

THE BASIC CONSTRUCTION FROM THE CONDITIONAL
EXPECTATION ON THE QUANTUM DOUBLE
OF A FINITE GROUP

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Abstract. Let G be a finite group and H a subgroup. Denote by $D(G; H)$ (or $D(G)$) the crossed product of $C(G)$ and $\mathbb{C}H$ (or $\mathbb{C}G$) with respect to the adjoint action of the latter on the former. Consider the algebra $\langle D(G), e \rangle$ generated by $D(G)$ and e , where we regard E as an idempotent operator e on $D(G)$ for a certain conditional expectation E of $D(G)$ onto $D(G; H)$. Let us call $\langle D(G), e \rangle$ the basic construction from the conditional expectation $E: D(G) \rightarrow D(G; H)$. The paper constructs a crossed product algebra $C(G/H \times G) \rtimes \mathbb{C}G$, and proves that there is an algebra isomorphism between $\langle D(G), e \rangle$ and $C(G/H \times G) \rtimes \mathbb{C}G$.

Keywords: conditional expectation; basic construction; quantum double; quasi-basis

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1. INTRODUCTION

Index theory for subfactors started with the breakthrough achieved by Jones ([8]). This theory was developed first for subfactors of type II_1 , i.e., those possessing a faithful normal tracial state. Jones' index theory has found important applications in conformal field theory and quantum field theory ([10], [12]) and in the study of the tensor categories arising from compact groups and quantum groups. For a nontechnical but broad overview of the subject including a lot of important connections with other areas, the reader can refer to [7].

Let M be a factor of type II_1 on the Hilbert space \mathcal{H} , and let N (contained in M) be a subfactor. By $\dim_M(\mathcal{H})$ we denote the coupling constant of M on \mathcal{H} . The Jones

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index $[M : N]$, as defined by Jones, is

$$[M : N] = \frac{\dim_N(\mathcal{H})}{\dim_M(\mathcal{H})}$$

if N' , the set of all bounded operators on \mathcal{H} commuting with every operator in \mathcal{H} , is finite. In particular, if G is an infinite conjugacy class group, namely, a discrete group all of whose conjugacy classes are infinite except that of the identity, and $H \leq G$, then the Jones index is exactly $[G : H]$, the index of H in G . Surprisingly, Jones answers this question: what are the possible values of the index $[M : N]$ for subfactors? The answer is $\{4 \cos^2 \pi/n : n \geq 3\} \cup [4, \infty]$. The values $[4, \infty]$ can be easily obtained. However, the situation between 1 and 4 is more difficult. In this case, what is called the basic construction plays an important role in constructing subfactors. More precisely, one can identify M with the algebra of left multiplication operators on $L^2(M, \text{tr})$ and consider the extension e_N to $L^2(M, \text{tr})$ of the conditional expectation onto N , where “tr” is the faithful normal normalized trace and $L^2(M, \text{tr})$ denotes the Hilbert space completion of M with respect to the inner product $\langle a, b \rangle = \text{tr}(b^*a)$ coming from the Gelfand-Naimark-Segal (G.N.S.) construction. One defines $M_1 \triangleq \langle M, e_N \rangle$ to be the von Neumann algebra generated by M and e_N , the Jones projection, on $L^2(M, \text{tr})$ (this is called the basic construction). Subsequently, Jones used the basic construction to obtain an increasing sequence of type II_1 factors, $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$, which is called the Jones tower, iteratively, by adding the Jones projections $\{e_n : n \geq 1, e_1 = e_N\}$, which satisfy the Temperley-Lieb relations. Eventually Jones used this structure to construct examples such that the values of the index can exhaust the set $\{4 \cos^2 \pi/n : n \geq 3\}$. Let J denote the $*$ -operation in $L^2(M, \text{tr})$, regarded as an isometric conjugate linear operator, i.e. $Jx = x^*$ for $x \in L^2(M, \text{tr})$; then $\langle M, e_N \rangle = JN'J$, which is a very elegant result.

