

COMPUTING THE DETERMINANTAL REPRESENTATIONS OF HYPERBOLIC FORMS

MAO-TING CHIEN, Taipei, HIROSHI NAKAZATO, Hirosaki

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Dedicated to the memory of Professor Miroslav Fiedler

Abstract. The numerical range of an $n \times n$ matrix is determined by an n degree hyperbolic ternary form. Helton-Vinnikov confirmed conversely that an n degree hyperbolic ternary form admits a symmetric determinantal representation. We determine the types of Riemann theta functions appearing in the Helton-Vinnikov formula for the real symmetric determinantal representation of hyperbolic forms for the genus $g = 1$. We reformulate the Fiedler-Helton-Vinnikov formulae for the genus $g = 0, 1$, and present an elementary computation of the reformulation. Several examples are provided for computing the real symmetric matrices using the reformulation.

Keywords: determinantal representation; hyperbolic form; Riemann theta function; numerical range

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1. INTRODUCTION

Let T be an $n \times n$ complex matrix. The numerical range of T is defined as the set

$$W(T) = \{\xi^* T \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

The range $W(T)$ is a convex set due to the famous Toeplitz-Hausdorff theorem. Kippenhahn [12] characterized $W(T)$ as the convex hull of the real affine part of the

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dual projective curve of $F_T(x, y, z) = 0$, where the real ternary form associated with T is given by

$$F_T(x, y, z) = \det(x\Re(T) + y\Im(T) + zI_n),$$

and $\Re(T) = (T + T^*)/2$, $\Im(T) = (T - T^*)/(2i)$. Obviously, the equation $F_T(x_0, y_0, z) = 0$ in z has only real roots for any $(x_0, y_0) \in \mathbb{R}^2$ and $F_T(0, 0, 1) \neq 0$. The form $F_T(x, y, z)$ possessing this real roots property is called *hyperbolic* with respect to $e = (0, 0, 1)$. Lax in [13] conjectured that an arbitrary ternary hyperbolic form $F(x, y, z)$ with respect to $e = (e_1, e_2, e_3) \in \mathbb{R}^3$, $e \neq 0$, admits a determinantal representation, i.e.,

$$F(x, y, z) = c \det(xM_1 + yM_2 + zM_3)$$

for some real symmetric matrices M_1, M_2, M_3 with positive definiteness of $e_1M_1 + e_2M_2 + e_3M_3$, and $c \neq 0$. Independently, Fiedler in [8] made a similar conjecture under a relaxing condition that M_1, M_2, M_3 are Hermitian instead. Fiedler in [7] proved that the Lax conjecture is true provided that $F(x, y, z) = 0$ is a rational curve. Recently, Helton and Vinnikov in [10] confirmed that the Lax conjecture is true by using Riemann's theta functions. Based on the confirmation of the Lax conjecture, the authors of this paper in [4] proved that the c -numerical range of an $n \times n$ matrix T is reduced to the classical numerical range of an $m \times m$ matrix A , such that $W_c(T) = W(A)$ for some $m \leq n!$, and Helton and Spitkovsky in [9] proved that any matrix T has a symmetric matrix S satisfying $W(T) = W(S)$.

The construction of real symmetric matrices from the Helton-Vinnikov theorem has attracted attention in studying the numerical range of matrices. One case, for instance, ask, whether the complex symmetric matrix S obtained by the Helton-Vinnikov formula from $F_T(x, y, z)$ is unitarily similar to a given matrix T . This question motivated us to compute explicitly the real symmetric matrices of the determinantal representation. In Section 2, we reformulate the formulae in [7], [10] for real symmetric matrices of the determinantal representations of hyperbolic forms with genus $g = 0$ or 1 . Notice that the entries of the symmetric matrices M_j in the Lax conjecture have to be real. The Riemann theta functions in the Helton-Vinnikov formula may produce imaginary symmetric matrices. We determine the types of Riemann theta functions which lead to real symmetric expressions in the elliptic curve case. In Sections 3 and 4, we present concrete examples of 3×3 and 4×4 matrices, and compute the real symmetric matrices using the reformulation which illustrate the means of the Helton-Vinnikov formula for studying the numerical range of matrices.

2. MAIN THEOREMS

Let $F(x, y, z)$ be an irreducible ternary form of degree $n \geq 3$. A point $P_0 = (x_0, y_0, z_0)$ of the complex projective curve

$$\mathcal{V}_{\mathbb{C}}(F) = \{[x, y, z] \in \mathbb{CP}^2 : F(x, y, z) = 0\}$$

is called a singular point if

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0) = \frac{\partial F}{\partial y}(x_0, y_0, z_0) = \frac{\partial F}{\partial z}(x_0, y_0, z_0) = 0.$$

We sometimes abbreviate the complex projective curve $\mathcal{V}_{\mathbb{C}}(F)$ as $F(x, y, z) = 0$. For a singular point $P_0 = (x_0, y_0, z_0)$, $z_0 \neq 0$, consider two functions

$$f(X, Y) = F(x_0 + X, y_0 + Y, z_0), \quad f_Y(X, Y) = F_Y(x_0 + X, y_0 + Y, z_0).$$

The Taylor series of these functions define an ideal (f, f_Y) of the ring $\mathbb{C}[[X, Y]]$ of formal power series in X, Y . We define

$$\delta(P_0) = \frac{1}{2} \left(\dim \left(\frac{\mathbb{C}[[X, Y]]}{(f, f_Y)} \right) - m + s \right),$$

where m is the multiplicity of P_0 and s is the number of irreducible analytic branches of the curve $\mathcal{V}_{\mathbb{C}}(F)$ near (x_0, y_0, z_0) . The number $\delta(P_0)$ is always a non-negative integer (cf. [14]). The genus of the curve $F(x, y, z) = 0$ is given by

$$g(F) = \frac{1}{2}(n-1)(n-2) - \sum_{j=1}^k \delta(P_j),$$

where P_1, \dots, P_k are singular points of the curve $F(x, y, z) = 0$. An irreducible curve is called a rational curve or an elliptic curve if its genus is $g = 0$ or $g = 1$, respectively. A rational curve has a rational function parametrization, and an elliptic curve can be parametrized by an elliptic function and its derivative (cf. [17]).

In the formulation of the Helton-Vinnikov theorem, the following two objects play a crucial role:

- (i) The Riemann theta functions on a complex torus \mathbb{C}^g/Γ , where Γ is a lattice in \mathbb{C}^g .
- (ii) The Abel-Jacobi map φ of an irreducible algebraic curve with genus g to its corresponding Abel-Jacobi variety \mathbb{C}^g/Γ .

