

POINTWISE CONVERGENCE TO THE INITIAL DATA
FOR NONLOCAL DYADIC DIFFUSIONS

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Abstract. We solve the initial value problem for the diffusion induced by dyadic fractional derivative s in \mathbb{R}^+ . First we obtain the spectral analysis of the dyadic fractional derivative operator in terms of the Haar system, which unveils a structure for the underlying “heat kernel”. We show that this kernel admits an integrable and decreasing majorant that involves the dyadic distance. This allows us to provide an estimate of the maximal operator of the diffusion by the Hardy-Littlewood dyadic maximal operator. As a consequence we obtain the pointwise convergence to the initial data.

Keywords: pointwise convergence; nonlocal diffusion; dyadic fractional derivatives; Haar base

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1. INTRODUCTION

The classical result of pointwise convergence to the initial data for the heat equation $\partial u / \partial t = \Delta u$, $u(x, 0) = f(x)$ can naturally be extended to fractional diffusions of the type $\partial u / \partial t = -(-\Delta u)^s$, $u(x, 0) = f(x)$, with $0 < s \leq 1$. This fact follows from the estimates for the Fourier transform of $e^{-|\xi|^{2s}}$ given by Blumenthal and Geotor in [2]. In fact when $s = 1$ we are in the classical local case. In this case the solution is given by convolution of the Weierstrass kernel with f . Hence its maximal function $\sup_{t>0} |u(x, t)|$ is bounded above by the Hardy-Littlewood maximal function $Mf(x)$. When $0 < s < 1$ the results in [2] prove that, even when the decay of the kernel is no longer gaussian, we have that it is bounded above by a constant times $(1 + |x|)^{-n-2s}$.

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So we still have the inequality

$$(1.1) \quad \sup_{t>0} |u(x, t)| \leq CMf(x).$$

We can rephrase the above for $0 < \sigma < 2$ without the use of the Fourier transform by saying that (1.1) holds for the solution of

$$\begin{cases} \frac{\partial u}{\partial t} = -D^\sigma u, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

given by $k_t * f(x) = u(x, t)$ with $k(x)$ the inverse Fourier transform of $e^{-|\xi|^\sigma}$, where

$$D^\sigma g(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{g(x) - g(y)}{|x - y|^{n+\sigma}} dy.$$

It is well known that $D^\sigma = (-\Delta)^{\sigma/2}$, see for instance [3]. Let us remark that Fourier analysis is used here to obtain the right estimates for the kernel given by the Fourier transform of $e^{-|\xi|^\sigma}$.

In this note we aim at proving pointwise convergence for the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = -D_{\text{dy}}^\sigma u, & x \in \mathbb{R}^+, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}^+, \end{cases}$$

where

$$D_{\text{dy}}^\sigma g(x) = \int_{\mathbb{R}^+} \frac{g(x) - g(y)}{\delta(x, y)^{1+\sigma}} dy,$$

and $\delta(x, y)$ is the measure of the smallest dyadic interval containing both x and y . We say that δ is the dyadic distance in \mathbb{R}^+ . The dyadic operator D_{dy}^σ was introduced in [1]. We show that

$$\sup_{t>0} |u(x, t)| \leq CM_{\text{dy}}f(x)$$

where M_{dy} denotes the dyadic Hardy-Littlewood maximal operator, i.e.

$$(1.2) \quad M_{\text{dy}}f(x) = \sup_{x \in I \in \mathcal{D}} \frac{1}{|I|} \int_I |f|,$$

where the supremum is taken over the family of all dyadic intervals of \mathbb{R}^+ containing x . In doing that we unveil the generalized Fourier analysis involved in D_{dy}^σ ($0 < \sigma < 1$). The basic fact is that the Haar functions are eigenfunctions of D_{dy}^σ . This allows to provide a structure for the underlying “heat kernel”. Finally we show that this kernel admits an integrable and decreasing majorant in terms of $\delta(x, y)$.

