

DOUBLY STOCHASTIC MATRICES AND THE BRUHAT ORDER

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Abstract. The Bruhat order is defined in terms of an interchange operation on the set of permutation matrices of order n which corresponds to the transposition of a pair of elements in a permutation. We introduce an extension of this partial order, which we call the stochastic Bruhat order, for the larger class Ω_n of doubly stochastic matrices (convex hull of $n \times n$ permutation matrices). An alternative description of this partial order is given. We define a class of special faces of Ω_n induced by permutation matrices, which we call Bruhat faces. Several examples of Bruhat faces are given and several results are presented.

Keywords: Bruhat order; doubly stochastic matrix; face

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1. INTRODUCTION

Let $\mathcal{S}_n$ denote the symmetric group of order n consisting of all permutations of $\{1, 2, \dots, n\}$. With each permutation $\sigma \in \mathcal{S}_n$, there is a corresponding $n \times n$ permutation matrix $P = [p_{ij}]$, where $p_{ij} = 1$ if and only if $j = \sigma(i)$. Let $\mathcal{P}_n$ denote the set of all $n \times n$ permutation matrices. The *Bruhat order* on $\mathcal{S}_n$ in terms of $\mathcal{P}_n$ is the partial order $\preceq_B$ defined as $P \preceq_B Q$ provided that P can be obtained from Q by a sequence of *backward interchanges*, that is, replacing 2×2 submatrices equal to L_2 with I_2 as shown below:

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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It follows that the identity matrix I_n is the unique minimal element (no backward interchanges possible) and the anti-identity matrix L_n is the unique maximal element ($\binom{n}{2}$ backward interchanges possible) of the Bruhat order on $\mathcal{P}_n$.

For an $m \times n$ matrix $A = [a_{ij}]$ we define the $m \times n$ matrix

$$\Sigma(A) = [\sigma_{ij}(A)], \quad \text{where } \sigma_{ij}(A) = \sum_{1 \leq k \leq i, 1 \leq l \leq j} a_{kl}, \quad 1 \leq i \leq m, \text{ and } 1 \leq j \leq n.$$

The Bruhat order on $\mathcal{P}_n$ may be characterized as follows. For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ we write $A \geq B$ (or $B \leq A$) to denote entrywise inequality.

The following result is known; see Theorem 2.1.5 in [1] or Lemma 7 of [7].

Theorem 1.1 ([1], [7]). *Let $P, Q \in \mathcal{P}_n$. Then $P \preceq_B Q$ if and only if $\Sigma(P) \geq \Sigma(Q)$.*

An improved version of this characterization was shown in [2]. The Bruhat order for the class of $(0, 1)$ -matrices with given row and column sums was investigated in [5], [6].

Recall that a square matrix is *doubly stochastic* provided it is nonnegative and each row and column sum is 1. We let Ω_n denote the set of doubly stochastic matrices of order n . Then Ω_n is a convex polytope of dimension $(n-1)^2$, often called the *Birkhoff polytope*, whose set of vertices is $\mathcal{P}_n$. Let $A_1, A_2 \in \Omega_n$. If $\Sigma(A_1) \geq \Sigma(A_2)$, we write $A_1 \preceq_B A_2$. This is a partial order on Ω_n , which we call the *stochastic Bruhat order*. Due to Theorem 1.1, the stochastic Bruhat order on Ω_n , when restricted to $\mathcal{P}_n$, reduces to the Bruhat order on $\mathcal{P}_n$.

The goal of this paper is to investigate properties of the stochastic Bruhat order and related subpolytopes of Ω_n .

A vector $x = (x_1, x_2, \dots, x_n)$ is *non-decreasing* if $x_1 \leq x_2 \leq \dots \leq x_n$. The *support* of an $m \times n$ matrix $A = [a_{ij}]$ is the set $\text{supp } A = \{(i, j) : a_{ij} \neq 0\}$. An $n \times n$ matrix A has *total support* if each of its nonzero elements lies in a nonzero diagonal of A (a permutation set of places occupied by nonzeros of A). The *convex hull* of a set S is denoted by $\text{conv } S$. We recall some notions and results from [4]. Let $P = [p_{ij}]$ be a permutation matrix of order n corresponding to a permutation $\sigma = (i_1, i_2, \dots, i_n)$ of $\{1, 2, \dots, n\}$. The *Bruhat shadow* $\mathcal{S}(P)$ of P is the $(0, 1)$ -matrix of order n whose support equals the union of the supports of all permutation matrices Q satisfying $Q \preceq_B P$, i.e., $\mathcal{S}(P)$ is the Boolean sum of these matrices. Define the *left-sequence*¹ of P as $l(P) = l_1, l_2, \dots, l_n$, where l_k is the largest integer in the set $\{i_1, i_2, \dots, i_k\}$ of integers ($k = 1, 2, \dots, n$). Similarly, we define the *right-sequence* $r(P) = r_1, r_2, \dots, r_n$

¹ The terminology left- and right- is due to the first k positions and last k positions, respectively, in the sequence σ .

of P , where r_k is the smallest integer in the set $\{i_k, i_{k+1}, \dots, i_n\}$. Then $r_k \leq k \leq l_k$ and $r_k \leq i_k \leq l_k$ for $k = 1, 2, \dots, n$.

Theorem 1.2 ([4]). *Let P be a permutation matrix of order n . Then its Bruhat shadow $\mathcal{S}(P) = [s_{kj}]$ is given by*

$$s_{kj} = \begin{cases} 1 & \text{if } r_k \leq j \leq l_k, \\ 0 & \text{otherwise,} \end{cases} \quad 1 \leq k \leq n, \text{ and } 1 \leq j \leq n.$$

The matrix $\mathcal{S}(P)$ has total support.

The definition of the left- and right-sequences implies that the matrix $\mathcal{S}(P)$ has a staircase pattern with $I_n \leq \mathcal{S}(P)$ and $P \leq \mathcal{S}(P)$. Here by a *staircase pattern* we mean that the 1's in each row and column are consecutive where the first (last) 1 in a row is in the same or earlier (later) column than the first (last) 1 in the following row. For example, if $\sigma = (5, 7, 1, 3, 2, 6, 4)$, we have $l(P) = 5, 7, 7, 7, 7, 7, 7$ and $r(P) = 1, 1, 1, 2, 2, 4, 4$, so

$$\mathcal{S}(P) = \begin{bmatrix} 1 & 1 & 1 & 1 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} \\ \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \mathbf{1} & 1 & 1 & 1 & 1 \\ 0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & \mathbf{1} & 1 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \end{bmatrix},$$

where the 1's of the permutation matrix corresponding to σ are in boldface.