Jones' index theory for subfactors of type II_1 has been generalized to general inclusions of factors by Kosaki ([11]) and Longo ([13]). Based on the work of Jones, Kosaki and Longo, Watatani ([16]) investigated Index Γ for a conditional expectation $\Gamma : B \rightarrow A$ of a C^* -algebra B onto a C^* -subalgebra A , and introduced the C^* -algebra basic construction $C^*\langle B, \gamma_A \rangle$ as follows: one can view B as a pre-Hilbert module over A by defining the A -valued inner product $\langle x, y \rangle = \Gamma(x^*y)$. Denote by \overline{B} the completion of B with respect to $\|\cdot\|_B$, where $\|x\|_B = \langle x, x \rangle^{1/2}$ for $x \in B$. Then \overline{B} is a Hilbert C^* -module over A . Denote by $\text{End}_A^*(\overline{B})$ the set of all adjointable operators from \overline{B} to \overline{B} . It is a C^* -algebra. In such a case, one can identify B with the algebra of left multiplication operators on \overline{B} . The conditional expectation Γ can be extended to a projection γ_A on \overline{B} via $\gamma_A(x) = \Gamma(x)$ for $x \in B$. The C^* -algebra $C^*\langle B, \gamma_A \rangle$ is generated by $\{x\gamma_A y : x, y \in B\}$.

However, in contrast to the basic construction for type II_1 factors, the C^* -algebra $C^*\langle B, \gamma_A \rangle$ does not have the concrete form, such as $\langle M, e_N \rangle = JN'J$, where M is a factor of type II_1 and $N \subseteq M$ is a subfactor. The reason for this phenomenon is that any factor of type II_1 admits a faithful trace which is a state, for which the G.N.S. construction may be performed, while for general algebras, the existence of such a functional is uncertain.

In this paper, we think about the basic construction for special algebras. Let G be a finite group and H a subgroup of G , denoted by $H \leq G$. Then $D(G; H)$ is defined as the crossed product of $C(G)$, the algebra of complex functions on G , and the group algebra $\mathbb{C}H$ with respect to the adjoint action of the latter on the former. In particular, if $H = G$, then $D(G; H) \triangleq D(G)$ is the quantum double of G . Then we consider the conditional expectation $E: D(G) \rightarrow D(G; H)$ defined in a natural way and give an appropriate form of the basic construction. In detail, considering the algebra $\langle D(G), e \rangle$ generated by $\{(g, h): g, h \in G\}$ and e , where we regard E as an idempotent operator e on $D(G)$, one finds that the algebra $\langle D(G), e \rangle$ is isomorphic to the crossed product algebra $C(G/H \times G) \rtimes \mathbb{C}G$. Moreover, by constructing a conditional expectation $\tilde{E}: \langle D(G), e \rangle \rightarrow D(G)$ of index-finite type, which is called the dual conditional expectation of $E: D(G) \rightarrow D(G; H)$, one finds that the index of \tilde{E} is exactly the index of E .

2. THE BASIC CONSTRUCTION FOR THE QUANTUM DOUBLE OF A FINITE GROUP

2.1. Conditional expectations between algebras. At the beginning of this section, we first briefly review some known results we will need later. Suppose that B is an algebra over \mathbb{C} and A a subalgebra with a common identity element 1. These definitions can be found in [16].

Definition 2.1. A linear map $\Gamma: B \rightarrow A$ is called a conditional expectation if it satisfies the following conditions: for all $a \in A$, $b \in B$,

- (1) $\Gamma(1) = 1$;
- (2) (bimodular property)

$$\Gamma(ab) = a\Gamma(b), \quad \Gamma(ba) = \Gamma(b)a.$$

Moreover, we say Γ is non-degenerate if $E(Bb) = 0$ or $E(bB) = 0$ implies $b = 0$ for $b \in B$.

Remark 2.2. If B is a C^* -algebra with a C^* -subalgebra A of B with a common identity element, we always assume that Γ is positive, i.e., $\Gamma(b^*b)$ is a positive element in A for any $b \in B$. Actually, Γ is a projection of norm one ([2]).