The paper is organized as follows. We introduce the general setting and state the main results in Section 2. In Section 3 we obtain the spectral analysis of the operator D_{dy}^σ in terms of the Haar system and prove Theorem 2.1. Section 4 is devoted to obtaining the maximal estimate contained in statement (2.4) of Theorem 2.2 and as a consequence we demonstrate the pointwise convergence to the initial data.

2. SETTING AND STATEMENT OF THE MAIN RESULTS

Let $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}^j$ be the family of all dyadic intervals in \mathbb{R}^+ organized in generations \mathcal{D}^j . If I belongs to \mathcal{D}^j , then $I = I_k^j = [(k-1)2^{-j}, k2^{-j})$ for some $k \in \mathbb{Z}^+$ and $|I| = 2^{-j}$, where the vertical bars denote Lebesgue measure in \mathbb{R} .

For each $I \in \mathcal{D}^j$ there exist 2 disjoint intervals I^+ and I^- in \mathcal{D}^{j+1} both contained in I , which are precisely the right and left halves of I , respectively. An “ancestor” of I is any $J \in \mathcal{D}$ such that $I \subseteq J$. Given I and I' in \mathcal{D} , we shall say that $J \in \mathcal{D}$ is the “first common ancestor” of I and I' if J is an ancestor of both I and I' and $J \subseteq J'$ for any common ancestor J' of I and I' .

The dyadic distance $\delta(x, y)$ from x to y , both in \mathbb{R}^+ , is defined as the measure of the smallest dyadic interval $J \in \mathcal{D}$ containing both x and y and $\delta(x, x) = 0$. Notice that for any two points x and y in \mathbb{R}^+ , $\delta(x, y)$ is well defined since for $|j|$ large enough and j negative the interval $[0, 2^{-j})$ is dyadic and contains x and y . As is easy to see $|x - y| \leq \delta(x, y)$ but $1/\delta(x, y)$ is still singular in the sense that $\int_{\mathbb{R}^+} dy/\delta(x, y) = \infty$ even when $\int_{B_\delta(x, 1)} dy/\delta(x, y)^{1-\varepsilon}$ and $\int_{\mathbb{R}^+ \setminus B_\delta(x, 1)} dy/\delta(x, y)^{1+\varepsilon}$ are both finite for $\varepsilon > 0$. See Lemma 3.1 in Section 3.

For $I \in \mathcal{D}$ we shall write h_I to denote the Haar function supported on I . In other words $h_I = |I|^{-1/2}(\chi_{I^-} - \chi_{I^+})$, where χ_E denotes the indicator function of the set E . The system $\mathcal{H} = \{h_I : I \in \mathcal{D}\}$, known as the Haar system, is an orthogonal basis for $L^2(\mathbb{R}^+)$ and an unconditional basis for $L^p(\mathbb{R}^+)$, $1 < p < \infty$ (see for example [4]). The span of the set of Haar functions will be denoted by $S(\mathcal{H})$, that is, if $f \in S(\mathcal{H})$ then there exists a finite subset \mathcal{F}_n of \mathcal{D} such that

$$f(x) = \sum_{I \in \mathcal{F}_n} \langle f, h_I \rangle h_I(x),$$

where $\langle f, h_I \rangle$ denotes the inner product $\int_{\mathbb{R}^+} f h_I dx$ as far as it is well defined.

The fractional dyadic derivative of order $\sigma \in (0, 1)$ of a function f is defined by

$$D_{\text{dy}}^\sigma f(x) = \int_{\mathbb{R}^+} \frac{f(x) - f(y)}{\delta(x, y)^{1+\sigma}} dy,$$

provided that the integral is absolutely convergent. This is the case if for example f is a bounded Lipschitz function with respect to δ . In particular, if f is Lipschitz in the classical sense, since $|x - y| \leq \delta(x, y)$.

We are now in position to state our main results.