2. DOUBLY STOCHASTIC MATRICES

Given a permutation matrix $Q \in \mathcal{P}_n$, let

$$(\preceq_B Q) = \{P \in \mathcal{P}_n : P \preceq_B Q\}.$$

Then $(\preceq_B Q)$ is a principal ideal of the Bruhat order on $\mathcal{P}_n$. Let

$$\Omega_n(\preceq_B Q) = \text{conv}(\preceq_B Q)$$

be the convex hull of $(\preceq_B Q)$, which is a subpolytope of Ω_n . Moreover, we define

$$\Omega_n(\geq \Sigma(Q)) = \{A \in \Omega_n : \Sigma(A) \geq \Sigma(Q)\}$$

and this set coincides with $\{A \in \Omega_n : A \preceq_B Q\}$.

Any $(0, 1)$ -matrix C of order n having total support induces a face of the Birkhoff polytope Ω_n as

$$\Omega_n^C := \{A \in \Omega_n : A \leq C\}.$$

In addition, any face of Ω_n arises from such a unique C in this way (see [3]). In particular, when Q is a permutation matrix, $\mathcal{S}(Q)$ has total support, so $\Omega_n^{\mathcal{S}(Q)}$ is a face of Ω_n .

Proposition 2.1. *Let $Q = [q_{ij}]$ be a permutation matrix of order n corresponding to the permutation $(i_1, i_2, \dots, i_n)$. Then*

$$(1) \quad \Omega_n(\preceq_B Q) \subseteq \Omega_n(\geq \Sigma(Q)) \subseteq \Omega_n^{\mathcal{S}(Q)}$$

and all these sets are polytopes.

Proof. We have that $\Omega_n(\geq \Sigma(Q))$ is a polytope, as it is a bounded polyhedron defined by the n^2 linear inequalities from $\Sigma(A) \geq \Sigma(Q)$ and the linear equations/inequalities defining the Birkhoff polytope. Since $\Omega_n(\geq \Sigma(Q))$ contains each $P \in \mathcal{P}_n$ satisfying $P \preceq_B Q$, the first inclusion in (1) follows from convexity.

Next, we show that $\Omega_n(\geq \Sigma(Q)) \subseteq \Omega_n^{\mathcal{S}(Q)}$. Let $A = [a_{ij}] \in \Omega_n(\geq \Sigma(Q))$ and $1 \leq k \leq n$. Since the ones in rows $1, 2, \dots, k$ of Q are in columns $i_1, i_2, \dots, i_k$, $\sigma_{kl_k}(A) \geq k$, where $l_k = \max\{i_1, i_2, \dots, i_k\}$. But $\sigma_{kn}(A) = k$, so we conclude that $a_{kj} = 0$ for $j > l_k$. Similarly, consider column k of A and let l be the largest index of the row that contains a one within columns $1, 2, \dots, k$. The staircase pattern of $\mathcal{S}(Q)$ now implies that all the ones in columns $1, 2, \dots, k$ of Q are in rows $1, 2, \dots, l$, so $\sigma_{lk}(A) \geq \sigma_{lk}(Q) = k$. Therefore $a_{ik} = 0$ for $i > l$. This shows that $A \leq \mathcal{S}(Q)$, so $A \in \Omega_n^{\mathcal{S}(Q)}$. $\square$

Note that if $A \in \Omega_n$, then the entries in the last row and the last column of $\Sigma(A)$ are $1, 2, \dots, n$.

Example 1. In this example we show that the first containment in Proposition 2.1 may be strict. Let

$$Q = \begin{bmatrix} & & & 1 & & \\ & 1 & & & & \\ & & & & 1 & \\ 1 & & & & & \\ & & & & & 1 \\ & & 1 & & & \end{bmatrix}.$$

Then

$$\Sigma(Q) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Let $A = [a_{ij}] \in \Omega_6$. If $\Sigma(A) \geq \Sigma(Q)$, then by Proposition 2.1 A has zeros as shown below:

$$A = \begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \\ & & & & & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix}.$$

Since A is assumed to be doubly stochastic, the only inequality in $\Sigma(A) \geq \Sigma(Q)$ that does not follow from the form of A is

$$\sigma_{22}(A) \geq \sigma_{22}(Q), \quad \text{that is } a_{11} + a_{12} + a_{21} + a_{22} \geq 1.$$

Let

$$A = \frac{1}{4} \begin{bmatrix} 3 & & 1 & & & \\ & 3 & & 1 & & \\ & 1 & 3 & & & \\ 1 & & & 3 & & \\ & & & & 3 & 1 \\ & & & & 1 & 3 \end{bmatrix}.$$

Since $\sigma_{22}(A) = 3/2 \geq 1$, it follows that A satisfies $\Sigma(A) \geq \Sigma(Q)$ and hence that $A \preceq_B Q$. However, A is not in the convex hull of $(\preceq_B Q)$ because any permutation matrix with a one in position $(1, 3)$ whose support is a subset of the support of A is of the form

$$P = \begin{bmatrix} & & 1 & & & \\ & & & 1 & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & * & * \\ & & & & * & * \end{bmatrix}$$

and is not in $(\preceq_B Q)$.

The previous example leads to the following question concerning a weaker property. Since $\Omega_n(\preceq_B)$ may not equal $\Omega_n(\geq \Sigma(Q))$, a weaker property is that $A \in \Omega_n(\geq \Sigma(Q))$ implies that there exists a permutation matrix P with $P \preceq_B Q$ and $\text{supp } P \subseteq \text{supp } A$. But even this may not be true as the following example shows.

Example 2. Consider the following permutation matrix Q where the zeros shown are those of the Bruhat shadow:

$$Q = \begin{bmatrix} & & & & & 1 & 0 \\ & & & & 1 & & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \\ & & & 1 & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ 1 & & & & & & \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} & & & 1/2 & & 1/2 & & \\ & & & & 1/2 & & 1/2 & \\ & & & & 1/2 & 1/2 & & \\ & & & & & & 1/2 & 1/2 \\ 1/2 & & & & & & & 1/2 \\ & 1/2 & 1/2 & & & & & \\ 1/2 & & 1/2 & & & & & \\ & 1/2 & & 1/2 & & & & \end{bmatrix}.$$

Clearly $A \in \Omega_8^{S(Q)}$, and one can verify that $A \in \Omega_8(\geq \Sigma(Q))$. Consider the permutation matrices

$$P_1 = \begin{bmatrix} & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 0 & 1 \\ 1 & & & & & & 0 \\ & 1 & 0 & & & & \\ 0 & & 1 & & & & \\ & 0 & 1 & & & & \end{bmatrix}, \quad P_2 = \begin{bmatrix} & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & 0 & 1 & \\ & & & & & 1 & 0 \\ 0 & & & & & & 1 \\ & 0 & 1 & & & & \\ 1 & & 0 & & & & \\ & 1 & 0 & & & & \end{bmatrix}.$$

Here $P_1, P_2 \in \Omega_8^{S(Q)}$ but $P_1 \not\preceq_B Q$ and $P_2 \not\preceq_B Q$ as

$$0 = \sigma_{25}(P_1) < \sigma_{25}(Q) = 1 \quad \text{and} \quad 0 = \sigma_{62}(P_2) < \sigma_{62}(Q) = 1.$$

Now, $A = \frac{1}{2}P_1 + \frac{1}{2}P_2$, and the only permutation matrices P satisfying $\text{supp } P \subseteq \text{supp } A$ are P_1 and P_2 . The last fact is easy to check directly. In fact, P_1 and P_2

have disjoint support and their union corresponds to a single cycle in the bipartite graph representation of the permutation matrices.