Definition 2.3. A finite family $\{(u_i, v_i): i = 1, 2, \dots, n\} \subseteq B \times B$ is called a quasi-basis for Γ if for all $x \in B$,

$$\sum_{i=1}^n u_i \Gamma(v_i x) = x = \sum_{i=1}^n \Gamma(x u_i) v_i.$$

Furthermore, if there exists a quasi-basis for Γ , we say Γ is of index-finite type. In this case we define the index of Γ by

$$\text{Index } \Gamma = \sum_{i=1}^n u_i v_i.$$

Remark 2.4. (1) If Γ is of index-finite type, then $\text{Index } \Gamma$ is in the center of B and does not depend on the choice of the quasi-basis. In fact, for any $b \in B$,

$$\begin{aligned} (\text{Index } \Gamma)b &= \sum_{i=1}^n u_i v_i b = \sum_{i=1}^n u_i \left(\sum_{j=1}^n \Gamma(v_i b u_j) v_j \right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n u_i \Gamma(v_i b u_j) \right) v_j = \sum_{j=1}^n b u_j v_j = b(\text{Index } \Gamma). \end{aligned}$$

This means $\text{Index } \Gamma$ is in the center of B . Let $\{(s_j, t_j): j = 1, 2, \dots, n\}$ be another quasi-basis for Γ , then

$$\sum_{i=1}^n u_i v_i = \sum_{i=1}^n \left(\sum_{j=1}^n s_j \Gamma(t_j u_i) \right) v_i = \sum_{j=1}^n s_j \left(\sum_{i=1}^n \Gamma(t_j u_i) v_i \right) = \sum_{j=1}^n s_j t_j,$$

which shows that the value $\text{Index } \Gamma$ does not depend on the choice of the quasi-basis.

(2) The existence of a quasi-basis guarantees that Γ is non-degenerate ([16]).

2.2. The quantum double and the basic construction. Suppose that G is a finite group with a unit u , and that H is a subgroup of G . Let $t_1 = u, t_2, \dots, t_k$ be a left coset representation of H in G , namely $G = \bigcup_{i=1}^k t_i H$ and $i \neq j$ induces that $t_i H \cap t_j H = \emptyset$. Let us begin with the following definitions.

Definition 2.5 ([5], [4]). $D(G; H)$ is the crossed product of $C(G)$ and the group algebra $\mathbb{C}H$, where $C(G)$ denotes the set of complex functions on G , with respect to the adjoint action of the latter on the former.

In particular, if $H = G$, then $D(G; H) \triangleq D(G)$ is the quantum double of G . For more information about $D(G)$, one can refer to [1], [3], [9]. The main difference between $D(G)$ and $D(G; H)$ is that the former is a quasi-triangular Hopf algebra while the latter is not ([14]).

It is easy to see that $\sum_{g \in G} (g, u) \triangleq I$ is the unit of $D(G; H)$, where u is the unit of G . There is a unique element $z_H = |H|^{-1} \sum_{h \in H} (e, h)$, called a cointegral, satisfying

$$az_H = z_H a = \varepsilon(a)z_H, \quad a \in D(G; H), \quad \text{and} \quad \varepsilon(z_H) = 1.$$

As a result, $D(G; H)$ is a semisimple finite dimensional algebra with a natural $*$ -structure ([15]). Consequently, it can be a C^* -algebra.

In [6], the authors investigate the index of conditional expectation of index-finite type from $D(G)$ onto the subalgebra $D(G; H)$. Consider

$$\begin{aligned} E: D(G) &\rightarrow D(G; H) \\ \sum_{g, h \in G} \lambda_{g, h}(g, h) &\mapsto \sum_{g \in G, h \in H} \lambda_{g, h}(g, h) \end{aligned}$$

where $\lambda_{g, h} \in \mathbb{C}$, and set $u_i = \sum_{\alpha \in G} (\alpha, t_i)$ and $v_i = \sum_{\beta \in G} (\beta, t_i^{-1})$. Then $\{(u_i, v_i): i = 1, 2, \dots, k\}$ is a quasi-basis for E , and therefore $\text{Index } E = kI$.

In the sequel we will consider the basic construction from the conditional expectation $E: D(G) \rightarrow D(G; H)$.