Theorem 2.1. *Let $0 < \sigma < 1$ be given. Then,*

(1) *for each $h_I \in \mathcal{H}$ we have*

$$(2.1) \quad D_{\text{dy}}^\sigma h_I(x) = b_\sigma |I|^{-\sigma} h_I(x),$$

with $b_\sigma = 1 + [2(2^\sigma - 1)]^{-1}$;

(2) *for $f \in S(\mathcal{H})$ the function u defined in $\mathbb{R}^+ \times \mathbb{R}^+$ by*

$$u(x, t) = \sum_{I \in \mathcal{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I(x)$$

solves the problem

$$(2.2) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) = -D_{\text{dy}}^\sigma u(x, t), & x \in \mathbb{R}^+, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}^+; \end{cases}$$

(3) *$u(x, t)$ can be written as an integral operator with the positive and finite kernel*

$$k_t(x, y) = \frac{1}{t^{1/\sigma}} \varphi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right),$$

where

$$\varphi(s) = \frac{1}{s} \left[-e^{-b_\sigma s^{-\sigma}} + \sum_{j \geq 1} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} \right]$$

for $t > 0$. In other words

$$(2.3) \quad u(x, t) = \int_{\mathbb{R}^+} k_t(x, y) f(y) dy.$$

Theorem 2.2. *Let $0 < \sigma < 1$, $1 < p < \infty$ and $f \in L^p(\mathbb{R}^+)$ be given. Then the integral in (2.3) is absolutely convergent for almost every $x \in \mathbb{R}^+$ and $u(\cdot, t)$ belongs to $L^p(\mathbb{R}^+)$ for each $t > 0$. Moreover,*

(1) *the function $u(x, t)$ satisfies*

$$(2.4) \quad \sup_{t > 0} |u(x, t)| \leq C M_{\text{dy}} f(x)$$

for some constant $C > 0$;

(2) *$u(\cdot, t)$ converges to f in L^p as t tends to 0;*

(3) *$\lim_{t \rightarrow 0^+} u(x, t) = f(x)$ for almost every $x \in \mathbb{R}^+$.*

3. THE DYADIC FRACTIONAL DIFFERENTIAL OPERATOR AND THE PROOF OF THEOREM 2.1

The first result in this section is an elementary lemma which reflects the one dimensional character of \mathbb{R}^+ equipped with the distance δ .

Lemma 3.1. *Let $0 < \varepsilon < 1$, and let I be a given dyadic interval in \mathbb{R}^+ . Then, for $x \in I$, we have*

$$\int_I \frac{dy}{\delta(x, y)^{1-\varepsilon}} = c_\varepsilon |I|^\varepsilon$$

and

$$\int_{\mathbb{R}^+ \setminus I} \frac{dy}{\delta(x, y)^{1+\varepsilon}} = C_\varepsilon |I|^{-\varepsilon},$$

where $c_\varepsilon = 2^{\varepsilon-1}(2^\varepsilon - 1)^{-1}$ and $C_\varepsilon = [2(2^\varepsilon - 1)]^{-1}$.

Proof. Observe that the open δ ball $B_\delta(x, r)$ is the largest dyadic interval I containing x with length less than r . Then for $I \in \mathcal{D}^j$ and $x \in I$ we have

$$\begin{aligned} \int_I \frac{dy}{\delta(x, y)^{1-\varepsilon}} &= \int_{B_\delta(x, 2^{-j+1})} \frac{dy}{\delta(x, y)^{1-\varepsilon}} = \sum_{k=j-1}^{\infty} \int_{\{y: 2^{-k-1} \leq \delta(x, y) < 2^{-k}\}} \frac{dy}{\delta(x, y)^{1-\varepsilon}} \\ &= \sum_{k=j-1}^{\infty} |\{y: \delta(x, y) = 2^{-k-1}\}| 2^{-(k+1)(\varepsilon-1)} \\ &= \frac{1}{2} \sum_{k=j-1}^{\infty} 2^{-(k+1)\varepsilon} = \frac{2^{\varepsilon-1}}{2^\varepsilon - 1} |I|^\varepsilon. \end{aligned}$$