An accurate numerical computation method of the Riemann theta functions for $g \geq 1$ and a program to calculate a basis of Γ for an algebraic curve can be found in [5] and [6], respectively. In this paper, we mainly deal with two cases: $g = 0$ and $g = 1$. The first reason is that the general theory of Abel functions and Riemann theta functions for $g \geq 2$ is rather complicated. In contrast to this, for $g = 1$, the complex torus \mathbb{C}^g/Γ has an abelian fundamental group, and the Riemann functions have a single main variable. Shortly, the case $g = 1$ is more treatable. The second reason is more important from the viewpoint of developing the theory of numerical range. In [3], the authors of this paper proved that any irreducible curve $\mathcal{V}_{\mathbb{C}}(F)$ associated with a weighted shift matrix has genus $g \geq 1$, and in [1], they showed that the j -invariant of an irreducible elliptic curve associated with a 3×3 or 4×4 matrix is real and greater than or equal to 1. There are many tools for computing Riemann theta functions on a Riemann surface with $g = 1$. We used Mathematica (cf. [18]) to implement the numerical computations.

In the rest of this paper, we assume a real ternary form $F(x, y, z)$ of degree n satisfying the following conditions:

- (F1) $F(x, y, z)$ is hyperbolic with respect to $e = (0, 0, 1)$ and $F(0, 0, 1) = 1$.
- (F2) $F(x, y, z)$ is irreducible.
- (F3) The n real intersection points of the complex projective curve $F(x, y, z) = 0$ and the line $x = 0$ are distinct non-singular points Q_1, \dots, Q_n with coordinates $Q_j = (0, 1, -\beta_j)$, where $\beta_j \neq 0$.

According to the determinantal representation theorem [7], [10], there exist real symmetric matrices B and C of dimension n such that

$$(2.1) \quad F(x, y, z) = \det(zI_n + yB + xC),$$

where $B = \text{diag}(\beta_1, \dots, \beta_n)$, and the diagonal entries c_{jj} of the real symmetric matrix C are given by

$$(2.2) \quad c_{jj} = \beta_j \frac{F_x(0, 1, -\beta_j)}{F_y(0, 1, -\beta_j)}.$$

The crucial problem is the construction of the off-diagonal entries of C . If $g = 0, 1$, we denote by Q'_j the point on the parameter space (the real line for $g = 0$, the complex torus for $g = 1$) corresponding to Q_j . In the expression (2.1), if we replace C by

$$\tilde{C} = \text{diag}(\eta_1, \eta_2, \dots, \eta_n) C \text{diag}(\eta_1, \eta_2, \dots, \eta_n),$$

($\eta_1, \eta_2, \dots, \eta_n = \pm 1$), we have another determinantal representation

$$F(x, y, z) = \det(zI_n + yB + x\tilde{C}).$$

The choice of the sign pattern of the off-diagonal entries of C is determined for the case $g = 0$, and is open for $g = 1$. We reformulate Fiedler formula ([7], Theorem 1), for the determinantal representation if $g = 0$.

Theorem 2.1. *Let $F(t, x, y)$ be a ternary form of degree n satisfying conditions (F1)–(F3). Assume the genus of the complex projective curve $F(x, y, z) = 0$ is 0. Then the off-diagonal entries of C in the determinantal representation (2.1) are given by*

$$(2.3) \quad c_{jk} = \varepsilon \frac{\beta_k - \beta_j}{Q'_k - Q'_j} \frac{1}{\sqrt{\left(d\left(\frac{R_1}{R_2}\right)(Q'_j)\right) \left(d\left(\frac{R_1}{R_2}\right)(Q'_k)\right)}},$$

where

$$x = R_1(s) = \frac{u(s)}{w(s)}, \quad y = R_2(s) = \frac{v(s)}{w(s)}$$

are real rational functions parametrizing the affine part $F(x, y, 1) = 0$, and $\varepsilon \in \{+1, -1\}$ satisfies $\varepsilon u'(Q'_j)v(Q'_j) > 0$ for all j .

Proof. It is shown in [7], Theorem 1, that we can choose $\varepsilon \in \{+1, -1\}$ such that $\varepsilon u'(Q'_j)v(Q'_j) > 0$ for all j . Further, we compute that

$$\frac{1}{d\left(\frac{R_1}{R_2}\right)(Q'_j)} = \frac{1}{d\left(\frac{u}{v}\right)(Q'_j)} = \frac{v(Q'_j)^2}{u'(Q'_j)v(Q'_j) - u(Q'_j)v'(Q'_j)} = \frac{v(Q'_j)}{u'(Q'_j)},$$

and hence the formula (2.3) essentially coincides with the formula obtained in [7], Theorem 1. \square

The formulation in Theorem 2.1 is just a slight modification of Fiedler formula. This reformulation is consistent with the formula pattern in Theorem 2.4 for the case $g = 1$.

The Helton-Vinnikov Formula in [10], Theorem 2.2 (see also [15], Theorem 6), for a hyperbolic form with genus g reads as follows:

Theorem 2.2. *Let $F(x, y, z)$ be a ternary form of degree n satisfying conditions (F1)–(F3). Assume the genus of the complex projective curve $F(x, y, z) = 0$ is $g \geq 1$. Then the off-diagonal entries of C in the determinantal representation (2.1) are given by*

$$c_{jk} = \frac{\beta_k - \beta_j}{\theta[\delta](0)} \frac{\theta[\delta](\varphi(Q_k) - \varphi(Q_j), S)}{E(\varphi(Q_k), \varphi(Q_j))} \frac{1}{\sqrt{d\left(\frac{x}{y}\right)(Q'_j)} \sqrt{d\left(\frac{x}{y}\right)(Q'_k)}},$$

where $\theta[\delta](\cdot, \cdot)$ is a Riemann theta function with an even characteristic δ , $E(\cdot, \cdot)$ is the prime form on the Jacobi-variety given as a constant multiple of a Riemann theta function $\theta[\varepsilon](\cdot, \cdot)$ with an odd characteristic ε , the two Riemann theta functions are defined for $(z, S) \in \mathbb{C}^g \times \mathcal{H}_g$, the matrix S is determined by the curve $\mathcal{V}_{\mathbb{C}}(F)$, φ is the Abel-Jacobi map from $\mathcal{V}_{\mathbb{C}}(F)$ into the Jacobian variety, and Q'_j is the point on the Riemann surface corresponding to Q_j . Symbol \mathcal{H}_g denotes the set of the $g \times g$ Riemann matrices, i.e., symmetric matrices whose imaginary parts are positive definite.

The Helton-Vinnikov formula in Theorem 2.2 involves computing the Riemann theta functions and Abel-Jacobi maps. The Riemann theta functions are explicit, but the non-explicitness arises because of the complexity in computation when the genus satisfies $g \geq 2$. For instance, we have a quartic curve with integral coefficients and $g = 2$ for which the computation of the Riemann matrix S is not possible by the usual software. We restrict our attention to the case $g = 1$, and reformulate Theorem 2.2 using Riemann theta functions with a single main variable, and the Weierstrass canonical forms of non-singular cubic curves.

