(\omega^{-1}(l))} = k$ , hence  $B^{h_{\phi(\omega^{-1}(1))} \dots h_{\phi(\omega^{-1}(q))}} = 0$ . Consequently  $B^{i_1 \dots i_q} = 0$  for all  $i_1, \dots, i_q \in \{1, \dots, n\}$  such that  $\{1, \dots, p\} \subset \{i_1, \dots, i_q\}$ . This means that  $L = 0$  on account of Lemmas 5.1 and 5.2, and the proof is complete.  $\square$

**Lemma 5.4.** *If  $B^{\omega^{-1}(1)\dots\omega^{-1}(q)} = 0$  for every  $\omega \in \Omega$ , then  $L = 0$ .*

*Proof.* We first show that if  $i_1, \dots, i_q \in \{1, \dots, n\}$  are such that  $i_k = k$  for  $k \leq p$  and  $i_k > p$  for  $k > p$ , then

$$(5.5) \quad B^{i_1 \dots i_q}(X) = C^{1 \dots q}(X^{i_{p+1}}, \dots, X^{i_q})$$

for every  $X \in A^n$ . The proof of (5.5) is by induction on the number  $N(i_{p+1}, \dots, i_q)$  of the elements of the set  $\{i_{p+1}, \dots, i_q\}$ . We fix  $g_1, \dots, g_q \in \{1, \dots, n\}$  such that  $g_k = k$  for  $k \leq p$  and  $g_k > p$  for  $k > p$ , and suppose (5.5) holds whenever  $N(i_{p+1}, \dots, i_q) > N(g_{p+1}, \dots, g_q)$ . Let  $R \subset \{p+1, \dots, q\}$  be such that for each  $k \in \{g_{p+1}, \dots, g_q\}$  there is one and only one  $l \in R$  such that  $k = g_l$ . Next, let  $h_1, \dots, h_n \in \{1, \dots, n\}$  be such that  $h_m = g_m$  for every  $m \in \{1, \dots, p\} \cup R$  and  $h_k \neq h_l$  for all  $k, l \in \{1, \dots, n\}$  such that  $k \neq l$ . Put

$$S_m = \begin{cases} \{h_m\} & \text{if } m \in \{1, \dots, p\} \cup R \cup \{q+1, \dots, n\}, \\ \{g_m, h_m\} & \text{if } m \in \{p+1, \dots, q\} \setminus R, \end{cases}$$

and define

$$f: \mathbb{R}^n \ni x \mapsto \left( \sum_{s_1 \in S_1} x^{s_1}, \dots, \sum_{s_n \in S_n} x^{s_n} \right) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this  $f$ ,  $t = e$  and  $u = e$ . Comparing the parts of both sides which contain  $e_1 \otimes \dots \otimes e_q$  we obtain  $\sum_{s_1 \in S_1} \dots \sum_{s_q \in S_q} B^{s_1 \dots s_q}(X) =$

$B^{1 \dots q} \left( \sum_{s_1 \in S_1} X^{s_1}, \dots, \sum_{s_n \in S_n} X^{s_n} \right)$  for every  $X \in A^n$ . This may be rewritten as

$$(5.6) \quad \sum_{s_{p+1} \in S_{p+1}} \dots \sum_{s_q \in S_q} B^{1 \dots p s_{p+1} \dots s_q}(X) = \sum_{s_{p+1} \in S_{p+1}} \dots \sum_{s_q \in S_q} C^{1 \dots q}(X^{s_{p+1}}, \dots, X^{s_q}).$$

But if  $s_{p+1} \in S_{p+1}, \dots, s_q \in S_q$  are such that there exists  $r \in \{p+1, \dots, q\} \setminus R$  with the property that  $s_r = h_r$ , then  $B^{1 \dots p s_{p+1} \dots s_q}(X) = C^{1 \dots q}(X^{s_{p+1}}, \dots, X^{s_q})$  on account of our assumption, because we have  $N(s_{p+1}, \dots, s_q) > N(g_{p+1}, \dots, g_q)$ . Subtracting all terms with such indices  $s_{p+1}, \dots, s_q$  from each side of (5.6) gives the equality  $B^{g_1 \dots g_q}(X) = C^{1 \dots q}(X^{g_{p+1}}, \dots, X^{g_q})$ , which completes the proof of (5.5).