The proof of the second identity follows the same lines. □

Let us notice that the indicator function of a dyadic interval $I \in \mathcal{D}$ is a Lipschitz function with respect to the distance δ . In fact $|\chi_I(x) - \chi_I(y)| \leq \delta(x, y)|I|^{-1}$. Hence for $0 < \sigma < 1$, the integral

$$\int_{\mathbb{R}^+} \frac{\chi_I(x) - \chi_I(y)}{\delta(x, y)^{1+\sigma}} dy$$

is absolutely convergent since by Lemma 3.1 for every $x \in \mathbb{R}^+$ and for any dyadic interval J containing x we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^+} \frac{\chi_I(x) - \chi_I(y)}{\delta(x, y)^{1+\sigma}} dy \right| &\leq \int_J \frac{|\chi_I(x) - \chi_I(y)|}{\delta(x, y)^{1+\sigma}} dy + \int_{J^c} \frac{|\chi_I(x) - \chi_I(y)|}{\delta(x, y)^{1+\sigma}} dy \\ &\leq \frac{1}{|I|} \int_J \frac{dy}{\delta(x, y)^\sigma} + 2 \int_{J^c} \frac{dy}{\delta(x, y)^{1+\sigma}} \\ &\leq \frac{1}{|I|} c_{1-\sigma} |J|^{1-\sigma} + 2C_\sigma |J|^{-\sigma}. \end{aligned}$$

Now, for $0 < \sigma < 1$ we are in position to define the operator D_{dy}^σ on the linear span $S(\mathcal{H})$ of the Haar system \mathcal{H} , by

$$(3.1) \quad D_{\text{dy}}^\sigma f(x) = \int_{\mathbb{R}^+} \frac{f(x) - f(y)}{\delta(x, y)^{1+\sigma}} dy.$$

Although in [1] the authors prove that Haar functions are eigenfunctions of D_{dy}^σ , we will give a simpler alternative proof.

P r o o f of Theorem 2.1. Notice that for $I, I' \in \mathcal{D}$ with $I \cap I' = \emptyset$, we have that

$$(3.2) \quad \delta(x, y) = C, \quad x \in I, \quad y \in I'.$$

Moreover, $C = |J|$, where J is the first common ancestor of I and I' .

Take $h_I \in \mathcal{H}$. Suppose first that $x \notin I$. Since h_I is supported on I , we have $h_I(x) = 0$. Hence

$$\int \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy = \int_{\mathbb{R}^+ \setminus I} \frac{-h_I(y)}{\delta(x, y)^{1+\sigma}} dy + \int_I \frac{-h_I(y)}{\delta(x, y)^{1+\sigma}} dy.$$

The first integral on the right hand side is zero since $h_I(y) \equiv 0$ for all $y \in \mathbb{R}^+ \setminus I$. For the second integral, since $x \notin I$ and $y \in I$, we apply (3.2) to obtain

$$\int_I \frac{-h_I(y)}{\delta(x, y)^{1+\sigma}} dy = -C^{-1-\sigma} \int_I h_I(y) dy = 0.$$

Therefore, we have proved (2.1) for $x \notin I$.

Suppose now that $x \in I$. Let us denote by I^* the child of I which contains x . Then

$$\int_I \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy = \int_{I^*} \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy + \int_{I \setminus I^*} \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy.$$

Since h_I is constant in each child of I , the integral over I^* vanishes. Note that in the integral over $I \setminus I^*$ we have $\delta(x, y) = |I|$, hence

$$\begin{aligned} (3.3) \quad \int_I \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy &= |I|^{-1-\sigma} \int_{I \setminus I^*} (h_I(x) - h_I(y)) dy \\ &= |I|^{-1-\sigma} \int_I (h_I(x) - h_I(y)) dy \\ &= |I|^{-1-\sigma} \left[\int_I h_I(x) dy - \int_I h_I(y) dy \right] \\ &= |I|^{-1-\sigma} h_I(x) |I| = |I|^{-\sigma} h_I(x). \end{aligned}$$

Finally, applying Lemma 3.1, we have that

$$(3.4) \quad \int_{\mathbb{R}^+ \setminus I} \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy = h_I(x) \int_{\mathbb{R}^+ \setminus I} \delta(x, y)^{-1-\sigma} dy = h_I(x) C_\sigma |I|^{-\sigma}.$$