In the previous example, $P_1, P_2 \in \Omega_8^{S(Q)}$ but $P_1, P_2 \not\leq_B Q$, and hence by Theorem 1.2, P_1, P_2 are not in $\Omega_n (\geq \Sigma(Q))$, which shows that the second containment in Proposition 2.1 can be proper, even with respect to permutation matrices.

Example 3. Let

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

There are four permutation matrices in $(\leq_B Q)$, namely the 3×3 permutation matrices with a zero in position $(3, 1)$. Hence it follows that

$$\Omega_3(\leq_B Q) = \Omega_3^{S(Q)} = \left\{ \begin{bmatrix} b+d & c & a \\ a+c & d & b \\ 0 & a+b & c+d \end{bmatrix} : a, b, c, d \geq 0, a+b+c+d=1 \right\}.$$

Let $A = [a_{ij}] \in \Omega_3$ satisfy

$$\Sigma(A) \geq \Sigma(Q) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Then $a_{11} + a_{21} \geq 1$, and hence $a_{11} + a_{21} = 1$ and $a_{31} = 0$. Thus, A is a convex combination of 3×3 permutation matrices with entry $(3, 1)$ equal to 0, that is, A is in $\Omega_3(\leq_B Q)$. Thus, in this case

$$\Omega_3(\leq_B Q) = \Omega_3 (\geq \Sigma(Q)) = \Omega^{S(Q)}.$$

□

Let $A_1, A_2 \in \Omega_n$. Our goal is to obtain a better understanding of the *stochastic Bruhat order* (recall $A_1 \leq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$).

Let $A = [a_{ij}] \in \Omega_n$. A *backward ε -interchange* of A is a replacement of a 2×2 submatrix of A with another 2×2 matrix as indicated below:

$$\begin{bmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{ij} + \varepsilon & a_{il} - \varepsilon \\ a_{kj} - \varepsilon & a_{kl} + \varepsilon \end{bmatrix}.$$

A *forward ε -interchange* is defined by

$$\begin{bmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{ij} - \varepsilon & a_{il} + \varepsilon \\ a_{kj} + \varepsilon & a_{kl} - \varepsilon \end{bmatrix}.$$

Here ε is assumed to satisfy $0 < \varepsilon \leq a_{il}, a_{kj}$ in the backward case, and $0 < \varepsilon \leq a_{ij}, a_{kl}$ in the forward case. If A' results from a doubly stochastic matrix by a backward ε -interchange in rows i_0, i_1 and columns j_0, j_1 , then $\Sigma(A')$ is given by

$$\sigma_{ij}(A') = \begin{cases} \sigma_{ij}(A) + \varepsilon & \text{if } i_0 \leq i < i_1 \text{ and } j_0 \leq j < j_1, \\ \sigma_{ij}(A) & \text{otherwise.} \end{cases}$$

Thus, if A' results from $A \in \Omega_n (\geq \Sigma(Q))$ by a sequence of backward ε -interchanges, then also $A' \in \Omega_n (\geq \Sigma(Q))$. Applying a forward ε -interchange, $\sigma_{ij}(A) + \varepsilon$ is replaced by $\sigma_{ij}(A) - \varepsilon$ in the expression above. Note that forward and backward ε -interchanges are inverse operations of each other.

Theorem 2.2. *Let $A_1, A_2 \in \Omega_n$. Then the following statements are equivalent:*

- (i) $A_1 \preceq_B A_2$,
- (ii) A_1 can be obtained from A_2 by a finite sequence of backward ε -interchanges; equivalently, A_2 can be obtained from A_1 by a finite sequence of forward ε -interchanges.

Proof. As shown above, (ii) implies (i), so we only need to prove that (i) implies (ii). Assume $A_1 \preceq_B A_2$, where $A_1 = [a_{ij}]$, $A_2 = [a'_{ij}]$. If $A_1 = A_2$, then there is nothing to be proved.

If $A_1 \neq A_2$, then there is at least one entry (i, j) such that $a_{ij} \neq a'_{ij}$. We define the sets of positions

$$\Delta_+ = \{(i, j): a_{ij} < a'_{ij}\},$$

$$I = \{(i, j): \sigma_{ij}(A_1) > \sigma_{ij}(A_2)\}.$$

Let i_0 be the first row in which A_1 and A_2 differ, and let j_1 be the largest j with $a_{i_0 j} \neq a'_{i_0 j}$. Clearly, $j_1 > 1$ because otherwise row i_0 of A_1 or A_2 would not have sum one. In the arguments that follow we use that $\sigma_{i_0 n}(A_1) = \sigma_{i_0 n}(A_2) = i_0$. We have $(i_0, j_1) \notin I$ because $a_{ij} = a'_{ij}$ for $i \leq i_0$ and $j > j_1$, so $\sigma_{i_0 j_1}(A_1) = \sigma_{i_0 j_1}(A_2)$. Since $\sigma_{i_0 j_1 - 1}(A_1) \geq \sigma_{i_0 j_1 - 1}(A_2)$, we conclude that $a_{i_0 j_1} < a'_{i_0 j_1}$ and hence $(i_0, j_1) \in \Delta_+$. Note that $(i_0, j_1 - 1) \in I$ because $(i_0, j_1) \in \Delta_+$.

Let $j_0 < j_1$ be the smallest index such that $(i_0, j) \in I$ for all $j_0 \leq j < j_1$ (j_0 exists, $j_1 - 1$ is one candidate). Now let $i_1 > i_0$ be the largest index such that $(i, j) \in I$ for all $j_0 \leq j < j_1$ and $i_0 \leq i < i_1$ (note that i_1 exists, $i_0 + 1$ is a candidate and there is no element in row n belonging to I). In row i_1 , there is a column $j_0 \leq x < j_1$ such that $(i_1, x) \notin I$, otherwise i_1 would be bigger.

