

BASIC EQUATIONS OF  $G$ -ALMOST GEODESIC  
MAPPINGS OF THE SECOND TYPE, WHICH HAVE  
THE PROPERTY OF RECIPROCITY

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*Abstract.* We study  $G$ -almost geodesic mappings of the second type  $\pi_2(e)$ ,  $\theta = 1, 2$  between non-symmetric affine connection spaces. These mappings are a generalization of the second type almost geodesic mappings defined by N. S. Sinyukov (1979). We investigate a special type of these mappings in this paper. We also consider  $e$ -structures that generate mappings of type  $\pi_2(e)$ ,  $\theta = 1, 2$ . For a mapping  $\pi_2(e, F)$ ,  $\theta = 1, 2$ , we determine the basic equations which generate them.

*Keywords:* non-symmetric affine connection; almost geodesic mapping;  $G$ -almost geodesic mapping; property of reciprocity; almost geodesic mapping of the second type

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## 1. INTRODUCTION

Let us consider two  $N$ -dimensional differentiable manifolds  $GA_N$  and  $G\bar{A}_N$  and a differentiable mapping  $f: GA_N \rightarrow G\bar{A}_N$ . We can consider these manifolds together with this *mapping system of local coordinates*. Namely, if  $f: M \in GA_N \rightarrow \bar{M} \in G\bar{A}_N$  and if  $(\mathcal{U}, \varphi)$  is the local chart around the point  $M$  then  $\varphi(M) = x = (x^1, \dots, x^N) \in E^N$ . In this case, we define mapping  $\bar{\varphi} = \varphi \circ f^{-1}$  for the coordinate mapping in  $G\bar{A}_N$ , and then

$$\bar{\varphi}(\bar{M}) = (\varphi \circ f^{-1})(f(M)) = \varphi(M) = x = (x^1, \dots, x^N) \in E^N.$$

The points  $M$  and  $\bar{M} = f(M)$  have the same local coordinates in this case. If the connection coefficients  $L_{jk}^i(x)$  and  $\bar{L}_{jk}^i(x)$  of the affine connections introduced

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in  $GA_N$  and  $G\bar{A}_N$ , respectively, are non-symmetric in lower indices then  $GA_N$  and  $G\bar{A}_N$  are *general affine connection spaces*.

One says that the reciprocal mapping  $f: GA_N \rightarrow G\bar{A}_N$  is *geodesic*, [17], [16] if geodesics of the space  $GA_N$  pass to geodesics of the space  $G\bar{A}_N$ . Generalizing the concept of a geodesic mapping between Riemannian spaces and symmetric affine connection ones, Sinyukov [18] introduced the following notions:

A curve  $l: x^h = x^h(t)$  is called an *almost geodesic line* if its tangential vector  $\lambda^h(t) = dx^h/dt \neq 0$  satisfies the equation

$$\bar{\lambda}_{(2)}^h = \bar{a}(t)\lambda^h + \bar{b}(t)\bar{\lambda}_{(1)}^h,$$

where  $\bar{\lambda}_{(1)}^h = \lambda_{||p}^h \lambda^p$ ,  $\bar{\lambda}_{(2)}^h = \bar{\lambda}_{(1)||p}^h \lambda^p$ . Here  $\bar{a}(t)$  and  $\bar{b}(t)$  are functions of a parameter  $t$  and  $||$  denotes the covariant derivative with regard to the connection in  $\bar{A}_N$ .

**Definition 1.1.** A mapping  $f$  of a symmetric affine connection space  $A_N$  onto a space  $\bar{A}_N$  is called an *almost geodesic mapping* if any geodesic line of the space  $A_N$  is mapped into an almost geodesic line of the space  $\bar{A}_N$ .

A lot of research papers and monographs [1]–[23] have been dedicated to the theory of geodesic mappings of Riemannian spaces, affine connected ones and their generalizations. Sinyukov [18] and Mikeš [1], [2], [12], [13], [23] gave some other significant contributions to the study of almost geodesic mappings of affine connected spaces and singled out three types  $\pi_1$ ,  $\pi_2$ ,  $\pi_3$  of almost geodesic mappings between affine connected spaces without torsion.

In a general affine connection space  $GA_N$ , with non-symmetric affine connection  $L$ , one can define four kinds of a covariant derivative [15], [14]. Let us denote a covariant derivative of a kind  $\theta$  ( $\theta = 1, \dots, 4$ ) with regard to affine connections of  $GA_N$  and  $G\bar{A}_N$  by  $|_{\theta}$  and  $||_{\theta}$ , respectively.