Hence, from (3.3) and (3.4) we obtain

$$\begin{aligned} D_{\text{dy}}^\sigma h_I(x) &= \int_I \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy + \int_{\mathbb{R}^+ \setminus I} \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\sigma}} dy \\ &= |I|^{-\sigma} h_I(x) + C_\sigma |I|^{-\sigma} h_I(x) = (1 + C_\sigma) |I|^{-\sigma} h_I(x). \end{aligned}$$

Thus we have proved (2.1) for $x \in I$, and the proof of (1) in Theorem 2.1 is complete.

In order to show (2), notice first that, from (1), the identity

$$D_{\text{dy}}^\sigma f(x) = \sum_{I \in \mathcal{D}} b_\sigma |I|^{-\sigma} \langle f, h_I \rangle h_I(x)$$

holds for every $f \in S(\mathcal{H})$. Hence for such an f , the function

$$(3.5) \quad u(x, t) = \sum_{I \in \mathcal{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I(x)$$

solves problem (2.2). In fact, the orthonormality of the Haar system shows that the series defining u has only a finite number of non vanishing terms, so that

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t) &= \sum_{I \in \mathcal{D}} -b_\sigma |I|^{-\sigma} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I(x) = - \sum_{I \in \mathcal{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle D_{\text{dy}}^\sigma h_I(x) \\ &= -D_{\text{dy}}^\sigma \left(\sum_{I \in \mathcal{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I \right)(x) = -D_{\text{dy}}^\sigma u(x, t). \end{aligned}$$

On the other hand, it is immediate that $u(x, 0) = f(x)$.

Finally, to prove (3), set for $t > 0$

$$(3.6) \quad k_t(x, y) = \sum_{I \in \mathcal{D}} e^{-b_\sigma |I|^{-\sigma} t} h_I(y) h_I(x).$$

Notice that for fixed positive x , $k_t(x, y)$ can be regarded as the function of y whose Haar coefficients are given by $c_I(x, t) = e^{-b_\sigma |I|^{-\sigma} t} h_I(x)$. This function of y belongs to $L^2(\mathbb{R}^+)$ since $|c_I(x, t)|^2 = e^{-2b_\sigma |I|^{-\sigma} t} |I|^{-1}$ whenever $x \in I$. Hence

$$\sum_{I \in \mathcal{D}} |c_I(x, t)|^2 = \sum_{\substack{I \in \mathcal{D} \\ I \ni x}} e^{-2b_\sigma |I|^{-\sigma} t} |I|^{-1} = \sum_{j \in \mathbb{Z}} e^{-2b_\sigma 2^{j\sigma} t} 2^j,$$

which is finite. Then for $f \in S(\mathcal{H})$ the integral $\int_{\mathbb{R}^+} k_t(x, y) f(y) dy$ is absolutely convergent. Therefore writing the integral as an inner product and noticing that

$$\left\langle f, \sum_{I \in \mathcal{D}} c_I(x, t) h_I \right\rangle = \sum_{I \in \mathcal{D}} c_I(x, t) \langle f, h_I \rangle,$$

we obtain from (3.5) that

$$\int_{\mathbb{R}^+} k_t(x, y) f(y) dy = u(x, t).$$

Notice first that for fixed x and y in \mathbb{R} , the only contribution to (3.6) are the terms in which I contains both x and y . We shall denote by I^0 the first common ancestor of x and y , and let l be such that $I^0 \in \mathcal{D}^l$. Also we shall denote by I^j the dyadic interval in \mathcal{D}^{l-j} containing I^0 . Then

$$\begin{aligned} k_t(x, y) &= \sum_{j \geq 0} e^{-b_\sigma |I^j|^{-\sigma} t} h_{I^j}(y) h_{I^j}(x) \\ &= e^{-b_\sigma |I^0|^{-\sigma} t} h_{I^0}(y) h_{I^0}(x) + \sum_{j \geq 1} e^{-b_\sigma |I^j|^{-\sigma} t} h_{I^j}(y) h_{I^j}(x). \end{aligned}$$