Let $F(t, x, y)$ be a ternary form satisfying conditions (F1)–(F3) with genus $g = 1$, i.e., $\mathcal{V}_{\mathbb{C}}(F)$ is an elliptic curve. Then there is a real birational transformation Φ for which $\Phi(\mathcal{V}_{\mathbb{C}}(F))$ is a non-singular cubic curve of the Weierstrass standard form

$$Y^2Z = 4X^3 - g_2X^2Z - g_3Z^3$$

for some real constants g_2, g_3 such that $g_2^3 - 27g_3^2 > 0$. The complex affine algebraic curve $Y^2 = 4X^3 - g_2X - g_3$ is parametrized as

$$X = \mathcal{P}(s; g_2, g_3), \quad Y = \mathcal{P}'(s; g_2, g_3),$$

where $\mathcal{P}(s; g_2, g_3)$ and $\mathcal{P}'(s; g_2, g_3)$ are the Weierstrass P -functions and its derivative with parameters g_2, g_3 satisfying the differential equation

$$\left(\frac{d\mathcal{P}'}{ds}\right)^2 = 4\mathcal{P}^3(s; g_2, g_3) - g_2\mathcal{P}(s; g_2, g_3) - g_3.$$

The meromorphic function $\mathcal{P}(s; g_2, g_3)$ on the Gaussian plane \mathbb{C} has two linearly independent half-periods ω_1 and ω_2 in the sense that

$$\mathcal{P}(s + 2\omega_1; g_2, g_3) = \mathcal{P}(s; g_2, g_3) \quad \text{and} \quad \mathcal{P}(s + 2\omega_2; g_2, g_3) = \mathcal{P}(s; g_2, g_3),$$

where ω_1 is a positive real number and ω_2 is a purely imaginary number with $\Im(\omega_2) > 0$ (cf. [1]). The τ -invariant of the curve $\mathcal{V}_{\mathbb{C}}(F)$ is defined by $\tau = \omega_2/\omega_1$.

The real affine part $F(x, y, 1) = 0$ of the curve $\mathcal{V}_{\mathbb{C}}(F)$ is then parametrized as

$$(2.4) \quad \{(x, y, 1) = (R_1(\mathcal{P}(u), \mathcal{P}'(u)), R_2(\mathcal{P}, \mathcal{P}'(u)), 1) : \Im(u) = 0, 0 < \Re(u) < 2\omega_1 \text{ or} \\ \Im(u) = \Im(\omega_2), 0 \leq \Re(u) \leq 2\omega_1\}$$

by real rational functions R_1, R_2 of \mathcal{P} and \mathcal{P}' over the torus \mathbb{T} . This parametrization $s \mapsto (x, y, 1)$ is the inverse of the Abel-Jacobi map $\varphi: \mathcal{V}_{\mathbb{C}}(F) \rightarrow \text{Jac}(X)$.

Denote by \mathcal{H} the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. The *Riemann theta function* is the holomorphic function on $\mathbb{C} \times \mathcal{H}$ defined by the exponential series

$$\theta(u, \tau) = \sum_{m \in \mathbb{Z}} \exp(\pi i(m^2 \tau + 2mu)),$$

which is quasi-periodic with respect to the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$:

$$\theta(u + m + \tau n, \tau) = \exp(\pi i(-2nu - n^2 \tau)) \theta(u, \tau)$$

for all integers m, n . We consider four Riemann theta functions $\theta[\varepsilon](u)$ with characteristics ε defined as

$$\theta[\varepsilon](u, \tau) = \exp(\pi i(a^2 \tau + 2au + 2ab)) \theta(u + \tau a + b, \tau)$$

for $\varepsilon = a + \tau b$ with

$$(a, b) = (0, 0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right).$$

Using the parameter $q = \exp(i\pi\tau)$, we have

$$\theta(u, [q]) = \theta(u, \tau) = \sum_{m \in \mathbb{Z}} q^{m^2} \exp(2m\pi i u).$$

The four Riemann theta functions are also denoted as

$$\begin{aligned} \theta_1(u, [q]) &= -\theta\left[\frac{1}{2}, \frac{1}{2}\right](u, \tau) = 2q^{1/4} \sum_{m=0}^{\infty} q^{(m+1)m} \sin((2m+1)\pi u), \\ \theta_2(u, [q]) &= \theta\left[\frac{1}{2}, 0\right](u, \tau) = 2q^{1/4} \sum_{m=0}^{\infty} q^{(m+1)m} \cos((2m+1)\pi u), \\ \theta_3(u, [q]) &= \theta[0, 0](u, \tau) = \theta(u, \tau) = 1 + 2 \sum_{m=1}^{\infty} q^{m^2} \cos(2m\pi u), \\ \theta_4(u, [q]) &= \theta\left[0, \frac{1}{2}\right](u, \tau) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos(2m\pi u). \end{aligned}$$

For references on the Weierstrass P -functions and Riemann theta functions, one may see, for instance, [11], [17].

Theorem 2.3. *The four Riemann theta functions θ_δ , $\delta = 1, 2, 3, 4$, are quasi-periodic, and the elliptic functions θ_δ/θ_1 , $\delta = 2, 3, 4$ have respective double periods $1, 2\tau$ ($\delta = 2$), $2, 2\tau$ ($\delta = 3$), $2, \tau$ ($\delta = 4$). Moreover, the function θ_4/θ_1 takes on real values on the real part of the Jacobian variety, and the functions $\theta_2/\theta_1, \theta_3/\theta_1$ take on real values or purely imaginary values depending on the two connected components of the real part of the Jacobi variety.*

Proof. Direct computations show that

$$\begin{aligned}\theta_1(u+1, [q]) &= -\theta_1(u, [q]), & \theta_1(u+\tau, [q]) &= -q^{-1} \exp(-2\pi i u) \theta_1(u, [q]), \\ \theta_2(u+1, [q]) &= -\theta_2(u, [q]), & \theta_2(u+\tau, [q]) &= q^{-1} \exp(-2\pi i u) \theta_2(u, [q]), \\ \theta_3(u+1, [q]) &= \theta_3(u, [q]), & \theta_3(u+\tau, [q]) &= q^{-1} \exp(-2\pi i u) \theta_3(u, [q]), \\ \theta_4(u+1, [q]) &= \theta_4(u, [q]), & \theta_4(u+\tau, [q]) &= -q^{-1} \exp(-2\pi i u) \theta_4(u, [q]).\end{aligned}$$

Thus the functions θ_δ/θ_1 , $\delta = 2, 3, 4$, are elliptic functions with double periods $1, 2\tau$ ($\delta = 2$), $2, 2\tau$ ($\delta = 3$), $2, \tau$ ($\delta = 4$).