For example, a tensor  $a_j^i$  in  $GA_N$  satisfies

$$a_{j|m}^i = a_{j,m}^i + L_{\alpha m}^i a_j^\alpha - L_{jm}^\alpha a_\alpha^i \quad \text{and} \quad a_{j|_m}^i = a_{j,m}^i + L_{m\alpha}^i a_j^\alpha - L_{mj}^\alpha a_\alpha^i.$$

Thus, in the case of a space with a non-symmetric affine connection we can define two kinds of almost geodesic lines and two kinds of almost geodesic mappings [20]–[19].

**Definition 1.2.** A curve  $l: x^h = x^h(t)$  on  $G\bar{A}_N$  is called [20]–[19] a *G-almost geodesic line of the first kind* if its tangent vector  $\lambda^h(t) = dx^h/dt \neq 0$  satisfies the equation

$$\bar{\lambda}_1^h{}_{(2)} = \bar{a}_1(t)\lambda^h + \bar{b}_1(t)\bar{\lambda}_1^h{}_{(1)},$$

where  $\bar{\lambda}_1^h{}_{(1)} = \lambda_{||_1}^h \lambda^\alpha$ ,  $\bar{\lambda}_1^h{}_{(2)} = \bar{\lambda}_{||_1}^h{}_{(1)} \lambda^\alpha$  and  $\bar{a}_1(t)$  and  $\bar{b}_1(t)$  are functions of a parameter  $t$ .

**Definition 1.3.** A curve  $l: x^h = x^h(t)$  is called a *G-almost geodesic line of the second kind* if its tangential vector  $\lambda^h(t) = dx^h/dt \neq 0$  satisfies the equation

$$\bar{\lambda}_{\frac{1}{2}}^h(t) = \bar{a}(t)\lambda^h + \bar{b}(t)\bar{\lambda}_{\frac{1}{2}}^h(t),$$

where  $\bar{\lambda}_{\frac{1}{2}}^h(t) = \lambda_{||\alpha}^h \lambda^\alpha$ ,  $\bar{\lambda}_{\frac{1}{2}}^h(t) = \bar{\lambda}_{\frac{1}{2}}^h(t)_{||\alpha} \lambda^\alpha$ ,  $\bar{a}(t)$  and  $\bar{b}(t)$  are functions of a parameter  $t$ .

**Definition 1.4.** A mapping  $f$  of the space  $GA_N$  onto a space  $G\bar{A}_N$  is called a *G-almost geodesic mapping of the first kind* if any geodesic line of the space  $GA_N$  turns into an almost geodesic line of the first kind of the space  $G\bar{A}_N$ .

**Definition 1.5.** A mapping  $f$  is called a *G-almost geodesic mapping of the second kind* if any geodesic line of the space  $GA_N$  turns into almost geodesic line of the second kind of the space  $G\bar{A}_N$ .

We can put

$$P_{ij}^h(x) = \bar{L}_{ij}^h(x) - L_{ij}^h(x),$$

where  $L_{ij}^h(x)$ ,  $\bar{L}_{ij}^h(x)$  are connection coefficients of the spaces  $GA_N$  and  $G\bar{A}_N$ ,  $N > 2$ , together with the mapping  $f$  system of local coordinates, and  $P_{ij}^h$  is a deformation tensor. From [20], it follows that the succeeding results hold:

**Theorem 1.1.** A mapping  $f$  of the space  $GA_N$  onto  $G\bar{A}_N$  is a *G-almost geodesic mapping of the first kind* if and only if the deformation tensor  $P_{ij}^h$  satisfies the conditions

$$(1.1) \quad (P_{\alpha\beta| \gamma}^h + P_{\delta\alpha}^h P_{\beta\gamma}^\delta) \lambda^\alpha \lambda^\beta \lambda^\gamma = b_1 P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + a_1 \lambda^h$$

identically, where  $a_1$  and  $b_1$  are functions.

**Theorem 1.2.** A mapping  $f$  of the space  $GA_N$  onto  $G\bar{A}_N$  is a *G-almost geodesic mapping of the second kind* if and only if the deformation tensor  $P_{ij}^h$  satisfies the conditions

$$(1.2) \quad (P_{\alpha\beta| \gamma}^h + P_{\alpha\delta}^h P_{\beta\gamma}^\delta) \lambda^\alpha \lambda^\beta \lambda^\gamma = b_2 P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + a_2 \lambda^h$$

identically, where  $a_2$  and  $b_2$  are functions.

We are going to present basic equations of *G-almost geodesic mappings* of the type  $\pi_2(e)$ ,  $\theta = 1, 2$ , between non-symmetric affine connection spaces  $G\mathbb{A}_N$  and  $G\bar{\mathbb{A}}_N$  in this paper.