Let us observe that, for every $j \geq 1$, x and y belong to the same child of I^j , so that $h_{I^j}(y) = h_{I^j}(x)$. Moreover,

$$h_{I^j}(y) h_{I^j}(x) = |I^j|^{-1}.$$

Hence,

$$k_t(x, y) = e^{-b_\sigma |I^0|^{-\sigma} t} h_{I^0}(y) h_{I^0}(x) + \sum_{j \geq 1} \frac{e^{-b_\sigma |I^j|^{-\sigma} t}}{|I^j|}.$$

Now, notice that $\delta(x, y) = |I^0|$ and that $|I^j| = 2^j |I^0|$. Also, since x and y belong to different children of I^0 , we have that $h_{I^0}(y) h_{I^0}(x) = -|I^0|^{-1}$. Then, we obtain that

$$\begin{aligned} k_t(x, y) &= -e^{-b_\sigma \delta(x, y)^{-\sigma} t} \delta(x, y)^{-1} + \sum_{j \geq 1} \frac{e^{-b_\sigma (2^j \delta(x, y))^{-\sigma} t}}{2^j \delta(x, y)} \\ &= \frac{1}{\delta(x, y)} \left[-e^{-b_\sigma \delta(x, y)^{-\sigma} t} + \sum_{j \geq 1} 2^{-j} e^{-b_\sigma (2^j \delta(x, y))^{-\sigma} t} \right]. \end{aligned}$$

Hence, defining $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$\varphi(s) = \frac{1}{s} \left[-e^{-b_\sigma s^{-\sigma}} + \sum_{j \geq 1} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} \right],$$

we have that

$$k_t(x, y) = \frac{1}{t^{1/\sigma}} \varphi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right).$$

Notice that the series defining $\varphi(s)$ converges for every positive s . On the other hand, since $e^{-b_\sigma s^{-\sigma}} < e^{-b_\sigma (2^j s)^{-\sigma}}$ for every $j \geq 1$, $\varphi(s)$ is strictly positive for every $s > 0$. Therefore the proof of (3) in Theorem 2.1 is complete. \square

4. MAXIMAL FUNCTION ESTIMATES AND THE PROOF OF THEOREM 2.2

To start with the analysis of the way in which the initial condition is attained, in this section we shall denote by K_t the operator with kernel k_t , i.e.

$$K_t f(x) := \int_{\mathbb{R}^+} k_t(x, y) f(y) dy = u(x, t).$$

In this section we aim at proving that the maximal operator K^* associated with $u(x, t)$ satisfies

$$K^* f(x) := \sup_{t>0} |K_t f(x)| = \sup_{t>0} |u(x, t)| \leq C M_{\text{dy}} f(x),$$

for every $f \in L^p(\mathbb{R}^+)$, where M_{dy} denotes the dyadic Hardy-Littlewood maximal operator defined in (1.2). In order to do this, we shall construct a decreasing function $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\psi \in L^1(0, \infty)$ with

$$|k_t(x, y)| \leq \frac{1}{t^{1/\sigma}} \psi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right).$$

Proof of Theorem 2.2. For $s > 0$, since $\sum_{j \geq 1} 2^{-j} = 1$ and $|e^{-x}| \leq 1$ for $x \in \mathbb{R}^+$, we have

$$\varphi(s) \leq \frac{1}{s} \sum_{j \geq 1} 2^{-j} [1 - e^{-b_\sigma s^{-\sigma}}].$$

Then using the Taylor expansion for the exponential function we obtain

$$(4.1) \quad \varphi(s) \leq \frac{1}{s} \sum_{j \geq 1} 2^{-j} \left[\frac{b_\sigma}{s^\sigma} \right] = \frac{b_\sigma}{s^{1+\sigma}}.$$