For the contradiction, let us suppose that $a_{ij} \geq a'_{ij}$ is in the rectangle given by $i_0 < i \leq i_1$ and $j_0 \leq j < j_1$.

		j_0		x		j_1
i_0	$\nwarrow$	I	I	I	I	Δ_+
	I	I	I	I	I	
	I	I	I	I	I	
i_1		I		$\odot$	I	

Using the minimality of j_0 we have $\sigma_{i_0 j_0 - 1}(A_1) = \sigma_{i_0 j_0 - 1}(A_2)$, of course if $j_0 \geq 2$ (if $j_0 = 1$, then we disregard rectangles with column $j_0 - 1$). Also recall that $\sigma_{i_1 j_0 - 1}(A_1) \geq \sigma_{i_1 j_0 - 1}(A_2)$ in general, and $\sigma_{i_0 x}(A_1) > \sigma_{i_0 x}(A_2)$ since $(i_0, x) \in I$. Then, by the above assumption, we have

$$\begin{aligned}
\sigma_{i_1 x}(A_1) &= \sigma_{i_1 j_0 - 1}(A_1) + \sigma_{i_0 x}(A_1) - \sigma_{i_0 j_0 - 1}(A_1) + \sum_{\substack{i_0 < i \leq i_1 \\ j_0 \leq j \leq x}} a_{ij} \\
&> \sigma_{i_1 j_0 - 1}(A_2) + \sigma_{i_0 x}(A_2) - \sigma_{i_0 j_0 - 1}(A_2) + \sum_{\substack{i_0 < i \leq i_1 \\ j_0 \leq j \leq x}} a'_{ij} \\
&= \sigma_{i_1 x}(A_2),
\end{aligned}$$

a contradiction since $(i_1, x) \notin I$. Thus, there is a $(i_*, j_*) \in \Delta_+$ contained in the rectangle defined by $i_0 < i_* \leq i_1$ and $j_0 \leq j_* < j_1$.

Now we can apply a backward ε -interchange to A_2 by adding the matrix

$$\begin{bmatrix} \varepsilon & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix}$$

to the submatrix $A_2[i_0, i_* | j_*, j_1]$ determined by rows i_0 and i_* , and columns j_* and j_1 with

$$\varepsilon = \min\{a'_{i_0 j_1} - a_{i_0 j_1}; a'_{i_* j_*} - a_{i_* j_*}; \sigma(A_1)_{ij} - \sigma(A_2)_{ij} \text{ for } i_0 \leq i < i_* \text{ and } j_* \leq j < j_1\}.$$

This operation creates a matrix A^* such that $\Sigma(A_1) \geq \Sigma(A^*) \geq \Sigma(A_2)$ with at least one entry of $\Sigma(A^*)$ strictly bigger than the corresponding entry of $\Sigma(A_2)$. Therefore $A_1 \preceq_B A^* \preceq_B A_2$.

Case 1: If $\varepsilon = \min\{\sigma(A_1)_{ij} - \sigma(A_2)_{ij}\}$, then we have strictly increased the number of entries where $\Sigma(A_1)$ and $\Sigma(A_2)$ agree, that is, some entries of I are removed.

Case 2: If $\varepsilon = a'_{i_0 j_1} - a_{i_0 j_1}$ (upper right corner), then (i_0, j_1) is no longer in Δ_+ . In the next step, we will take a new element of Δ_+ in column $j_2 < j_1$, or there will be no more elements in row i_0 in Δ_+ . In any case, the position $(i_0, j_1 - 1)$ will

no longer belong to the set I , and again we strictly increased the number of entries where $\Sigma(A_1)$ and $\Sigma(A_2)$ agree.

Case 3: If $\varepsilon = a'_{i_* j_*} - a_{i_* j_*}$, then (i_*, j_*) is no longer in Δ_+ . But there could be another position $(i_{**}, j_{**}) \in \Delta_+$ in the rectangle $i_0 < i \leq i_1$ and $j_0 \leq j < j_1$. We repeat applying backwards ε -interchanges until (i_0, j_1) is eliminated from Δ_+ , which eliminates $(i_0, j_1 - 1)$ from I .

Since each backward ε -interchange brings A_2 closer to A_1 by decreasing $|I|$, eventually we will have $I = \emptyset$ and then A_1 is reached, as desired.

A_2 can be obtained from A_1 by forward ε -interchanges in the reverse order. $\square$

The previous proof gives an algorithm for bringing A_2 to A_1 when $A_1 \preceq_B A_2$ holds. We illustrate this algorithm by an example.

Example 4. Consider the matrices A_1 and A_2 below such as $A_1 \preceq_B A_2$ and I and Δ_+ are shown schematically:

$$A_1 = \frac{1}{10} \begin{bmatrix} 3 & 1 & 2 & 3 & 1 \\ 3 & 2 & 4 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \\ 2 & 1 & 0 & 3 & 4 \\ 0 & 3 & 0 & 3 & 4 \end{bmatrix}, \quad A_2 = \frac{1}{10} \begin{bmatrix} 2 & 1 & 3 & 0 & 4 \\ 4 & 2 & 2 & 2 & 0 \\ 2 & 2 & 5 & 1 & 0 \\ 1 & 1 & 0 & 4 & 4 \\ 1 & 4 & 0 & 3 & 2 \end{bmatrix};$$

$$I\Delta_+ = \begin{bmatrix} I & I & \Delta & I & \Delta \\ \Delta & & I & I^\Delta & \\ & I & I^\Delta & I & \\ I & I & I & I^\Delta & \\ \Delta & \Delta & & & \end{bmatrix}.$$

The first modification to bring A_2 to A_1 consists of a backward ε -interchange using position $(i_0, j_1) = (1, 5) \in \Delta_+$. There are two positions in the rectangle $1 < i \leq 5$ and $4 \leq j < 5$ belonging to Δ_+ . We choose $(2, 4) \in \Delta_+$ and apply $\varepsilon(1, 2 : 4, 5 : 2/10)$, the backward ε -interchange in rows 1 and 2, and columns 4 and 5, for $\varepsilon = \frac{2}{10} = \frac{1}{10} \min\{3, 2, 3\}$. This leads us to the matrix

$$B_1 = \frac{1}{10} \begin{bmatrix} 2 & 1 & 3 & 2 & 2 \\ 4 & 2 & 2 & 0 & 2 \\ 2 & 2 & 5 & 1 & 0 \\ 1 & 1 & 0 & 4 & 4 \\ 1 & 4 & 0 & 3 & 2 \end{bmatrix}; \quad \begin{bmatrix} I & I & \Delta & I & \Delta \\ \Delta & & I & I & \Delta \\ & I & I^\Delta & I & \\ I & I & I & I^\Delta & \\ \Delta & \Delta & & & \end{bmatrix}$$

with the property $A_1 \preceq_B B_1 \preceq_B A_2$. This operation does not get us closer to A_1 (Case 3 of the proof) in the sense that I remains the same. So we choose the next position in the rectangle that belongs to Δ_+ , so $(i_0, j_1) = (4, 4)$. We apply $\varepsilon(1, 4|4, 5|1/10)$ and obtain

$$B_2 = \frac{1}{10} \begin{bmatrix} 2 & 1 & 3 & \boxed{3} & \boxed{1} \\ 4 & 2 & 2 & 0 & 2 \\ 2 & 2 & 5 & 1 & 0 \\ 1 & 1 & 0 & \boxed{3} & \boxed{5} \\ 1 & 4 & 0 & 3 & 2 \end{bmatrix} ; \quad \begin{bmatrix} I & I & \Delta & & \\ \Delta & & I & I & \Delta \\ & I & I^\Delta & I & \\ I & I & I & I & \Delta \\ \Delta & \Delta & & & \end{bmatrix} .$$