## 2. $G$ -ALMOST GEODESIC MAPPINGS OF THE SECOND TYPE

Sinyukov (see [18]) introduced almost geodesic mapping of the second type  $\pi_2$  for affine connection spaces without torsion with the condition

$$b = \frac{b_{\gamma\delta}\lambda^\gamma\lambda^\delta}{\sigma_\alpha\lambda^\alpha},$$

where  $\sigma_\alpha\lambda^\alpha \neq 0$  and  $b_{\gamma\delta}$  is a twice covariant tensor.

Analogously, a  $G$ -almost geodesic mapping of the first kind of a non-symmetric affine connection space is an almost geodesic mapping of the second type  $\pi_2$  if the function  $b_1$  satisfies the condition

$$b_1 = \frac{b_{\gamma\delta}\lambda^\gamma\lambda^\delta}{\sigma_\alpha\lambda^\alpha},$$

where  $\sigma_\alpha\lambda^\alpha \neq 0$  and  $b_{\gamma\delta}$  is a twice covariant tensor.

Let

$$P_{\alpha\beta}^h\lambda^\alpha\lambda^\beta = 2\sigma_\alpha\lambda^\alpha F_\beta^h\lambda^\beta + 2\psi_\alpha\lambda^\alpha\lambda^\beta.$$

Then

$$(P_{\alpha\beta}^h - 2\sigma_\alpha F_\beta^h - 2\psi_\alpha\delta_\beta^h)\lambda^\alpha\lambda^\beta \equiv 0,$$

wherefrom

$$P_{ij}^h = \psi_i\delta_j^h + \psi_j\delta_i^h + \sigma_i F_j^h + \sigma_j F_i^h.$$

Here,  $\psi_i$  and  $\sigma_i$  are vectors,  $F_j^i$  is a tensor,  $\underline{ij}$  denotes a symmetrization with division,  $\bar{ij}$  denotes an anti-symmetrization with division and  $\delta_i^h$  is the Kronecker symbol. We can put  $P_{ij}^h = \xi_{ij}^h$ .

Then

$$(2.1) \quad P_{ij}^h = \psi_i\delta_j^h + \psi_j\delta_i^h + \sigma_i F_j^h + \sigma_j F_i^h + \xi_{ij}^h.$$

In the equation (2.1), magnitudes  $\psi_i$ ,  $\sigma_i$  are covariant vectors,  $F_i^h$  is a tensor and  $\xi_{ij}^h$  is an anti-symmetric tensor.

After substituting the equation (2.1) in the equation (1.1), we conclude that

$$(2.2) \quad F_{i|j}^h + F_{j|i}^h + F_\delta^h F_i^\delta \sigma_j + F_\delta^h F_j^\delta \sigma_i + \xi_{\delta i}^h F_j^\delta + \xi_{\delta j}^h F_i^\delta = \mu_i F_j^h + \nu_j F_i^h + \nu_i \delta_j^h + \nu_j \delta_i^h,$$

where  $\mu_i$  and  $\nu_i$  are covariant vectors.

Conditions (2.1) and (2.2) are the *basic equations* of the mapping  $\pi_2$ .

A *G*-almost geodesic mapping of the second kind is a *G*-almost geodesic mapping of the second type  $\pi_2$  if it satisfies the following condition for the function  $b_2$  in (1.2):

$$b_2 = \frac{b_{\gamma\delta} \lambda^\gamma \lambda^\delta}{\sigma_\alpha \lambda^\alpha},$$

where  $\sigma_\alpha \lambda^\alpha \neq 0$  and  $b_{\gamma\delta}$  is a twice covariant tensor.

Using the method from the previous case, we get

$$(2.3) \quad \begin{aligned} F_{i|j}^h + F_{j|i}^h + F_\delta^h F_i^\delta \sigma_j + F^h \delta F_j^\delta \sigma_i + \xi_{i\delta}^h F_j^\delta + \xi_{j\delta}^h F_i^\delta \\ = \mu_i F_j^h + \mu_j F_i^h + \nu_i \delta_j^h + \nu_j \delta_i^h, \end{aligned}$$

where  $\mu_i, \nu_i$  are covariant vectors.

Conditions (2.1) and (2.3) are the *basic equations* of *G*-almost geodesic mappings of the type  $\pi_2$ .