For $0 < s < 1$, fix $0 < \varepsilon < 1$ and define $n(s) := \lfloor -\varepsilon \log_2 s \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function. Then we can split the series defining φ in the following way:

$$(4.2) \quad \varphi(s) = \frac{-e^{-b_\sigma s^{-\sigma}}}{s} + \frac{1}{s} \sum_{j=1}^{n(s)} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} + \frac{1}{s} \sum_{j>n(s)} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}}.$$

The absolute value of the first term on the right hand side of (4.2) is clearly bounded. The second term is also bounded since for $j \leq n(s)$ we have that $e^{-b_\sigma(2^j s)^{-\sigma}} \leq e^{-b_\sigma s^{-(1-\varepsilon)\sigma}}$. Then

$$\frac{1}{s} \sum_{j=1}^{n(s)} 2^{-j} e^{-b_\sigma(2^j s)^{-\sigma}} \leq \frac{e^{-b_\sigma s^{-(1-\varepsilon)\sigma}}}{s} \sum_{j=1}^{n(s)} 2^{-j} \leq \frac{e^{-b_\sigma s^{-(1-\varepsilon)\sigma}}}{s}.$$

For the third term we can see that

$$\frac{1}{s} \sum_{j>n(s)} 2^{-j} e^{-b_\sigma(2^j s)^{-\sigma}} \leq \frac{1}{s} \sum_{j>n(s)} 2^{-j} \leq \frac{1}{s} 2^{-n(s)} \leq \frac{2}{s} s^\varepsilon = \frac{2}{s^{1-\varepsilon}}.$$

Therefore, for $0 < s < 1$ we have that

$$(4.3) \quad \varphi(s) \leq \frac{C}{s^{1-\varepsilon}}.$$

So, from (4.1) and (4.3), $\varphi(s) \leq \psi(s)$ for every $s \in \mathbb{R}^+$ with

$$\psi(s) = C \begin{cases} 1/s^{1-\varepsilon} & \text{if } 0 < s < 1, \\ 1/s^{1+\sigma} & \text{if } s \geq 1, \end{cases}$$

for some positive constant C . Hence,

$$\begin{aligned} |K_t f(x)| &\leq \int_{\mathbb{R}^+} |k_t(x, y)| |f(y)| dy \leq \int_{\mathbb{R}^+} \frac{1}{t^{1/\sigma}} \psi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right) |f(y)| dy \\ &= \sum_{j=-\infty}^{\infty} \frac{1}{t^{1/\sigma}} \int_{\{y: t^{1/\sigma} 2^j \leq \delta(x, y) < t^{1/\sigma} 2^{j+1}\}} \psi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right) |f(y)| dy \\ &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) \frac{1}{t^{1/\sigma} 2^{j+1}} \int_{B_\delta(x, t^{1/\sigma} 2^{j+1})} |f(y)| dy. \end{aligned}$$

Since $|B_\delta(x, r)| < r$ and each B_δ is a dyadic interval, we have

$$\begin{aligned} |K_t f(x)| &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) \frac{1}{|B_\delta(x, t^{1/\sigma} 2^{j+1})|} \int_{B_\delta(x, t^{1/\sigma} 2^{j+1})} |f(y)| dy \\ &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) M_{\text{dy}} f(x) = 4 M_{\text{dy}} f(x) \sum_{j=-\infty}^{\infty} \int_{\{y: 2^{j-1} \leq y < 2^j\}} \psi(2^j) dy \\ &\leq 4 M_{\text{dy}} f(x) \int_{\mathbb{R}^+} \psi(y) dy \leq 4 \|\psi\|_{L^1} M_{\text{dy}} f(x). \end{aligned}$$

Therefore, taking supremum in t we obtain

$$\sup_{t>0} |K_t f(x)| \leq 4\|\psi\|_{L^1} M_{\text{dy}} f(x),$$

which completes the proof of (1) in Theorem 2.2.