Suppose that τ is a purely imaginary number. Then the four functions $\theta_\delta(u, [q])$ take on real values on the real line. On the line $\Im(z) = \Im(\tau)/2$, we have

$$\frac{\theta_4}{\theta_1}\left(u + \frac{\tau}{2}\right) = \frac{\theta_1}{\theta_4}(u),$$

and

$$\frac{\theta_2}{\theta_1}\left(u + \frac{\tau}{2}\right) = -i \frac{\theta_3}{\theta_4}(u), \quad \frac{\theta_3}{\theta_1}\left(u + \frac{\tau}{2}\right) = -i \frac{\theta_2}{\theta_4}(u)$$

for any $u \in \mathbb{R}$. Hence, $\theta_2/\theta_1, \theta_3/\theta_1$ take on either real or purely imaginary values on the real part of the Jacobi variety. \square

Using the notation of Theorem 2.3, we reformulate the Helton-Vinnikov Formula in [10], Theorem 2.2, (cf. [15], Theorem 6) for $g = 1$, and determine the types of Riemann theta functions which lead to real symmetric determinantal representations.

Theorem 2.4. *Let $F(t, x, y)$ be a ternary form of degree n satisfying conditions (F1)–(F3). Assume the genus of the complex projective curve $F(x, y, z) = 0$ is 1, and $x = R_1(\mathcal{P}(u), \mathcal{P}'(u))$, $y = R_2(\mathcal{P}, \mathcal{P}'(u))$ parametrize the elliptic curve $\mathcal{V}_{\mathbb{C}}(F)$ in (2.4). Let $Q'_j = \varphi(Q_j)$ be the point of the torus \mathbb{T} corresponding to the point*

$Q_j \in \mathcal{V}_{\mathbb{C}}(F)$. For $\delta = 2, 3$, the matrix C in the determinantal representation (2.1) is real symmetric, and its off-diagonal entries are given by

$$(2.5) \quad c_{jk} = \frac{(\beta_k - \beta_j)\theta'_1(0)}{2\omega_1\theta_\delta(0)} \frac{\theta_\delta((Q'_k - Q'_j)/2\omega_1)}{\theta_1((Q'_k - Q'_j)/2\omega_1)} \frac{1}{\sqrt{d(\frac{R_1}{R_2})(Q'_j)}\sqrt{d(\frac{R_1}{R_2})(Q'_k)}}.$$

Proof. We use a non-normalized Jacobi variety $\mathbb{C}/(2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2)$ in place of the normalized Jacobi variety $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for $\tau = \omega_2/\omega_1$. According to this frame, we easily use the Weierstrass P -function to express the inverse of the Abel-Jacobi map φ . By this parameter change, a new factor $1/(2\omega_1)$ appears in the formulation (2.5). For $g = 1$, the Riemann theta function with an odd characteristic is uniquely given by $\theta_1(\cdot)$. As the prime form $E(\cdot, \cdot)$, it generates the term $\theta_1((Q'_k - Q'_j)(2\omega_1)^{-1})/\theta'_1(0)$ in (2.5). The Riemann theta function $\theta[\delta](\cdot)$ with an even characteristic appearing in (2.5), is given by θ_2, θ_3 or θ_4 .

We claim that $\delta = 4$ produces an imaginary C in (2.1). For a real parameter θ , we consider the equation $F(-\cos\theta, -\sin\theta, z) = 0$ in z . By the hyperbolicity of $F(x, y, z)$, this equation has n real roots $z_j(\theta)$ counting multiplicities. In particular, for $\theta = -\pi/2$, the n distinct real roots are $Q_j = (0, 1, -\beta_j)$, $j = 1, 2, \dots, n$. By the Helton-Vinnikov theorem and Rellich's theorem, the roots $z_j(\theta)$ of the equation depend analytically on θ . Every real point of the curve $\mathcal{V}_{\mathbb{C}}(F)$ is joined to some Q_j . By a birational transformation, each point Q_j is mapped to Q'_j on a non-singular cubic curve, and the curve $z_j(\theta)$ is mapped to the real part of the cubic curve consisting of a pseudo line and an oval. The image of $z_j(\theta)$ covers the real part of the cubic curve except for a finite number of points. There are $j \neq k$ for which Q'_j lies on the pseudo line and Q'_k lies on the oval. We use the same symbol Q'_j for the point on the non-normalized torus $\mathbb{C}/(2\omega_1\mathbb{C} + 2\omega_2\mathbb{Z})$ corresponding to the point Q'_j on the cubic curve. Then $\Im(Q'_j) = 0$ and $\Im(Q'_k) = \Im(\omega_2)$, and thus $\sqrt{d(x/y)(Q'_j)}\sqrt{d(x/y)(Q'_k)}$ is purely imaginary. Hence, the entries c_{jk} in (2.5) are real if and only if the ratio $\theta_\delta((Q'_k - Q'_j)(2\omega_1)^{-1})/\theta_1((Q'_k - Q'_j)(2\omega_1)^{-1})$ of a purely imaginary value. This happens only for $\delta = 2, 3$, by Theorem 2.3, since $\Im(Q'_k - Q'_j) = \Im(\omega_2)$. The case $\delta = 4$ results in complex entries of C . \square

Remarks. 1. Applying the formulae mentioned in the proof of Theorem 2.3, we find that the function θ_δ/θ_1 on the normalized torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is defined up to multiplicative constants ± 1 .

2. For $g = 1$, the advantage of the formulation (2.5) is that the torus \mathbb{C}^g/Γ is a one-dimensional analytic manifold which is realized as a complex projective curve $\mathcal{V}_{\mathbb{C}}(G)$ for some ternary form G using a birational transformation.

3. Plaumann et al. mentioned in [15], page 270, that they do not know why sometimes the off-diagonal entries of C are wrong by a constant factor when applying

the Helton-Vinnikov formula. The authors of this paper are not able to fix their problem, but the formulation in Theorem 2.4 for $g = 1$ has no such trouble.

4. It is also shown in [15], Theorem 7, that for a smooth curve $\mathcal{V}_{\mathbb{C}}(f)$ there are 2^g real positive definite representations. We prove in Theorem 2.4 that for an elliptic curve ($g = 1$) with singular points (non-smooth), there are $2 = 2^g$ real positive definite representations.