**Remark 2.1.** If  $\sigma_i \equiv 0$  in the equation (2.1) then almost geodesic mappings are reduced to the geodesic ones. On the other hand, if  $\psi_i \equiv 0$ , then this mapping is a canonical almost geodesic one (see [21]). In the case  $\sigma_i \equiv 0$  and  $\psi_i \equiv 0$ , we have a trivial almost geodesic mapping. We are working with nontrivial almost geodesic mappings only in the sequel.

### 3. THE PROPERTY OF RECIPROCITY OF *G*-ALMOST GEODESIC MAPPINGS OF THE SECOND TYPE

A mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_1$  has the *property of reciprocity*, if its inverse mapping  $f^{-1}: G\bar{A}_N \rightarrow GA_N$  (provided it exists) is of the  $\pi_2$  type, and  $f^{-1}$  corresponds to the same tensor  $F_i^h$ , see also [21]. Since the inverse mapping  $f^{-1}: G\bar{A}_N \rightarrow GA_N$  satisfies

$$\bar{P}_{ij}^h = -P_{ij}^h,$$

we can put the following in the equation (2.1):

$$\bar{\psi}_i = -\psi_i, \quad \bar{\sigma}_i = -\sigma_i, \quad \bar{F}_i^h = F_i^h, \quad \bar{\xi}_{ij}^h = -\xi_{ij}^h.$$

A mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_2$  has the property of reciprocity if and only if the tensor  $F_i^h$  of the space  $G\bar{A}_N$  satisfies the equation of the form (2.2), i.e.,

$$(3.1) \quad F_{(i|j)}^h - F_\alpha^h F_{(i}^\alpha \sigma_{j)} - \xi_{\alpha(i}^h F_{j)}^\alpha = \bar{\mu}_{(i} F_{j)}^h + \bar{\nu}_{(i} \delta_{j)}^h,$$

where  $(ij)$  is a symmetrization without division with respect to  $i$  and  $j$ , and  $\parallel_1$  is a covariant derivative of the first kind in  $G\bar{A}_N$ . Inserting a covariant derivative of the first kind in  $GA_N$  into the equation (3.1) we get

$$F_\alpha^h F_{(i}^\alpha \sigma_{j)} + \xi_{\alpha(i}^h F_{j)}^\alpha = \bar{\mu}_{(i} F_{j)}^h + \bar{\nu}_{(i} \delta_{j)}^h,$$

where vectors  $\bar{\mu}_i, \bar{\nu}_i$  are expressed by  $\mu_i, \nu_i, \bar{\mu}_i, \bar{\nu}_i, \psi_i, \sigma_i, F_i^h$ . Since  $\sigma \neq 0$ , we get

$$(3.2) \quad F_\alpha^h F_i^\alpha = p \delta_i^h + q F_i^h,$$

where  $p$  and  $q$  are functions.

Based on the facts given above, we have:

**Theorem 3.1.** *The relation (3.2) expresses the necessary and sufficient condition for a mapping  $\pi_2: GA_N \rightarrow G\bar{A}_N$  to have the property of reciprocity.*

The equations (2.1) and (2.2) are invariant under the mapping  $\pi_2$  of a tensor

$$\tilde{F}_i^h = r F_i^h + s \delta_i^h, \quad r \neq 0.$$

Then we have

$$\tilde{F}_\alpha^h \tilde{F}_i^\alpha = \tilde{p} \delta_i^h + \tilde{q} \tilde{F}_i^h,$$

where

$$\tilde{p} = r^2 p - s^2 - srq, \quad \tilde{q} = 2s + rq.$$

Here we can select invariants  $r$  and  $s$  such that

$$\tilde{q} \equiv 0, \quad \tilde{p} = \tilde{e} \quad (= \pm 1, 0).$$

In this case, we have

$$\tilde{F}_\alpha^h \tilde{F}_i^\alpha = \tilde{e} \delta_i^h.$$

Based on the facts given above, we can put

$$(3.3) \quad F_\alpha^h F_i^\alpha = e \delta_i^h, \quad e = \pm 1, 0.$$

Substituting the equation (3.3) into the condition (2.2), we get

$$(3.4) \quad F_{(i|j)}^h + \xi_{\alpha(i}^h F_{j)}^\alpha = \mu_{(i} F_{j)}^h + \nu_{(i} \delta_{j)}^h.$$

Hence, a  $G$ -almost geodesic mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_2$  which has the property of reciprocity is determined by the equations (2.1), (3.3) and (3.4) (see [21]). This mapping is denoted by  $\pi_2(e)$ .