In order to prove (2), notice first that for $g \in S(\mathcal{H})$ and $1 < p < \infty$ we have the L^p convergence of $K_t g$ to g . Take $f \in L^p$, then for a function g in $S(\mathcal{H})$ we have that

$$\begin{aligned} \|K_t f - f\|_{L^p} &\leq \|K_t(f - g)\|_{L^p} + \|K_t g - g\|_{L^p} + \|g - f\|_{L^p} \\ &\leq C\|M_{\text{dy}}(f - g)\|_{L^p} + \|K_t g - g\|_{L^p} + \|g - f\|_{L^p} \\ &\leq \tilde{C}\|f - g\|_{L^p} + \|K_t g - g\|_{L^p}. \end{aligned}$$

So, from the above remark and the density of $S(\mathcal{H})$ in L^p we obtain the desired result.

Finally, as usual, the pointwise convergence to the initial data in (3) is an immediate consequence of the boundedness properties of the maximal operator associated with u and the pointwise convergence in a dense subset of L^p ($1 < p < \infty$). We will sketch a brief proof for the sake of completeness.

Since we already know that $K_t f \rightarrow f$ in the L^p sense as $t \rightarrow 0^+$, in order to prove the pointwise convergence, define

$$E = \{f \in L^p : \lim_{t \rightarrow 0^+} K_t f \text{ exists for almost every } x \in \mathbb{R}^+\}.$$

Notice that $S(\mathcal{H}) \subseteq E \subseteq L^p$. Since $S(\mathcal{H})$ is dense in L^p , we only need to prove that E is a closed subset of L^p . Let $\{f_n\}$ be a sequence contained in E such that f_n converges in L^p to a function f . To see that $f \in E$ it is enough to prove that for all $\varepsilon > 0$ we have

$$(4.4) \quad |E_\varepsilon| := \left| \left\{ x : \limsup_{t \rightarrow 0^+} K_t f(x) - \liminf_{t \rightarrow 0^+} K_t f(x) > \varepsilon \right\} \right| = 0.$$

For every n we can write

$$\begin{aligned} |E_\varepsilon| &\leq \left| \left\{ x : \limsup_{t \rightarrow 0^+} K_t f_n(x) - \liminf_{t \rightarrow 0^+} K_t f_n(x) > \frac{\varepsilon}{3} \right\} \right| \\ &\quad + \left| \left\{ x : \limsup_{t \rightarrow 0^+} K_t(f - f_n)(x) > \frac{\varepsilon}{3} \right\} \right| + \left| \left\{ x : \liminf_{t \rightarrow 0^+} K_t(f - f_n)(x) < -\frac{\varepsilon}{3} \right\} \right|. \end{aligned}$$

The first term is zero since $f_n \in E$. For the other two terms we use the weak type inequality boundedness on L^p of the maximal operator K^* which follows from the item (1). Notice that for every function g we have that

$$\left| \liminf_{t \rightarrow 0^+} K_t g(x) \right| \leq K^* g(x) \quad \text{and} \quad \left| \limsup_{t \rightarrow 0^+} K_t g(x) \right| \leq K^* g(x).$$

Then, since K^* is weakly bounded on L^p , we obtain

$$\left| \left\{ x: \limsup_{t \rightarrow 0^+} K_t(f - f_n)(x) > \frac{\varepsilon}{3} \right\} \right| \leq \frac{C}{\varepsilon^p} \|f_n - f\|_{L^p}^p$$

and

$$\left| \left\{ x: \liminf_{t \rightarrow 0^+} K_t(f - f_n)(x) < -\frac{\varepsilon}{3} \right\} \right| \leq \frac{C}{\varepsilon^p} \|f_n - f\|_{L^p}^p.$$

Hence,

$$|E_\varepsilon| \leq \frac{C}{\varepsilon^p} \|f_n - f\|_{L^p}^p.$$

When n tends to infinity we have (4.4). Then E is closed and therefore $E = L^p$. This means that for every $f \in L^p$ we have that

$$\lim_{t \rightarrow 0^+} u(x, t) = \lim_{t \rightarrow 0^+} K_t f \quad \text{exists for a.e. } x \in \mathbb{R}^+.$$

But we already know that $u(x, t) \rightarrow f(x)$ when $t \rightarrow 0^+$ in L^p , hence (3) follows, which completes the proof. \square

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