Let now  $i_1, \dots, i_q \in \{1, \dots, n\}$  be such that for every  $k \in \{1, \dots, p\}$  there is one and only one  $l \in \{1, \dots, q\}$  for which  $i_l = k$ . There are  $g_1, \dots, g_q \in \{1, \dots, n\}$  such that  $g_k = k$  for  $k \leq p$  and  $g_k > p$  for  $k > p$ , as well as  $\omega \in \Omega$  such that  $g_k = i_{\omega(k)}$  for every  $k \in \{1, \dots, q\}$ . Applying (5.5) to  $\omega \circ L$  instead of  $L$  and using (5.3) we obtain for every  $X \in A^n$

$$\begin{aligned} B^{i_1 \dots i_q}(X) &= \omega_A^{-1}(B_\omega^{g_1 \dots g_q}(X)) = \omega_A^{-1}(C_\omega^{1 \dots q}(X^{g_{p+1}}, \dots, X^{g_q})) = \omega_A^{-1}(B_\omega^{1 \dots q}(Y)) \\ &= B^{\omega^{-1}(1) \dots \omega^{-1}(q)}(Y), \end{aligned}$$

where  $Y$  is an element of the set  $A^n$  such that  $Y^{p+1} = X^{g_{p+1}}, \dots, Y^q = X^{g_q}$ . This means that  $L = 0$  on account of Lemma 5.3 and the proof is complete.  $\square$

**Lemma 5.5.** *If  $B^{\omega_{k_1 \dots k_p}^{-1}(1) \dots \omega_{k_1 \dots k_p}^{-1}(q)} = 0$  for all  $k_1, \dots, k_p \in \{1, \dots, q\}$  such that  $k_1 < \dots < k_p$ , then  $L = 0$ .*

*Proof.* Let  $\omega$  be an arbitrary permutation of the set  $\{1, \dots, q\}$ . Then there are  $k_1, \dots, k_p \in \{1, \dots, q\}$  such that  $k_1 < \dots < k_p$  and the permutations  $\sigma$  and  $\tau$  of the sets  $\{1, \dots, p\}$  and  $\{p+1, \dots, q\}$ , respectively, such that  $\omega = \omega_{k_1 \dots k_p} \circ (\sigma \cup \tau)^{-1}$ , where

$$(\sigma \cup \tau)(m) = \begin{cases} \sigma(m) & \text{if } m \in \{1, \dots, p\}, \\ \tau(m) & \text{if } m \in \{p+1, \dots, q\}. \end{cases}$$

Put

$$f: \mathbb{R}^n \ni x \mapsto (x^{\sigma(1)}, \dots, x^{\sigma(p)}, x^{\tau(p+1)}, \dots, x^{\tau(q)}, x^{q+1}, \dots, x^n) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this  $f$ ,  $t = e$  and  $u = \text{sgn } \sigma e$ . Comparing the parts of both sides which contain  $e_{\omega_{k_1 \dots k_p}^{-1}(1)} \otimes \dots \otimes e_{\omega_{k_1 \dots k_p}^{-1}(q)}$  we obtain

$$B^{\omega^{-1}(1) \dots \omega^{-1}(q)}(X) = \text{sgn } \sigma B^{\omega_{k_1 \dots k_p}^{-1}(1) \dots \omega_{k_1 \dots k_p}^{-1}(q)}((T^A f)(X))$$

for every  $X \in A^n$ . But  $B^{\omega_{k_1 \dots k_p}^{-1}(1) \dots \omega_{k_1 \dots k_p}^{-1}(q)} = 0$ , so using Lemma 5.4 completes the proof.  $\square$