In the case of the  $G$ -almost geodesic mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_2$  which has the property of reciprocity, it is determined by the equations

$$(3.5) \quad \begin{aligned} P_{ij}^h &= \psi_i \delta_j^h + \psi_j \delta_i^h + \sigma_i F_j^h + \sigma_j F_i^h + \xi_{ij}^h, \\ F_{(i|j)}^h - \xi_{\alpha(i} F_{j)}^\alpha &= \mu_{(i} F_{j)}^h + \nu_{(i} \delta_{j)}^h, \\ F_\alpha^h F_i^\alpha &= e \delta_i^h, \quad e = \pm 1, 0. \end{aligned}$$

This mapping is denoted by  $\pi_2(e)$ .

#### 4. ON $e$ -STRUCTURES THAT DETERMINE $G$ -ALMOST GEODESIC MAPPINGS OF TYPE $\pi_2(e)$ OF FIRST AND SECOND KINDS

**Definition 4.1.** A tensor  $F_i^h$  which satisfies the conditions (3.3) and (3.4) is called an  $e$ -structure which determines a  $G$ -almost geodesic mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_2(e)$ .

**Theorem 4.1.** An  $e$ -structure  $F_i^h$  determines a  $G$ -almost geodesic mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_2(e)$ ,  $e = \pm 1$ , if and only if it satisfies the conditions

$$(4.1) \quad F_{(i|j)}^h + \xi_{\alpha(i} F_{j)}^\alpha = \mu_{(i} F_{j)}^h - \mu_\alpha F_{(i} \delta_{j)}^h,$$

$$(4.2) \quad F_\alpha^h F_i^\alpha = e \delta_i^h.$$

**Proof.** Based on the covariant derivative of the first kind of the condition (4.2) in the direction  $x^j$ , we get

$$(4.3) \quad F_{\alpha|j}^h F_i^\alpha + F_{i|j}^\alpha F_\alpha^h = 0.$$

After the symmetrization of the equation (4.3) with respect to the indices  $i$  and  $j$ , we have

$$(4.4) \quad F_{\alpha|j}^h F_i^\alpha + F_{\alpha|i}^h F_j^\alpha + F_{(i|j)}^\alpha F_\alpha^h = 0.$$

Based on the equations (3.4) and (4.4), we conclude that

$$F_{\alpha|i}^h F_j^\alpha + F_{\alpha|j}^h F_i^\alpha + e \delta_{(i}^h \mu_{j)} + F_{(i}^h \nu_{j)} + F_\alpha^h F_{(i}^\beta \xi_{j)\beta}^\alpha = 0.$$

Composing the previous relation with  $F_k^j$ , one obtains

$$(4.5) \quad eF_{k|_1}^h + F_{\alpha|_1}^h F_i^\alpha F_k^\beta + e\delta_i^h \mu_\alpha F_k^\alpha + e\mu_i F_k^h + F_i^h \nu_\alpha F_k^\alpha + e\delta_k^h \nu_i \\ + F_i^h \nu_\alpha F_k^\alpha + e\delta_k^h \nu_i + F_\alpha^h F_i^\beta F_k^\gamma \xi_{\gamma\beta}^\alpha + eF_\alpha^h \xi_{ik}^\alpha = 0.$$

After symmetrizing of the equation (4.5) by indices  $i$  and  $k$ , we infer

$$(4.6) \quad eF_{(i|_1}^h + F_{(\alpha|_1}^h F_{i)}^\alpha F_k^\beta + e\delta_{(i}^h F_{k)}^\alpha \mu_\alpha + e\mu_{(i} F_{k)}^h + \nu_\alpha F_{(i}^h F_{k)}^\alpha + e\delta_{(i}^h \nu_{k)} = 0.$$

From the equation (3.4), we have

$$(4.7) \quad eF_{(i|_1}^h + F_{(\alpha|_1}^h F_{i)}^\alpha F_k^\beta = F_i^h (\nu_\beta F_k^\beta + e\mu_k) + F_k^h (\nu_\alpha F_i^\alpha + e\mu_i) \\ + e\delta_i^h (\mu_\beta F_k^\beta + \nu_k) + e\delta_k^h (\mu_\alpha F_i^\alpha + \nu_i).$$

Now, from the equations (4.6) and (4.7) we obtain

$$F_i^h (F_k^\alpha \nu_\alpha + e\mu_k) + F_k^h (F_i^\alpha \nu_\alpha + e\mu_i) + e\delta_i^h (F_k^\alpha \mu_k + \nu_k) + e\delta_k^h (F_i^\alpha \mu_\alpha + \nu_i) = 0.$$

By examining the last equality, we conclude that

$$(4.8) \quad F_i^\alpha \mu_\alpha + \nu_i = 0, \quad \text{i.e.} \quad \nu_i = -F_i^\alpha \mu_\alpha.$$

After substituting (4.8) into (3.4), we obtain the relation (4.1) is valid.  $\square$

Analogously, in the case of  $G$ -almost geodesic mappings of the type  $\pi_2(e)$  of the second kind we obtain

**Definition 4.2.** A tensor  $F_i^h$  which satisfies the conditions (3.5) is an  $e$ -structure which determines a  $G$ -almost geodesic mapping of the type  $\pi_2(e)$ .