This operation (Cases 1 and 2 in the proof) decreases $|I|$ by one. Next we have $(i_0, j_1) = (1, 3) \in \Delta_+$, and $j_0 = 1$ and $i_1 = 2$. We apply $\varepsilon(1, 2|1, 3|1/10)$ and obtain

$$B_3 = \frac{1}{10} \begin{bmatrix} \boxed{3} & 1 & \boxed{2} & 3 & 1 \\ \boxed{3} & 2 & \boxed{3} & 0 & 2 \\ 2 & 2 & 5 & 1 & 0 \\ 1 & 1 & 0 & 3 & 5 \\ 1 & 4 & 0 & 3 & 2 \end{bmatrix} ; \quad \begin{bmatrix} & & & & \\ & & I & I & \Delta \\ & I & I^\Delta & I & \\ I & I & I & I & \Delta \\ \Delta & \Delta & & & \end{bmatrix} .$$

Next we have $(i_0, j_1) = (2, 5) \in \Delta_+$ and choose the unique position of Δ_+ in the rectangle $2 < i \leq 5$ and $3 \leq j < 5$ and obtain

$$B_3 = \frac{1}{10} \begin{bmatrix} \boxed{3} & 1 & \boxed{2} & 3 & 1 \\ \boxed{3} & 2 & \boxed{3} & 0 & 2 \\ 2 & 2 & 5 & 1 & 0 \\ 1 & 1 & 0 & 3 & 5 \\ 1 & 4 & 0 & 3 & 2 \end{bmatrix} ; \quad \begin{bmatrix} & & & & \\ & & I & I & \Delta \\ & I & I^\Delta & I & \\ I & I & I & I & \Delta \\ \Delta & \Delta & & & \end{bmatrix} .$$

We apply $\varepsilon(2, 3|3, 5|1/10)$ and obtain

$$B_4 = \frac{1}{10} \begin{bmatrix} 3 & 1 & 2 & 3 & 1 \\ 3 & 2 & \boxed{4} & 0 & \boxed{1} \\ 2 & 2 & \boxed{4} & 1 & \boxed{1} \\ 1 & 1 & 0 & 3 & 5 \\ 1 & 4 & 0 & 3 & 2 \end{bmatrix} ; \quad \begin{bmatrix} & & & & \\ & & & & \\ & I & I & I & \Delta \\ I & I & I & I & \Delta \\ \Delta & \Delta & & & \end{bmatrix} .$$

Next we have $(i_0, j_1) = (3, 5) \in \Delta_+$ and apply $\varepsilon(3, 5|2, 5|1/10)$ obtaining

$$B_5 = \frac{1}{10} \begin{bmatrix} 3 & 1 & 2 & 3 & 1 \\ 3 & 2 & 4 & 0 & 1 \\ 2 & \boxed{3} & 4 & 1 & \boxed{0} \\ 1 & 1 & 0 & 3 & 5 \\ 1 & \boxed{3} & 0 & 3 & \boxed{3} \end{bmatrix} ; \quad \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ I & I & I & I & \Delta \\ \Delta & & & & \end{bmatrix} .$$

After that we have $(i_0, j_1) = (4, 5) \in \Delta_+$ and apply $\varepsilon(4, 5|1, 5|1/10)$ to finally reach

$$A_1 = \frac{1}{10} \begin{bmatrix} 3 & 1 & 2 & 3 & 1 \\ 3 & 2 & 4 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \\ \boxed{2} & 1 & 0 & 3 & \boxed{4} \\ \boxed{0} & 3 & 0 & 3 & \boxed{4} \end{bmatrix} .$$

We remark that one can see from Example 4 that Case 3 in the proof of Theorem 2.2 is in fact needed. The first step allows us to choose (i_*, j_*) as $(2, 4)$ or $(4, 4)$, but neither of these single choices will decrease set I .

3. BRUHAT FACES

A *Bruhat face* of Ω_n is a face $\mathcal{F}$ of Ω_n for which there exists a permutation matrix Q such that the set of vertices of $\mathcal{F}$ is $(\preceq_B Q)$; equivalently,

$$\{P \in \mathcal{P}_n : P \preceq \mathcal{S}(Q)\} = (\preceq_B Q).$$

We then write $\mathcal{F} = \mathcal{F}(Q)$ and say that Q *induces or generates the Bruhat face* $\mathcal{F}(Q)$. If $\mathcal{F}(Q)$ is a Bruhat face, then the $(0, 1)$ -matrix determining that face is the shadow $\mathcal{S}(Q)$ of Q . Thus, for a Bruhat face $\mathcal{F}$ the three sets in Proposition 2.1 coincide.

Following [4] we define the *Bruhat convex hull* of a $(0, 1)$ -matrix $A = [a_{ij}]$ as the $(0, 1)$ -matrix whose support is the union of all sets $\{(r, s) : i' \leq r \leq i \text{ and } j \leq s \leq j'\}$ such that $a_{ij} = a_{i'j'} = 1$ with $i' < i$ and $j < j'$. Let B be a matrix with staircase pattern and let S be its support. Let $(i, j) \in S$ and let B' be the Bruhat convex hull of the matrix with support $S \setminus \{(i, j)\}$. Then (i, j) is in an *extreme position* in B if $B \neq B'$. One might think that if each 1 in Q is in an extreme position, then Q induces a Bruhat face. However, this is not the case as the following example shows.

Example 5. Consider the permutation matrix

$$Q = \begin{bmatrix} & & 1 & & & \\ & & & & 1 & \\ 1 & & & & & \\ & & & & & 1 \\ & 1 & & & & \\ & & & 1 & & \end{bmatrix} \quad \text{with } \mathcal{S}(Q) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Q does not induce a Bruhat face. To see this, consider the following permutation matrix P which lies in $\Omega_n^{\mathcal{S}(Q)}$:

$$P = \begin{bmatrix} 1 & & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & & 1 \end{bmatrix}.$$

Here $P \not\leq_B Q$ as $\sigma_{33}(P) = 1 < 2 = \sigma_{33}(Q)$. Moreover, $Q \not\leq_B P$. Actually, both P and Q are maximal elements in the Bruhat order among permutation matrices in the face $\Omega_n^{\mathcal{S}(Q)}$.