*Proof of Theorem 4.1.* For every  $D \in \text{ED}_p^q(A)$  and every  $X \in A^n$  we have

$$\begin{aligned} & \overline{D}_{\mathbb{R}^n}(e)(X) \\ &= \sum_{\phi \in \Phi} \sum_{i_{p+1}=1}^n \dots \sum_{i_q=1}^n \text{sgn } \phi D(X^{i_{p+1}}, \dots, X^{i_q}) \otimes e_{\phi(1)} \otimes \dots \otimes e_{\phi(p)} \otimes e_{i_{p+1}} \otimes \dots \otimes e_{i_q}, \end{aligned}$$

where  $\Phi$  is the group of permutations of  $\{1, \dots, p\}$ . From this formula we see that for all  $k_1, \dots, k_p, l_1, \dots, l_p \in \{1, \dots, q\}$  such that  $k_1 < \dots < k_p$  and  $l_1 < \dots < l_p$  the coordinate of  $\omega_{k_1 \dots k_p}^{-1} \circ \overline{D}$  at  $e_{\omega_{l_1 \dots l_p}^{-1}(1)} \otimes \dots \otimes e_{\omega_{l_1 \dots l_p}^{-1}(q)}$  equals either

$$(\omega_{l_1 \dots l_q}^{-1})_A(D(X^{p+1}, \dots, X^q)) \quad \text{if } (k_1, \dots, k_p) = (l_1, \dots, l_p)$$

or 0 if  $(k_1, \dots, k_p) \neq (l_1, \dots, l_p)$ . It follows immediately that if  $D_{k_1 \dots k_p} \in \text{ED}_p^q(A)$  for all  $k_1, \dots, k_p \in \{1, \dots, q\}$  such that  $k_1 < \dots < k_p$ , then the coordinate of  $\sum_{1 \leq k_1 < \dots < k_p \leq q} \omega_{k_1 \dots k_p}^{-1} \circ \overline{D}_{k_1 \dots k_p}$  at  $e_{\omega_{l_1 \dots l_p}^{-1}(1)} \otimes \dots \otimes e_{\omega_{l_1 \dots l_p}^{-1}(q)}$  is equal to

$$(\omega_{l_1 \dots l_q}^{-1})_A(D_{l_1 \dots l_p}(X^{p+1}, \dots, X^q)).$$

For all  $l_1, \dots, l_p \in \{1, \dots, q\}$  such that  $l_1 < \dots < l_p$  put

$$D_{l_1 \dots l_p} = (\omega_{l_1 \dots l_p})_A \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)},$$

where  $C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}$  is defined by the coordinate  $B^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}$  of  $L$  as in (5.1).

By the above, the proof will be completed as soon as we can show that

$$(5.7) \quad (\omega_{l_1 \dots l_p})_A \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)} \in \text{ED}_p^q(A)$$

for all  $l_1, \dots, l_p \in \{1, \dots, q\}$  such that  $l_1 < \dots < l_p$ . Indeed, if we apply Lemma 5.5 to  $L - \sum_{1 \leq k_1 < \dots < k_p \leq q} \omega_{k_1 \dots k_p}^{-1} \circ \overline{D}_{k_1 \dots k_p}$  instead of  $L$  in this case, we obtain the desired equality  $L - \sum_{1 \leq k_1 < \dots < k_p \leq q} \omega_{k_1 \dots k_p}^{-1} \circ \overline{D}_{k_1 \dots k_p} = 0$ .

Therefore it remains to prove (5.7).

Let  $i, j \in \{1, \dots, p\}$  be such that  $i < j$ . Put

$$f: \mathbb{R}^n \ni x \mapsto \left( x^1, \dots, x^{i-1}, x^i + \frac{(x^j)^2}{2}, x^{i+1}, \dots, x^n \right) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this  $f$ ,  $t = e$ ,  $u = e$ . Comparing the parts of both sides which contain

$$e_{\omega_{l_1 \dots l_p}^{-1}(1)} \otimes \dots \otimes e_{\omega_{l_1 \dots l_p}^{-1}(l_j-1)} \otimes e_{\omega_{l_1 \dots l_p}^{-1}(l_i)} \otimes e_{\omega_{l_1 \dots l_p}^{-1}(l_j+1)} \otimes \dots \otimes e_{\omega_{l_1 \dots l_p}^{-1}(q)}$$