**Theorem 4.2.** An  $e$ -structure  $F_i^h$  determines a  $G$ -almost geodesic mapping of the type  $\pi_2(e)$ ,  $e = \pm 1$ , if and only if it satisfies the conditions

$$(4.9) \quad F_{(i|_2}^h - \xi_{\alpha(i}^h F_{j)}^\alpha = \mu_{(i} F_{j)}^h - \mu_\alpha F_{(i}^h \delta_{j)}^h,$$

$$(4.10) \quad F_\alpha^h F_i^\alpha = e\delta_i^h.$$



**Theorem 4.3.** An  $e$ -structure  $F_i^h$  which determines a  $G$ -almost geodesic mapping of the type  $\pi_2(e)$ ,  $e = \pm 1$ , satisfies the following conditions:

$$(4.11) \quad F_{i|j|k}^h + \xi_{ijk}^h = \mu_{j|k} F_i^h + \mu_{i|k} F_j^h + \mu_{i|j} F_k^h \\ - \mu_{\alpha|j} F_k^\alpha \delta_i^h + \mu_{\alpha|i} F_k^\alpha \delta_j^h + \mu_{\alpha|i} F_j^\alpha \delta_k^h + \theta_{ijk}^h,$$

where  $[i, j]$  is an anti-symmetrization without division,

$$\begin{aligned} \theta_{ijk}^h &= \theta_{ijk}^h + \theta_{ikj}^h - \theta_{jki}^h - R_{\alpha ij}^h F_k^\alpha + R_{kij}^\alpha F_\alpha^h - R_{\alpha ik}^h F_j^\alpha + R_{jik}^\alpha F_\alpha^h \\ &\quad + L_{[ij]}^\alpha F_{k|\alpha}^h + L_{[ik]}^\alpha F_{j|\alpha}^h, \\ \theta_{ijk}^h &= \mu_i F_{j|k}^h + \mu_i F_{i|k}^h - \mu_\alpha F_{i|k}^\alpha \delta_j^h - \mu_\alpha F_{j|k}^\alpha \delta_i^h - \xi_{\alpha i}^h F_{j|k}^\alpha - \xi_{\alpha j}^h F_{i|k}^\alpha, \\ \xi_{ijk}^h &= \xi_{\alpha[i|k]}^h F_j^h + \xi_{\alpha[i|j]}^h F_k^h + \xi_{\alpha(j|k)}^h F_i^h, \end{aligned}$$

and

$$R_{ijk}^h = L_{ij,k}^h - L_{ik,j}^h + L_{ij}^\alpha L_{\alpha k}^h - L_{ik}^\alpha L_{\alpha j}^h$$

is a curvature tensor of the first kind (see [15]).

**Proof.** Taking the covariant derivative of the first kind of (4.1) in the direction of  $x^k$ , we get

$$(4.12) \quad F_{i|jk}^h + F_{j|ik}^h + \xi_{\alpha i|k}^h F_j^\alpha + \xi_{\alpha j|k}^h F_i^\alpha \\ = \mu_{i|k} F_j^h + \mu_{j|k} F_i^h - \mu_{\alpha|k} F_i^\alpha \delta_j^h - \mu_{\alpha|k} F_j^\alpha \delta_i^h + \theta_{ijk}^h.$$

Alternating this equation with respect to  $i$  and  $k$  and using the first Ricci identity, we get

$$(4.13) \quad F_{i|jk}^h - F_{k|ji}^h + \xi_{\alpha i|k}^h F_j^\alpha - \xi_{\alpha k|i}^h F_j^\alpha + \xi_{\alpha j|k}^h F_i^\alpha - \xi_{\alpha j|i}^h F_k^\alpha \\ = \mu_{i|k} F_j^h - \mu_{k|i} F_j^h + \mu_{j|k} F_i^h - \mu_{j|i} F_k^h \\ - \mu_{\alpha|k} F_i^\alpha \delta_j^h + \mu_{\alpha|i} F_k^\alpha \delta_j^h - \mu_{\alpha|k} F_j^\alpha \delta_i^h + \mu_{\alpha|i} F_j^\alpha \delta_k^h + \theta_{ijk}^h,$$

where

$$\theta_{ijk}^h = \theta_{ijk}^h - \theta_{kji}^h - R_{\alpha ik}^h F_j^\alpha + R_{jik}^\alpha F_\alpha^h + L_{[ik]}^\alpha F_{j|\alpha}^h.$$