In Theorems 2.1 and 2.4, we use a parametrization of an irreducible projective algebraic curve $F(x, y, z) = 0$. An irreducible curve $F(x, y, z) = 0$ is transformed into an algebraic curve $G(x, y, z) = 0$ for which every singular point $(x_0, y_0, z_0) \neq (0, 0, 0)$ of $G(x, y, z) = 0$ has pairwise distinct tangents by successive Cremona transformations (cf. [16], Theorem 7.4). Such a birational transformation preserves the genus of the curve. We assume that $G(x, y, z)$ is an irreducible homogeneous polynomial of degree n , the curve $G(x, y, z) = 0$ has ordinary multiple points of multiplicities m_1, \dots, m_k and has no singular points other than the ordinary ones. Then the genus g of $G(x, y, z) = 0$ is given by

$$(2.6) \quad g = \frac{1}{2}(n-1)(n-2) - \frac{1}{2} \sum_{j=1}^k m_j(m_j-1)$$

(cf. [16]). The number g can be evaluated by the function ‘genus’ of algcurves package in Maple. For $g = 0$, the method of constructing a parametrization of the curve $G(x, y, z) = 0$ as $x = u(s)$, $y = v(s)$, $z = w(s)$ of degree at most n is given in [16], pages 67–68. For $g = 1$, the curve $G(x, y, z) = 0$ is transformed into the Weierstrass canonical form

$$-y^2z + 4x^3 - g_2xz^2 - g_3 = 0$$

with $g_2^3 - 27g_3^2 \neq 0$. The affine curve $G(x, y, 1) = 0$ is then expressed as

$$x = R_1(\mathcal{P}, \mathcal{P}'), \quad y = R_2(\mathcal{P}, \mathcal{P}')$$

by some rational functions R_1, R_2 of two variables (cf. [16], page 72, [17], pages 489–493). The Riemann theta functions $\theta_\delta(u, [q])$, $\delta = 1, 2, 3, 4$, can be numerically computed using Mathematica function ‘EllipticTheta $[\delta, \pi u, q]$ ’ (cf. [18]).

3. COMPUTING RATIONAL CURVES

We explain the formula in Theorem 2.1 by practical computation on an algebraic curve with genus $g = 0$. Consider a typical roulette curve defined by a trigonometric polynomial

$$\varphi(\theta) = \exp(2i\theta) + \frac{4}{5} \exp(-i\theta).$$

The determinantal representation of this curve has been studied in [2]. We apply Theorem 2.1 to find the real symmetric matrices B and C . By using a parameter $s = \tan(\theta)/2$, this roulette curve is parametrized as

$$x = \Re(\varphi(\theta)) = \frac{u(s)}{w(s)}, \quad y = \Im(\varphi(\theta)) = \frac{v(s)}{w(s)},$$

where

$$\begin{aligned} u(s) &= \frac{1}{5}(s^2 + 6s + 3)(s^2 - 6s + 3), \\ v(s) &= -\frac{4}{5}(7s^2 - 3)s, \\ w(s) &= (s^2 + 1)^2, \end{aligned}$$

and the roulette curve as an affine curve $F(x, y, 1) = 0$ is parametrized as

$$\begin{aligned} L_1(x, s) &= -(s^2 + 1)^2 x + \frac{1}{5}(s^2 + 6s + 3)(s^2 - 6s + 3) = 0, \\ L_2(y, s) &= -(s^2 + 1)^2 y - \frac{4}{5}(7s^2 - 3)s = 0. \end{aligned}$$

By taking the resultant of $L_1(x, s)$ and $L_2(y, s)$ with respect to s , we obtain the equation $F(x, y, 1) = 0$ of the roulette curve which, in homogeneous form, is expressed as

$$F(x, y, z) = \frac{15,625}{729}(x^2 + y^2)^2 - \frac{20,000}{729}(x^3 z - 3xy^2 z) - \frac{550}{27}(x^2 + y^2)z^2 + z^4.$$

Solving the equation $F(0, 1, -\beta_j) = 0$, we find that the matrix B is given by

$$B = \text{diag} \left(\frac{5}{9}(-3 + 2\sqrt{6}), -\frac{5}{9}(3 + 2\sqrt{6}), \frac{5}{9}(3 + 2\sqrt{6}), -\frac{5}{9}(-3 + 2\sqrt{6}) \right).$$

The corresponding points $Q'_j = s$ of the real line are characterized as

$$(s^2 + 6s + 3)(s^2 - 6s + 3) = 0, \quad -\frac{v(s)}{w(s)} = -\frac{1}{\beta_j}.$$

It follows that

$$Q'_1 = 3 + \sqrt{6}, \quad Q'_2 = 3 - \sqrt{6}, \quad Q'_3 = -3 + \sqrt{6}, \quad Q'_4 = -3 - \sqrt{6}.$$

We conclude by (2.3) that the matrix C and its entries are

$$C = \begin{bmatrix} -c_{22} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{13} \\ c_{13} & c_{23} & c_{22} & c_{12} \\ c_{14} & c_{13} & c_{12} & -c_{22} \end{bmatrix},$$

where

$$\begin{aligned} c_{22} &= \frac{25\sqrt{2}}{9\sqrt{3}}, & c_{12} &= -\frac{5\sqrt{10}}{9\sqrt{3}}, & c_{13} &= \frac{5\sqrt{5}}{9\sqrt{6}}, \\ c_{14} &= -\frac{5}{9\sqrt{6}}\sqrt{73+28\sqrt{6}}, & c_{23} &= \frac{5}{9\sqrt{6}}\sqrt{73-28\sqrt{6}}. \end{aligned}$$

4. COMPUTING ELLIPTIC CURVES

In the paper [1], the so-called j -invariant of an irreducible elliptic curve associated with the following 4×4 matrix is explicitly formulated. The 4×4 cyclic weighted shift matrix is

$$S = \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \\ a_1 & 0 & 0 & 0 \end{bmatrix},$$

where $a_1 = \sqrt{2}k(1-s^2)/(1+s^2)$, $a_2 = \sqrt{2}k(2s)/(1+s^2)$ for $0 < k$, $0 < s < \sqrt{2}$. Then

$$(s^2+1)^4 F_S(x, y, z) = (s^2+1)^4 z^4 - 2k^2(s^2+1)^4 (x^2+y^2)z^2 + 16k^4 s^2 (s^2-1)^2 (x^2+y^2)^2.$$

This form is hyperbolic with respect to $(0, 0, 1)$, and has two ordinary double points at $(0, 1, 0)$ and $(1, 0, 0)$. Accordingly, by the genus formula (2.6), $g(F_S) = 1$.