we obtain for every  $X \in A^n$

$$\begin{aligned} & Z_{X^j}^{l_j} \circ C^{\omega_l^{-1}(1) \dots \omega_l^{-1}(q)} \\ & + Z_{X^j}^{l_i} \circ C^{\omega_l^{-1}(1) \dots \omega_l^{-1}(l_i-1) \omega_l^{-1}(l_j) \omega_l^{-1}(l_i+1) \dots \omega_l^{-1}(l_j-1) \omega_l^{-1}(l_i) \omega_l^{-1}(l_j+1) \dots \omega_l^{-1}(q)} = 0 \end{aligned}$$

where  $l = l_1 \dots l_p$ . Consider the consequent of (3.1) with

$$f: \mathbb{R}^n \ni x \mapsto (x^1, \dots, x^{i-1}, x^j, x^{i-1}, \dots, x^{j-1}, x^i, x^{j+1}, \dots, x^n) \in \mathbb{R}^n,$$

$t = e$ ,  $u = -e$ . Comparing the parts of both sides which contain  $e_{\omega_{l_1 \dots l_p}^{-1}(1)} \otimes \dots \otimes e_{\omega_{l_1 \dots l_p}^{-1}(q)}$  we obtain

$$\begin{aligned} & C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(l_i-1) \omega_{l_1 \dots l_p}^{-1}(l_j) \omega_{l_1 \dots l_p}^{-1}(l_i+1) \dots \omega_{l_1 \dots l_p}^{-1}(l_j-1) \omega_{l_1 \dots l_p}^{-1}(l_i) \omega_{l_1 \dots l_p}^{-1}(l_j+1) \dots \omega_{l_1 \dots l_p}^{-1}(q)} \\ & = -C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}. \end{aligned}$$

Therefore

$$Z_{X^j}^{l_j} \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)} = Z_{X^j}^{l_i} \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}.$$

Combining the both sides of this equality with  $(\omega_{l_1 \dots l_p})_A$  yields

$$(5.8) \quad Z_{X^j}^j \circ (\omega_{l_1 \dots l_p})_A \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)} = Z_{X^j}^i \circ (\omega_{l_1 \dots l_p})_A \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}.$$

Since  $X^j$  in (5.8) may be any element of  $A$ ,  $(\omega_{l_1 \dots l_p})_A \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}$  satisfies (3.2).

Let now  $k \in \{p+1, \dots, q\}$ . Put  $U = \{x \in \mathbb{R}^n : x^k > 0\}$  and

$$f: U \ni x \mapsto \left(x^1, \dots, x^{k-1}, \frac{(x^k)^2}{2}, x^{k+1}, \dots, x^n\right) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this  $f$ ,  $t = e$  and  $u = e$ . Comparing the parts of both sides which contain  $e_{\omega_{l_1 \dots l_p}^{-1}(1)} \otimes \dots \otimes e_{\omega_{l_1 \dots l_p}^{-1}(q)}$  we obtain for every  $X \in T^A U$

$$(5.9) \quad Z_{X^k}^{\omega_{l_1 \dots l_p}(k)} (C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}(X^{p+1}, \dots, X^q)) \\ = C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)} \left(X^{p+1}, \dots, X^{k-1}, \frac{(X^k)^2}{2}, X^{k+1}, \dots, X^q\right).$$

In the same manner, with  $U$  replaced by  $\{x \in \mathbb{R}^n : x^k < 0\}$ , we see that (5.9) also holds for every  $X \in T^A \{x \in \mathbb{R}^n : x^k < 0\}$ , and so, by continuity, for every  $X \in A^n$ . Combining the both sides of (5.9) with  $(\omega_{l_1 \dots l_p})_A$  yields

$$(5.10) \quad Z_{X^k}^k ((\omega_{l_1 \dots l_p})_A (C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}(X^{p+1}, \dots, X^q))) \\ = (\omega_{l_1 \dots l_p})_A \left(C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)} \left(X^{p+1}, \dots, X^{k-1}, \frac{(X^k)^2}{2}, X^{k+1}, \dots, X^q\right)\right).$$

Now the polarization of (5.10) with respect to  $X^k$  leads to the conclusion that  $(\omega_{l_1 \dots l_p})_A \circ C^{\omega_{l_1 \dots l_p}^{-1}(1) \dots \omega_{l_1 \dots l_p}^{-1}(q)}$  satisfies (3.3). This completes the proof.  $\square$

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