Let us interchange indices  $j$  and  $k$  in (4.13). Then we have

$$\begin{aligned}
 (4.14) \quad & F_{i|_1 k j}^h - F_{j|_1 k i}^h + \xi_{\alpha i|_1 j}^h F_k^\alpha - \xi_{\alpha j|_1 i}^h F_k^\alpha + \xi_{\alpha k|_1 j}^h F_i^\alpha - \xi_{\alpha k|_1 i}^h F_j^\alpha \\
 & = \mu_{i|_1 j} F_k^h - \mu_{j|_1 i} F_k^h + \mu_{k|_1 j} F_i^h - \mu_{k|_1 i} F_j^h \\
 & \quad - \mu_{\alpha|_1 j} F_i^\alpha \delta_k^h + \mu_{\alpha|_1 i} F_j^\alpha \delta_k^h - \mu_{\alpha|_1 j} F_k^\alpha \delta_i^h + \mu_{\alpha|_1 i} F_k^\alpha \delta_j^h + \theta_{i k j}^3.
 \end{aligned}$$

Adding the equations (4.12) and (4.14) together with some other calculations proves the equation (4.11) holds.  $\square$

We are going to proceed with the study of conditions on the  $e$ -structure that generates  $G$ -almost geodesic mappings of the type  $\pi_2(e)$ ,  $e = \pm 1$ .

**Definition 4.3.** A  $G$ -almost geodesic mapping  $f: GA_N \rightarrow G\bar{A}_N$  of the type  $\pi_2(e)$  ( $\theta = 1, 2$ ), which satisfies the condition  $F_\alpha^\alpha = 0$  is a  $G$ -almost geodesic mapping of the type  $\pi_2(e, F)$  ( $\theta = 1, 2$ ).

Perform a contraction by indices  $h$  and  $i$  in the algebraic condition (4.2). Then we have the equation

$$F_\beta^\alpha F_\alpha^\beta = eN.$$

Let us take its second covariant derivative of the first kind in the directions  $x^i$  and  $x^k$ :

$$(4.15) \quad F_\beta^\alpha F_{\alpha|_1 j k}^\beta + F_{\beta|_1 j}^\alpha F_{\alpha|_1 k}^\beta = 0.$$

After the composing the equation (4.11) with  $F_k^i$  and using the result (4.15), we get

$$\begin{aligned}
 (4.16) \quad & -2F_{\beta|_1 j}^\alpha F_{\alpha|_1 k}^\beta + F_\beta^\alpha \xi_{\alpha j k}^\beta = \mu_{(j|_1 k)} eN - F_\beta^\beta \mu_{\alpha|_1 (j} F_{k)}^\alpha + \mu_{(\alpha|_1 \beta)} F_k^\alpha F_j^\beta \\
 & \quad - e\mu_{(j|_1 k)} + F_{\beta|_1 \alpha k j}^\alpha \theta_{1}^\beta.
 \end{aligned}$$

Using the condition  $F_\alpha^\alpha = 0$ , from (4.16) we have

$$(4.17) \quad e(N-1)\mu_{(j|_1 k)} + \mu_{(\alpha|_1 \beta)} F_j^\alpha F_k^\beta = \theta_{1 j k}^4,$$

where we denoted  $\theta_{1 j k}^4 = F_{\beta|_1 \alpha k j}^\alpha \theta_{1}^\beta + 2F_{\beta|_1 j}^\alpha F_{\alpha|_1 k}^\beta - F_\beta^\alpha \xi_{\alpha j k}^\beta$ . Composing (4.17) with  $F_j^j F_k^k$ , we obtain

$$(4.18) \quad e(N-1)\mu_{(\alpha|_1 \beta)} F_j^\alpha F_k^\beta + \mu_{(j|_1 k)} = \theta_{1 \alpha \beta}^4 F_j^\alpha F_k^\beta.$$