The curve $F_S(x, y, z) = 0$ intersects the line $x = 0$ at four distinct points $(0, 1, -\beta_j)$ with

$$\beta_1 = \sqrt{2}k \frac{1-s^2}{1+s^2}, \quad \beta_2 = -\beta_1, \quad \beta_3 = \sqrt{2}k \frac{2s}{1+s^2}, \quad \beta_4 = -\beta_3.$$

Then the diagonal matrix is $B = \text{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$. The quartic form $F_S(x, y, z)$ has a rather simple symmetric determinantal representation

$$F(x, y, z) = \det(zI_4 + yB + xA_1),$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & \varepsilon a_{13} & a_{14} \\ 0 & 0 & \varepsilon a_{14} & a_{13} \\ \varepsilon a_{13} & a_{14} & 0 & 0 \\ \varepsilon a_{14} & a_{13} & 0 & 0 \end{bmatrix}$$

with

$$a_{13} = \frac{k(1+2s-s^2)}{\sqrt{2}(1+s^2)}, \quad a_{14} = \frac{k(1-2s-s^2)}{\sqrt{2}(1+s^2)}, \quad \varepsilon = \pm 1.$$

For $k = 1/\sqrt{2}$ and $s = 1/5$, we have $a_{13} = \varepsilon 17/26$, $a_{14} = 7/26$.

Another symmetric determinantal representation is given by

$$F_S(x, y, z) = \det(zI_4 + yB + xA_2),$$

where

$$A_2 = \begin{bmatrix} 0 & \varepsilon a_{12} & \eta a_{13} & 0 \\ \varepsilon a_{12} & 0 & 0 & -\eta a_{13} \\ \eta a_{13} & 0 & 0 & \varepsilon a_{34} \\ 0 & -\eta a_{13} & \varepsilon a_{34} & 0 \end{bmatrix},$$

with

$$\begin{aligned} a_{12} &= \frac{2\sqrt{2}ks(1-2s-s^2)}{(1+2s-s^2)(1+s^2)}, \\ a_{13} &= \frac{2\sqrt{2}k\sqrt{s(1-s^2)}}{1+2s-s^2}, \\ a_{34} &= -\frac{\sqrt{2}k(1-s^2)(1-2s-s^2)}{(1+2s-s^2)(1+s^2)}, \end{aligned}$$

$\varepsilon, \eta = \pm 1$. For $k = 1/\sqrt{2}$ and $s = 1/5$, we have $a_{12} = 35/221$, $a_{34} = -84/221$, $a_{13} = 2\sqrt{30}/17$.

Now, we explain the computation of the formula in Theorem 2.4. To parametrize the curve $\mathcal{V}_{\mathbb{C}}(F_S)$ using elliptic functions, we introduce new variables U, V, W by

$$U = k(x^2 - y^2), \quad V = \frac{z(z - kx + ky)}{k}, \quad W = (z + kx - ky)(x + y).$$

The inverse of this birational transformation is given by

$$x = \frac{1}{2k}(2U^2 + UV - 3UW + W^2), \quad y = \frac{1}{2k}(2U^2 - UV - 3UW + W^2), \quad z = V(W - U).$$

The quartic curve $F_S(x, y, z) = 0$ is birationally transformed into the non-singular cubic curve $G(U, V, W) = 0$ where

$$\begin{aligned} G(U, V, W) &= (s^2 + 1)^4 W^3 - 4(s^2 + 1)^4 U W^2 + (5s^8 + 4s^6 + 62s^4 + 4s^2 + 5) U^2 W \\ &\quad - 2(s^4 - 6s^2 + 1) U^3 - (s^2 + 1)^4 V^2 W. \end{aligned}$$

We perform numerical computations for $k = 1/\sqrt{2}, s = 1/5$. The points Q_j on the curve $F_S(x, y, z) = 0$ are transformed into the points

$$\begin{aligned} [Q_1] &= \left(1, -\frac{50}{169} + \frac{5\sqrt{2}}{13}, 1 + \frac{5\sqrt{2}}{13}\right), & [Q_2] &= \left(1, -\frac{50}{169} - \frac{5\sqrt{2}}{13}, 1 - \frac{5\sqrt{2}}{13}\right), \\ [Q_3] &= \left(1, -\frac{288}{169} + \frac{12\sqrt{2}}{13}, 1 + \frac{12\sqrt{2}}{13}\right), & [Q_4] &= \left(1, -\frac{288}{169} - \frac{12\sqrt{2}}{13}, 1 - \frac{12\sqrt{2}}{13}\right) \end{aligned}$$

on the curve

$$G(U, V, W) = 16(28,561W^3 - 114,244UW^2 + 128,405U^2W - 28,322U^3 - 28,561V^2W) = 0.$$

By the transformation

$$U = -\tilde{U} + \frac{128,405}{84,966}W, \quad V = \frac{119}{169\sqrt{2}}\tilde{V},$$

the cubic curve $G(U, V, W) = 0$ turns into the Weierstrass canonical form

$$-\tilde{V}^2W + 4\tilde{U}^3 - g_2\tilde{U}W^2 - g_3W^3 = 0$$

with

$$g_2 = \frac{6,780,988,321}{601,601,763}, \quad g_3 = -\frac{556,790,665,176,719}{76,673,543,092,587}.$$

Thus the affine algebraic curve $F_S(x, y, 1) = 0$ is parametrized as

$$\begin{aligned} x = R_1(u) &= \frac{1}{10,110,954(84,966\mathcal{P}(u) - 43,439)\mathcal{P}'(u)} \\ &\quad \times (-5,055,477\sqrt{2}(84,966\mathcal{P}(u) - 128,405)\mathcal{P}'(u) \\ &\quad + 676(42,483\mathcal{P}(u) - 42,961)(84,966\mathcal{P}(u) - 43,439)), \\ y = R_2(u) &= \frac{1}{10,110,954(84,966\mathcal{P}(u) - 43,439)\mathcal{P}'(u)} \\ &\quad \times (5,055,477\sqrt{2}(84,966\mathcal{P}(u) - 128,405)\mathcal{P}'(u) \\ &\quad + 676(4,2483\mathcal{P}(u) - 42,961)(84,966\mathcal{P}(u) - 43,439)). \end{aligned}$$

The half-periods of the Weierstrass P -function are approximately

$$\omega_1 = 1.849,847,0, \quad \omega_2 = 0.921,393,5i.$$

Their ratio $\tau = \omega_2/\omega_1$ is approximately 0.498,091,74i.