We now consider which permutation matrices Q generate Bruhat faces. If $Q \in \mathcal{P}_n$ induces a Bruhat face, then no other permutation matrix induces the same Bruhat face. This is because if $Q' \in \mathcal{P}_n$ induces the same Bruhat face, then $\mathcal{S}(Q) = \mathcal{S}(Q')$, $Q' \preceq_B Q$ and $Q \preceq_B Q'$, so $Q = Q'$.

Clearly, if $Q \in \mathcal{P}_n$ and $Q' \in \mathcal{P}_m$ each induces a Bruhat face, then the direct sum $Q \oplus Q'$ induces a Bruhat face.

For a nonnegative $n \times n$ matrix $A = [a_{ij}]$ with non-decreasing rows and columns let

$$\Delta(A) = \{(i, j): a_{ij} > \max\{a_{i-1,j}, a_{i,j-1}\},$$

where we let $a_{0i} = a_{i0} = 0$, $1 \leq i \leq n$. Define for $1 \leq i \leq n$, $1 \leq j \leq n$

$$\gamma_{ij}(Q) = \min\{\sigma_{ij}(P): P \leq \mathcal{S}(Q), P \text{ a permutation matrix}\},$$

so $\gamma_{ij}(Q) \leq \sigma_{ij}(Q)$. Let $\Gamma(Q) = [\gamma_{ij}(Q)]$ be the corresponding $n \times n$ matrix with these numbers as its entries. Below we give a simple and efficient method for computing these numbers. $\Gamma(Q)$ is nonnegative and has non-decreasing rows and columns. This is also the case for matrix $\Sigma(Q)$. The *term rank* of a $(0, 1)$ -matrix A is the maximum cardinality of a set of ones in A such that no two are in the same row or column.

Theorem 3.1. *Let Q be a permutation matrix of order n . Then the following statements are equivalent:*

- (i) Q induces a Bruhat face.
- (ii) $\Gamma(Q) = \Sigma(Q)$.
- (iii) $\Delta(\Gamma(Q)) = \Delta(\Sigma(Q))$.
- (iv) *For each $i, j \leq n$ the term rank of the matrix obtained from $\mathcal{S}(Q)$ by replacing its leading $i \times j$ submatrix with a zero matrix is $n - \sigma_{ij}(Q)$.*

Proof. If Q induces a Bruhat face, then $\Gamma(Q) = \Sigma(Q)$ (for if $\gamma_{ij}(Q) < \sigma_{ij}(Q)$ for some i, j , then there would exist a $P \leq \mathcal{S}(Q)$ with $P \not\leq_B Q$). Conversely, if $\Gamma(Q) = \Sigma(Q)$, then every permutation matrix P with $P \leq \mathcal{S}(Q)$ also satisfies $\Sigma(P) \geq \Sigma(Q)$. This shows the equivalence of (i) and (ii).

Clearly, (ii) implies (iii). Next, assume (iii) holds. In each of matrices $\Gamma(Q)$ and $\Sigma(Q)$ the first row consists of a sequence of zeros followed by a sequence of ones. Moreover, the transition from 0 to 1 occurs in the same column j ; this follows from the assumption $\Delta(\Gamma(Q)) = \Delta(\Sigma(Q))$ because this set contains a unique element $(1, j)$ for some $j \leq n$. The second row of $\Gamma(Q)$ and $\Sigma(Q)$ consists of a sequence of 0's followed by a sequence of 1's and finally a sequence of 2's. Using again assumption (iii) and the fact that the first row of $\Gamma(Q)$ and $\Sigma(Q)$ coincide, we conclude that the second row of these two matrices coincide. We may proceed by induction and conclude that $\Gamma(Q) = \Sigma(Q)$ holds.

Finally, (ii) and (iv) are equivalent as (iv) means that the minimum number of ones in the leading $i \times j$ submatrix of a permutation matrix $P \leq \mathcal{S}(Q)$ is $\sigma_{ij}(Q)$. $\square$

Recall that a backward interchange in a permutation matrix P is replacing a 2×2 submatrix which is equal to L_2 by I_2 . Note that the resulting matrix P' satisfies $\mathcal{S}(P') \leq \mathcal{S}(P)$. Let $1 \leq i, j \leq n$, and let k be such that $\max\{i + j - n, 0\} \leq k \leq \min\{i, j\}$. Define the $n \times n$ permutation matrix

$$(2) \quad P^{(i,j,k,n)} = \begin{bmatrix} I_k & O & O & O \\ O & O_{i-k,j-k} & I_{i-k} & O \\ O & I_{j-k} & O & O \\ O & O & O & I_{n-i-j+k} \end{bmatrix}.$$

Theorem 3.2. *Let Q be a permutation matrix and $1 \leq i \leq n$, $1 \leq j \leq n$. Let $k = \gamma_{ij}(Q)$. Then $P^{(i,j,k,n)} \leq \mathcal{S}(Q)$ and*

$$\sigma_{ij}(P^{(i,j,k,n)}) = \gamma_{ij}(Q),$$

and thus $P^{(i,j,k,n)}$ minimizes $\sigma_{ij}(P)$ among all permutation matrices P satisfying $P \leq \mathcal{S}(Q)$.

Proof. Let $P = [p_{rs}] \in \mathcal{P}_n$ be such that $P \leq \mathcal{S}(Q)$ and $\sigma_{ij}(P) = \gamma_{ij}(Q) = k$. Assume that $k \geq 1$. If $p_{l1} = 1$ with $l > i$, choose (r, s) with $p_{rs} = 1$ and $r \leq i, s \leq j$. Then make a backward interchange for rows r, l and columns 1, s . The new matrix, still called P for simplicity, also has k ones in the leading $i \times j$ submatrix. If $r = 1$, we now have $p_{11} = 1$. Otherwise, when $p_{r1} = 1$ for some $r > 1$, make a backward interchange involving positions $(r, 1)$ and the position of the unique 1 in row 1. After this, the new updated matrix P satisfies $p_{11} = 1$. We may now delete the first row and column, and repeat this procedure for the remaining $k - 1$ ones in the leading $i \times j$ submatrix. After this we have

$$p_{11} = p_{22} = \dots = p_{kk} = 1.$$

So, even if $k = 0$, the leading $i \times j$ submatrix of P now coincides with that of $P^{(i,j,k,n)}$, and P has the following structure

$$P = \begin{bmatrix} I_k & O & O \\ O & O_{i-k,j-k} & A_{23} \\ O & A_{32} & A_{33} \end{bmatrix}.$$

Each column of A_{32} contains a 1 and with backward interchanges we may assure that each 1 in this submatrix is to the right of each 1 in its previous rows. This is possible due to the staircase structure and does not affect the number of ones in the leading $i \times j$ submatrix of P . Moreover, for each row in A_{32} which is zero, there must be a 1 in the same row in A_{33} . This fact makes it possible to perform backward interchanges until the leading $(j - k) \times (j - k)$ submatrix of A_{32} equals I_{j-k} . After this we have

$$P = \begin{bmatrix} I_k & O & O \\ O & O_{i-k,j-k} & A_{24} \\ O & I_{j-k} & O \\ O & O & A_{44} \end{bmatrix}.$$