Now, from (4.17) and (4.18) we obtain

$$(4.19) \quad \mu_{(i|j)} = \overset{5}{\theta}_{ij},$$

where  $\overset{5}{\theta}_{ij} = N^{-1}(2 - N)^{-1}[\overset{4}{\theta}_{\alpha\beta}F_i^\alpha F_j^\beta - e(N - 1)\overset{4}{\theta}_{ij}]$ . Let us take the covariant derivative of the first kind of the (4.19) in the direction of  $x^k$ :

$$(4.20) \quad \mu_{i|jk} + \mu_{j|ik} = \overset{5}{\theta}_{ij|k}$$

and alternate this equation with respect to the indices  $i$  and  $k$ . Then we have

$$\mu_{i|jk} - \mu_{k|ji} - \overset{\alpha}{R}_{ijk}\mu_\alpha - L_{[jk]}^\alpha \mu_{i|\alpha} = \overset{5}{\theta}_{ij|k} - \overset{5}{\theta}_{kj|i}.$$

Switching indices  $j$  and  $k$ , we obtain

$$\mu_{i|kj} - \mu_{j|ki} - \overset{\alpha}{R}_{ikj}\mu_\alpha - L_{[kj]}^\alpha \mu_{i|\alpha} = \overset{5}{\theta}_{ik|j} - \overset{5}{\theta}_{jk|i}.$$

After adding this result to (4.20), we get

$$(4.21) \quad \mu_{i|(jk)} + \mu_{j|[ki]} = \overset{\alpha}{R}_{ikj}\mu_\alpha + L_{[kj]}^\alpha \mu_{i|\alpha} + \overset{5}{\theta}_{ik|j} - \overset{5}{\theta}_{jk|i} + \overset{5}{\theta}_{ij|k}.$$

Finally, we obtain a system of differential equations of the Cauchy type with covariant derivatives with respect to unknown functions  $\mu_i$ ,  $\mu_{ij}$ ,  $F_i^h$  and  $F_{ij}^h$ :

$$(4.22) \quad \begin{aligned} F_{i|j}^h &= F_{ij}^h, \\ F_{i(j|k)}^h &= \overset{6}{\theta}_{ijk}^h, \\ \mu_{i|j} &= \mu_{ij}, \\ \mu_{i|(jk)} + \mu_{j|[ki]} &= \overset{7}{\theta}_{ijk}^h, \end{aligned}$$

where

$$\begin{aligned} \overset{6}{\theta}_{ijk}^h &= -\overset{h}{\xi}_{ijk}^h + \mu_{(j|k)}F_i^h + \mu_{[i|k]}F_j^h + \mu_{[i|j]}F_k^h - \mu_{\alpha|[j}F_k^\alpha\delta_i^h \\ &\quad + \mu_{\alpha|[i}F_k^\alpha\delta_j^h + \mu_{\alpha|[i}F_j^\alpha\delta_k^h + \overset{1}{\theta}_{ikj}^h \end{aligned}$$

and

$$\theta_{ijk}^7 = R_{ikj}^\alpha \mu_\alpha + L_{[kj]}^\alpha \mu_{i|}^\alpha + \theta_{ik|j}^5 - \theta_{jk|i}^5 + \theta_{ij|k}^5.$$

On the other hand, functions  $\mu_i$ ,  $\mu_{ij}$ ,  $F_i^h$  and  $F_{ij}^h$  satisfy the algebraic formulas

$$(4.23) \quad \begin{aligned} F_{(i|j)}^h + \xi_{\alpha(i)}^h F_j^\alpha &= \mu_{(i} F_{j)}^h - \mu_\alpha F_{(i}^\alpha \delta_{j)}^h, \\ F_\alpha^h F_i^\alpha &= e \delta_i^h, \quad \mu_{(ij)} = \theta_{ij}^5. \end{aligned}$$

The system (4.22) has at most one solution for initial conditions (4.23). Initial conditions are limited by the algebraic ones (4.23). It can be easily seen that the initial conditions have at most

$$\frac{1}{2}N(N^2 - 1)$$

independent parameters. In this way, the following theorems are proved.

**Theorem 4.4.** *The equations (4.22) and (4.23) give an algebraic differential equation system of the Cauchy type in covariant derivatives with respect to the unknown functions  $\mu_i$ ,  $\mu_{ij}$ ,  $F_i^h$  and  $F_{ij}^h$ . This system generates all  $e$ -structures  $F_i^h$  determining  $G$ -almost geodesic mappings of the type  $\pi_2(e, F)$ ,  $e = \pm 1$ .*

**Theorem 4.5.** *Let  $GA_n$  be a non-symmetric affine connection space. A family of all  $e$ -structures  $F_i^h$  which determine a  $G$ -almost geodesic mapping of the type  $\pi_2(e, F)$ ,  $e = \pm 1$ , depends on at most  $N(N^2 - 1)/2$  real parameters.*

Analogously, we can consider the case of  $G$ -almost geodesic mappings of the type  $\pi_2(e)$ ,  $e = \pm 1$ .

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