The cubic curve $-\tilde{V}^2 + 4\tilde{U}^3 - g_2\tilde{U} - g_3 = 0$ is parametrized as

$$\tilde{U} = \mathcal{P}(u, \{g_2, g_3\}), \quad \tilde{V} = \mathcal{P}'(u, \{g_2, g_3\}).$$

The cubic and the line $\tilde{V} = 0$ intersect at

$$(\tilde{U}_1, 0) = \left(\frac{42,961}{42,483}, 0\right), \quad (\tilde{U}_2, 0) = \left(\frac{78,719}{84,966}, 0\right), \quad (\tilde{U}_3, 0) = \left(-\frac{16,4641}{84,966}, 0\right).$$

These three points correspond respectively to points $\omega_1, \omega_1 + \omega_2, \omega_2$ on the torus $\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})$. The points Q_j are transformed into the points

$$\begin{aligned} \tilde{Q}_1 &= \left(\frac{7,739}{84,966} + \frac{65\sqrt{2}}{119}, \frac{28,470}{14,161} - \frac{16,900\sqrt{2}}{14,161}, 1\right), \\ \tilde{Q}_2 &= \left(\frac{7,739}{84,966} - \frac{65\sqrt{2}}{119}, -\frac{28,470}{14,161} - \frac{16,900\sqrt{2}}{14,161}, 1\right), \\ \tilde{Q}_3 &= \left(\frac{249,071}{84,966} - \frac{156\sqrt{2}}{119}, -\frac{142,584}{14,161} + \frac{97,344\sqrt{2}}{14,161}, 1\right), \\ \tilde{Q}_4 &= \left(\frac{249,071}{84,966} + \frac{156\sqrt{2}}{119}, \frac{142,584}{14,161} + \frac{97,344\sqrt{2}}{14,161}, 1\right) \end{aligned}$$

on the cubic curve $-\tilde{V}^2W + 4\tilde{U}^3 - g_2\tilde{U}W^2 - g_3 = 0$.

The two points \tilde{Q}_3, \tilde{Q}_4 lie on the pseudo line of the real part of the cubic curve, and the two points \tilde{Q}_1, \tilde{Q}_2 lie on the oval of the real part of the cubic curve. Under the elliptic curve group operation

$$(\mathcal{P}(u_1), \mathcal{P}'(u_1)) + (\mathcal{P}(u_2), \mathcal{P}'(u_2)) = (\mathcal{P}(u_1 + u_2), \mathcal{P}'(u_1 + u_2)),$$

the points \tilde{Q}_j satisfy

$$2\tilde{Q}_1 = 2\tilde{Q}_2 = 2\tilde{Q}_3 = 2\tilde{Q}_4 = \left(\frac{128,405}{84,966}, \frac{169\sqrt{2}}{119}\right).$$

We also have

$$2\left(\frac{128,405}{84,966}, \frac{169\sqrt{2}}{119}\right) = \left(\frac{42,961}{42,483}, 0\right).$$

Then we find that the point of the torus $\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})$ corresponding to $2\tilde{Q}_j$ is $3/2\omega_1$. Each difference $\tilde{Q}_j - \tilde{Q}_k$ ($j \neq k$) satisfies

$$2(\tilde{Q}_j - \tilde{Q}_k) = 0$$

with respect to the elliptic curve group structure. By computing the tangent line passing through \tilde{Q}_j, \tilde{Q}_k , we find that

$$\begin{aligned}\tilde{Q}_2 - \tilde{Q}_1 &= \tilde{Q}_4 - \tilde{Q}_3 = (\tilde{U}_1, 0), \\ \tilde{Q}_3 - \tilde{Q}_2 &= \tilde{Q}_4 - \tilde{Q}_1 = (\tilde{U}_2, 0), \\ \tilde{Q}_3 - \tilde{Q}_1 &= \tilde{Q}_4 - \tilde{Q}_2 = (\tilde{U}_3, 0).\end{aligned}$$

Using these relations, we find that the respective points on the torus $\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})$ and the normalized torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ are

$$Q'_1 = \frac{3}{4}\omega_1 + \omega_2, \quad Q'_2 = \frac{7}{4}\omega_1 + \omega_2, \quad Q'_3 = \frac{3}{4}\omega_1, \quad Q'_4 = \frac{7}{4}\omega_1,$$

and

$$Q''_1 = \frac{3}{8} + \frac{1}{2}\tau, \quad Q''_2 = \frac{7}{8} + \frac{1}{2}\tau, \quad Q''_3 = \frac{3}{8}, \quad Q''_4 = \frac{7}{8}.$$

Then the even Riemann theta functions $\theta_2, \theta_3, \theta_4$ on the normalized Jacobi variety $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ satisfy the equations

$$\begin{aligned}\theta_2(Q''_2 - Q''_1) &= \theta_2(Q''_4 - Q''_3) = \theta_2\left(\frac{1}{2}\right) = 0, \\ \theta_3(Q''_1 - Q''_4) &= \theta_3\left(-\frac{1}{2} + \frac{\tau}{2}\right) = 0, \\ \theta_3(Q''_2 - Q''_3) &= \theta_3\left(\frac{1}{2} + \frac{\tau}{2}\right) = 0, \\ \theta_4(Q''_1 - Q''_3) &= \theta_4(Q''_2 - Q''_4) = \theta_4\left(\frac{\tau}{2}\right) = 0.\end{aligned}$$

The elliptic functions θ_δ/θ_1 over the normalized Jacobi variety take on the following approximate values at the points $Q''_j - Q''_k$:

$$\begin{aligned}\frac{\theta_2}{\theta_1}(Q''_3 - Q''_1) &= \frac{\theta_2}{\theta_1}(Q''_4 - Q''_2) = \frac{\theta_2}{\theta_1}\left(-\frac{\tau}{2}\right) = 2.428,571,4i \approx \frac{17}{7}i, \\ \frac{\theta_2}{\theta_1}(Q''_4 - Q''_1) &= \frac{\theta_2}{\theta_1}(Q''_3 - Q''_2) = \frac{\theta_2}{\theta_1}\left(\pm\frac{1}{2} - \frac{\tau}{2}\right) = 0.411,764,71i \approx \frac{7}{17}i, \\ \frac{\theta_3}{\theta_1}(Q''_2 - Q''_1) &= \frac{\theta_3}{\theta_1}(Q''_4 - Q''_3) = \frac{\theta_3}{\theta_1}\left(\frac{1}{2}\right) = 0.414,778,33, \\ \frac{\theta_3}{\theta_1}(Q''_3 - Q''_1) &= \frac{\theta_3}{\theta_1}(Q''_4 - Q''_2) = \frac{\theta_3}{\theta_1}\left(-\frac{\tau}{2}\right) = 2.410,926,4i, \\ \frac{\theta_4}{\theta_1}(Q''_2 - Q''_1) &= \frac{\theta_4}{\theta_1}(Q''_4 - Q''_3) = \frac{\theta_4}{\theta_1}\left(\frac{1}{2}\right) = 1.007,318,8, \\ \frac{\theta_4}{\theta_1}(Q''_3 - Q''_2) &= \frac{\theta_4}{\theta_1}\left(-\frac{1}{2} - \frac{\tau}{2}\right) = -0.992,734,38, \\ \frac{\theta_4}{\theta_1}(Q''_4 - Q''_1) &= \frac{\theta_4}{\theta_1}\left(\frac{1}{2} - \frac{\tau}{2}\right) = 0.992,734,38.\end{aligned}$$

The numerical values of $\theta_1(0)$, $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$ are given respectively by

$$3.693,259,2, \quad 1.411,753,8, \quad 1.422,086,2, \quad 0.585,564,89.$$

The values $d(R_1/R_2)(Q'_j)$ are given numerically by

$$d\left(\frac{R_1}{R_2}\right)(Q'_1) = d\left(\frac{R_1}{R_2}\right)(Q'_2) = -d\left(\frac{R_1}{R_2}\right)(Q'_3) = -d\left(\frac{R_1}{R_2}\right)(Q'_4) = 1.414,213,6.$$

We then find that both the values

$$\frac{\theta'_1(0)}{2\omega_1 \theta_2(0)} \frac{1}{\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_1)} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_3)}}, \quad \frac{\theta'_1(0)}{2\omega_1 \theta_2(0)} \frac{1}{\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_1)} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_4)}},$$

are approximated by $-0.500,000,000 \, i$.