Now, each row of A_{24} contains a 1 and with backward interchanges involving A_{24} and A_{44} we may assure that each 1 in this submatrix A_{24} is to the right of each 1 in its previous rows. Moreover, for each column in A_{24} which is zero, there must be a 1 in the same column in A_{44} . We may then use backward interchanges, so that the leading $(i - k) \times (i - k)$ submatrix of A_{24} equals I_{i-k} . Now backward interchanges on the lower right submatrix get us to $P = P^{(i,j,k,n)}$ as desired. $\square$

Corollary 3.3. For every $Q \in \mathcal{P}_n$ and $1 \leq i \leq n, 1 \leq j \leq n$

$$\gamma_{ij}(Q) = \min\{k: P^{(i,j,k,n)} \leq \mathcal{S}(Q)\}.$$

This corollary leads to a simple and efficient algorithm for computing $\gamma_{ij}(Q)$ for given i, j and $Q \in \mathcal{P}_n$: start with $k = \max\{i + j - n, 0\}$ and increase k by 1 until $P^{(i,j,k,n)} \leq \mathcal{S}(Q)$; then $k = \gamma_{ij}(Q)$. Combining this with Theorem 3.1 (ii) or (iii) we obtain a simple, and polynomial-time, algorithm for deciding if Q induces a Bruhat face. By (iv) of Theorem 3.1, the usual matching algorithm for bipartite graphs also gives a polynomial-time algorithm.

Example 6. Consider again Example 5, and let $i = j = 3$. Then

$$P^{(3,3,0,6)} = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \end{bmatrix}, \quad P^{(3,3,1,6)} = \begin{bmatrix} 1 & & & & & \\ & & & & & \\ & & & 1 & & \\ & & & & 1 & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \end{bmatrix}.$$

As $P^{(3,3,0,6)} \not\leq \mathcal{S}(Q)$, we conclude that $\gamma_{33}(Q) = 1$. As noted before, $\sigma_{33}(Q) = 2$, so Q does not induce a Bruhat face.

Define the *backward direct sum* $P_1 \oplus_b P_2$ of two square matrices P_1 and P_2 as the matrix

$$P_1 \oplus_b P_2 = \begin{bmatrix} O & P_1 \\ P_2 & O \end{bmatrix}.$$

More generally, for k square matrices P_i , $1 \leq i \leq k$, we define

$$P_1 \oplus_b \dots \oplus_b P_k = (P_1 \oplus_b \dots \oplus_b P_{k-1}) \oplus_b P_k.$$

Corollary 3.4. Let r, s, t be nonnegative integers such as $r + s + t = n$. Then the permutation matrix

$$(3) \quad Q = I_r \oplus_b L_s \oplus_b I_t$$

induces a Bruhat face whose shadow is given by $r_i = 1$ for $1 \leq i \leq r + s + 1$, $r_i = i - r - s$ for $r + s + 1 < i \leq n$ and $l_i = r + s + i$ for $1 \leq i < r$, $l_i = n$ for $r \leq i \leq n$.

Proof. By Theorem 1.2, the shadow of Q is as described in the statement of the corollary and thus is the $n \times n$ $(0, 1)$ -matrix which has zeros in its upper triangular right corner where I_r has zeros, zeros in its lower triangular left corner where I_t has zeros, and ones everywhere else. Using this characterization of the shadow of Q , the following calculations are straightforward to verify.

Let $1 \leq i, j \leq n$. We prove that $\sigma_{ij}(Q) = \gamma_{ij}(Q)$, and discuss different cases:

Case 1: $1 \leq i \leq r, 1 \leq j \leq n - r$. Then $\sigma_{ij}(Q) = 0 = \gamma_{ij}(Q)$.

Case 2: $1 \leq i \leq r, n - r < j \leq n$. Then $\sigma_{ij}(Q) = \min\{i, j - n + r\}$ and due to the staircase pattern of $\mathcal{S}(Q)$, this coincides with $\gamma_{ij}(Q)$.

Case 3: $1 \leq i \leq n - t, 1 \leq j \leq t$. Then $\sigma_{ij}(Q) = 0 = \gamma_{ij}(Q)$.

Case 4: $n - t < i \leq n, 1 \leq j \leq t$. Then $\sigma_{ij}(Q) = \min\{i - n + t, j\}$ and due to the staircase pattern of $\mathcal{S}(Q)$, this coincides with $\gamma_{ij}(Q)$.

Case 5: $r < i \leq n, t < j \leq n$. Then $\sigma_{ij}(Q) = \max\{i + j - n, 0\}$. On the other hand, any $P \in \mathcal{P}_n$ contains at most $n - i$ ones in rows $i + 1, i + 2, \dots, n$ and at most $n - j$ ones in columns $j + 1, j + 2, \dots, n$. Therefore such P contains at least $n - (n - i) - (n - j) = i + j - n$ in its leading $i \times j$ submatrix. So $\gamma_{ij}(Q) \geq i + j - n$. Since $\gamma_{ij}(Q) \geq 0$, this shows that $\sigma_{ij}(Q) = \max\{i + j - n, 0\} \leq \gamma_{ij}(Q)$, but then equality must hold here (as the opposite inequality holds by definition of $\gamma_{ij}(Q)$).

This proves that $\Sigma(Q) = \Gamma(Q)$, and the theorem follows. $\square$

Using Corollary 3.4 and the fact that the property of inducing a Bruhat face is preserved under taking direct sums, one may construct several permutation matrices that induce Bruhat faces, as illustrated in the next example.

Example 7. (i) I_n and L_n induce Bruhat faces; see Corollary 3.4 with $r = n, s = t = 0$ and $s = n, r = t = 0$, respectively. Therefore the direct sum $I_s \oplus L_r \oplus I_t$ also induces a Bruhat face.

(ii) $Q = I_r \oplus_b I_t$, where $r + t = n$, induces a Bruhat face ($s = 0$ in the corollary). In particular, with $t = 1$ one obtains a Hessenberg matrix, for instance

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(iii) The matrix $P^{(i,j,k,n)}$ in (2) induces a Bruhat face because

$$P^{(i,j,k,n)} = I_k \oplus (I_{i-k} \oplus_b I_{j-k}) \oplus I_{n-i-j+k},$$

so it is the direct sum of identity matrices and the matrix in (3) with $s = 0$.