Now, for the main diagonals of C , it can be easily deduced from (2.2) that $c_{11} = c_{22} = c_{33} = c_{44} = 0$ for $\delta = 2, 3, 4$. We have the equations

$$\begin{aligned} \beta_3 - \beta_1 &= \frac{12}{13} - \frac{5}{13} = \frac{7}{13}, & \beta_4 - \beta_2 &= -(\beta_3 - \beta_1) = -\frac{7}{13}, \\ \beta_4 - \beta_1 &= -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}, & \beta_3 - \beta_2 &= -(\beta_4 - \beta_1) = \frac{17}{13}, \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\theta_\delta}{\theta_1}\right)(Q''_3 - Q''_1) &= \left(\frac{\theta_\delta}{\theta_1}\right)(Q''_4 - Q''_2), \quad \delta = 2, 3, 4, \\ \left(\frac{\theta_\delta}{\theta_1}\right)(Q''_4 - Q''_1) &= \varepsilon_\delta \left(\frac{\theta_\delta}{\theta_1}\right)(Q''_3 - Q''_2), \end{aligned}$$

where $\varepsilon_2 = \varepsilon_3 = 1$, $\varepsilon_1 = -1$, and

$$\begin{aligned} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_1)} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_3)} &= \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_2)} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_4)} \\ &= \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_1)} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_4)} = \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_2)} \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_3)}. \end{aligned}$$

For $\delta = 2, 3$, we have that $c_{23} = -c_{14}$, $c_{24} = -c_{13}$.

Suppose that $\delta = 2$. Then

$$\theta_2(Q''_2 - Q''_1) = \theta_2(Q''_4 - Q''_3) = 0$$

and $c_{12} = c_{34} = 0$. This implies that $c_{24} = -c_{13}$, $c_{23} = -c_{13}$, and

$$\begin{aligned} c_{13} &= \frac{i}{2} \times (\beta_3 - \beta_1) \times \frac{\theta_2}{\theta_1}(Q''_3 - Q''_1) = \frac{i}{2} \times \frac{7}{13} \times \frac{17i}{7} = -\frac{17}{26}, \\ c_{14} &= \frac{i}{2} \times (\beta_4 - \beta_1) \times \frac{\theta_2}{\theta_1}(Q''_4 - Q''_1) = \frac{i}{2} \times \left(-\frac{17}{13}\right) \times \frac{7i}{17} = \frac{7}{26}. \end{aligned}$$

The formula (2.5) then produces a real symmetric matrix

$$C = \begin{bmatrix} 0 & 0 & -\frac{17}{26} & \frac{7}{26} \\ 0 & 0 & -\frac{7}{26} & \frac{17}{26} \\ -\frac{17}{26} & -\frac{7}{26} & 0 & 0 \\ \frac{7}{26} & \frac{17}{26} & 0 & 0 \end{bmatrix}$$

admitting the representation $F_S(x, y, z) = \det(zI_4 + yB + xC)$.

Suppose that $\delta = 3$. Then $\theta_3(Q''_4 - Q''_1) = \theta_3(Q''_3 - Q''_2) = 0$ and $c_{14} = c_{23} = 0$. By the relations

$$\frac{\theta_3}{\theta_1}(Q''_4 - Q''_3) = \frac{\theta_3}{\theta_1}(Q''_2 - Q''_1)$$

and

$$\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_1)}\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_2)} = \sqrt{2}, \quad \sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_3)}\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_4)} = -\sqrt{2},$$

we have

$$c_{12} = -\frac{\beta_2 - \beta_1}{\beta_4 - \beta_3}c_{34} = -\frac{5}{12}c_{34}.$$

Numerical computation, yields that

$$\frac{\theta'_1(0)}{2\omega_1\theta_3(0)} \times \frac{\theta_3(Q''_4 - Q''_3)}{\theta_1(Q''_4 - Q''_3)} \times \frac{1}{\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_3)}\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_4)}} \approx -0.205,882,35,$$

which is approximately $-7/34$, and thus

$$c_{34} = (\beta_4 - \beta_3) \times \frac{7}{34} = \frac{13}{24} \times \frac{7}{34} = \frac{84}{221}.$$

We also have

$$\frac{\theta'_1(0)}{2\omega_1\theta_3(0)} \times \frac{\theta_3(Q''_3 - Q''_1)}{\theta_1(Q''_3 - Q''_1)} \times \frac{1}{\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_1)}\sqrt{d\left(\frac{R_1}{R_2}\right)(Q'_3)}} \approx -1.196,704,7,$$

which is approximately $-26\sqrt{30}/119$, and the value leads to

$$c_{13} = (\beta_3 - \beta_1) \times \left(-\frac{26\sqrt{30}}{119}\right) = -\frac{2\sqrt{30}}{17}.$$

The formula (2.5) produces another real symmetric matrix

$$C = \begin{bmatrix} 0 & -\frac{35}{221} & -2\frac{\sqrt{30}}{17} & 0 \\ -\frac{35}{221} & 0 & 0 & \frac{2\sqrt{30}}{7} \\ -2\frac{\sqrt{30}}{17} & 0 & 0 & \frac{84}{221} \\ 0 & \frac{2\sqrt{30}}{17} & \frac{84}{221} & 0 \end{bmatrix}$$

satisfying the representation $F(x, y, z) = \det(zI_4 + yB + xC)$.

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Authors' addresses: Mao-Ting Chien (corresponding author), Department of Mathematics, Soochow University, 70 Linshi Road, Taipei 11102, Taiwan, e-mail: mtchien@scu.edu.tw; Hiroshi Nakazato, Department of Mathematical Sciences, Faculty of Science and Technology, Hirosaki University, 1-bunkyocho Hirosaki-shi Aomori-ken 036-8561, Japan, e-mail: nakahr@hirosaki-u.ac.jp.