For $n \leq 3$ one can check that every permutation matrix induces a Bruhat face (since it can be obtained from I_r and L_s (with $r + s = n$) by taking direct sum or backward direct sum). An example of a matrix which is not obtained using the

constructions above, but still induces a Bruhat face is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $n = 4$ the only permutation matrix that does not induce a Bruhat face is

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

But this is for the obvious reason that P is obtained by an “internal” backward interchange from L_4 , and this does not change the Bruhat shadow. We say that a permutation matrix is *shadow-maximal* if it allows no forward interchange within its Bruhat shadow (replacing a submatrix I_2 by L_2). Clearly, a necessary condition for a matrix to induce a Bruhat face is that it is shadow-maximal. Therefore a permutation of the form $I_r \oplus_b P \oplus_b I_s$ induces a Bruhat face if and only if P is the L permutation. But this condition (being shadow-maximal) is not sufficient. The matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is shadow-maximal, but it does not induce a Bruhat face. Indeed, the matrix $Q = L_4 \oplus I_1$ is within the Bruhat shadow of P , but we have $P \not\leq_B Q$ and $Q \not\leq_B P$. Also, in general, backward direct sums of permutations do not induce Bruhat faces, see the 4×4 matrix $I_1 \oplus_b I_2 \oplus_b I_1$ shown above.

A class of permutation matrices that induce Bruhat faces is discussed next. Let $\pi = (i_1, i_2, \dots, i_n)$ be a permutation of $\{1, 2, \dots, n\}$. Then i_k, i_{k+1} is a *descent* of π if $i_k > i_{k+1}$; we also say that a descent occurs at position k . Here $1 \leq k \leq n - 1$. A permutation is a *grassmanian* provided it has exactly one descent. We say that a permutation matrix Q is a grassmanian when its corresponding permutation is a grassmanian; if the unique descent of the permutation occurs at position k , then Q has a unique descent at row k . For example, with $n = 12$, $\sigma = (3, 6, 7, 9, 10, 1, 2, 4, 5, 8, 11, 12)$ is a grassmanian whose unique descent occurs at $k = 5$. Another example is the matrix $P^{(i,j,k,n)}$ defined in (2). The permutation

matrix corresponding to σ also with the zeros defining its shadow is:

$$Q = \begin{bmatrix} & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & & & & & 1 & 0 & 0 \\ 1 & & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & & \\ 0 & 0 & 0 & 1 & & & & & & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \end{bmatrix}.$$

Thus, the permutation matrices P with $P \leq \mathcal{S}(Q)$ are those whose 1's are 1's of Q or are in the empty positions. In the proof of the next theorem it may be helpful to refer to this example.

Theorem 3.5. *Let $Q = [q_{ij}]$ be an $n \times n$ grassmanian permutation matrix. Then Q induces a Bruhat face, so*

$$\Omega_n(\preceq_B Q) = \Omega_n(\geq \Sigma(Q)) = \Omega_n^{S(Q)}.$$

Proof. Let P be a permutation matrix with $P \preceq_B Q$. Then by definition, $P \leq \mathcal{S}(Q)$. Now suppose that $P \leq \mathcal{S}(Q)$ and P corresponds to the permutation $(i_1, i_2, \dots, i_n)$. To complete the proof we show that $P \preceq_B Q$ or equivalently, by Theorem 1.1, that $\Sigma(P) \geq \Sigma(Q)$.

Since Q is a grassmanian permutation matrix, it has a unique descent, say at row k . Since P is a permutation matrix and $P \leq \mathcal{S}(Q)$, it follows that

$$\sigma_{ij}(P) \geq \sigma_{ij}(Q) \quad \text{if either } 1 \leq i \leq k \text{ or } 1 \leq j < i_1.$$

Now assume that $i > k$ and $j \geq i_1$. We claim that the term rank of the matrix $\mathcal{S}(Q)_{ij}$ obtained from $\mathcal{S}(Q)$ by replacing its leading $i \times j$ submatrix with a zero matrix is at most $n - \sigma_{ij}(Q)$, that is, n minus the number of 1's of Q in its leading $i \times j$ submatrix. This follows from the assumption that Q is grassmanian, since we can then cover all the 1's of $\mathcal{S}(Q)_{ij}$ with $(n - j)$ columns $j + 1, j + 2, \dots, n$ (so each containing a 1 of Q) and $(j - \sigma_{ij}(Q))$ rows $u > i$ which contain a 1 in columns $1, 2, \dots, j$. Thus, $\mathcal{S}(Q)_{ij}$ has term rank $n - \sigma_{ij}(Q)$ proving the claim. Hence, any permutation matrix $P \leq \mathcal{S}(Q)$ contains $\sigma_{ij}(Q)$ 1's in its leading $i \times j$ submatrix. Then (see also (iv) in Theorem 3.1) we conclude that $\Sigma(P) \geq \Sigma(Q)$ and hence $P \preceq_B Q$. $\square$

We note that if a permutation matrix has more than one descent, it may, or may not, induce a Bruhat face. For instance, the permutation matrix in Example 5 does not induce a Bruhat face and it has two descents. The permutation matrix

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has two descents and induces a Bruhat face, namely Ω_3 itself.

Finally, we mention that the Bruhat order may be extended to the class $\mathcal{N}(R, S)$ of nonnegative matrices with row sum vector R and column sum vector S , the class of transportation matrices, and a study of this partial order is ongoing work. An interesting topic is to study the convex hull of $(\preceq Q)$, where Q is an extreme point, by linear constraints and determine its extreme points.

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References

- [1] *A. Björner, F. Brenti*: Combinatorics of Coxeter Groups. Graduate Texts in Mathematics 231, Springer, New York, 2005. [zbl](#) [MR](#)
- [2] *A. Björner, F. Brenti*: An improved tableau criterion for Bruhat order. Electron. J. Comb. 3 (1996), Research paper R22, 5 pages; printed version J. Comb. 3 (1996), 311–315. [zbl](#) [MR](#)
- [3] *R. A. Brualdi*: Combinatorial Matrix Classes. Encyclopedia of Mathematics and Its Applications 108, Cambridge University Press, Cambridge, 2006. [zbl](#) [MR](#)
- [4] *R. A. Brualdi, G. Dahl*: The Bruhat shadow of a permutation matrix. Mathematical Papers in Honour of Eduardo Marques de Sá. Textos de Matemática. Série B 39, Universidade de Coimbra, Coimbra, 2006, pp. 25–38. [zbl](#) [MR](#)
- [5] *R. A. Brualdi, L. Deaett*: More on the Bruhat order for $(0, 1)$ -matrices. Linear Algebra Appl. 421 (2007), 219–232. [zbl](#) [MR](#)
- [6] *R. A. Brualdi, S.-G. Hwang*: A Bruhat order for the class of $(0, 1)$ -matrices with row sum vector R and column sum vector S . Electron. J. Linear Algebra (electronic only) 12 (2004/2005), 6–16. [zbl](#) [MR](#)
- [7] *P. Magyar*: Bruhat order for two flags and a line. J. Algebr. Comb. 21 (2005), 71–101. [zbl](#) [MR](#)

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