Recall that a homomorphic map $T: D(G) \rightarrow D(G)$ is said to be a right $D(G; H)$ -module homomorphism if $T(xy) = T(x)y$ for $x \in D(G), y \in D(G; H)$. Denote by $\text{End}_{D(G; H)} D(G)$ the set of all right $D(G; H)$ -module homomorphisms from $D(G)$ to $D(G)$. Then it is a nonzero algebra. In such a case, one can regard $D(G)$ as the algebra of left multiplication operators on $D(G)$, while the conditional expectation E as an idempotent operator on $D(G)$, denoted by e .

Lemma 2.6. *As operators on $D(G)$, e and $(g, h) \in D(G)$ satisfy the following covariant relations:*

- (1) $e(g, h)e = E(g, h)e$;
- (2) $h \in H$ if and only if $e(g, h) = (g, h)e$.

Proof. To prove the first part, just observe that for $(m, n) \in D(G)$,

$$(e(g, h)e)(m, n) = e((g, h)E(m, n)) = E(g, h)E(m, n) = (E(g, h)e)(m, n).$$

For the second part, assume that $h \in H$, then

$$(e(g, h))(m, n) = E((g, h)(m, n)) = (g, h)E(m, n) = ((g, h)e)(m, n)$$

for $(m, n) \in D(G)$. Thus (g, h) commutes with e . Conversely, if $(g, h) \in D(G)$ and $e(g, h) = (g, h)e$, then

$$(g, h) = (g, h)e \sum_{\alpha \in G} (\alpha, u) = e(g, h) \sum_{\alpha \in G} (\alpha, u) = E(g, h),$$

which shows that $h \in H$. □

Definition 2.7. Let $\langle D(G), e \rangle$ be the subalgebra of $\text{End}_{D(G;H)} D(G)$ generated by $\{(g, h): g, h \in G\}$ and e . We call it the basic construction from the conditional expectation $E: D(G) \rightarrow D(G; H)$.

Remark 2.8. The set $\{(g_1, h_1)e(g_2, h_2): g_i, h_i \in G, i = 1, 2\}$ can generate the algebra $\langle D(G), e \rangle$. Clearly, $e = \sum_{\alpha, \beta \in G} (\alpha, u)e(\beta, u)$. And for $(g, h) \in D(G)$, we have that

$$\begin{aligned} \sum_{i=1}^k (g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h)(m, n) &= \delta_{gh, hm} \sum_{i=1}^k (g, t_i)E(t_i^{-1}gt_i, t_i^{-1}hn) \\ &= \delta_{gh, hm}(g, t_{i_0})(t_{i_0}^{-1}gt_{i_0}, t_{i_0}^{-1}hn) \\ &= \delta_{gh, hm}(g, hn) \\ &= (g, h)(m, n), \end{aligned}$$

where in the second equation we use the fact that for $hn \in G$, there exists t_{i_0} such that $hn \in t_{i_0}H$. This implies that $(g, h) = \sum_{i=1}^k (g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h)$, and that the element of $D(G)$ can be linearly expressed by $\{(g_1, h_1)e(g_2, h_2): g_i, h_i \in G, i = 1, 2\}$.

3. THE CONSTRUCTION OF THE CROSSED PRODUCT ALGEBRA AND THE ISOMORPHISM THEOREM

Let us continue to assume that G is a finite group with a subgroup H . Denote by G/H the set of all left cosets of H , i.e. $G/H = \{[t_1], [t_2], \dots, [t_k]\}$. Let $C(G/H \times G)$ and $\mathbb{C}G$ be the algebra of complex functions on $G/H \times G$ and the group algebra over the \mathbb{C} , respectively.

The set $\{\chi_{[t_i]}: i = 1, 2, \dots, k\}$ is a linear basis of $C(G/H)$ where

$$\chi_{[t_i]}[t_j] = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

while the set $\{F_g: g \in G\}$ is a linear basis of $C(G)$ where

$$F_g(h) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g. \end{cases}$$

Since G is a finite group and $C(G/H \times G) \cong C(G/H) \otimes C(G)$, hence $\{(t_i, g): i = 1, 2, \dots, k; g \in G\}$ can be viewed as a linear basis of $C(G/H \times G)$, where we use (t_i, g) in place of $(\chi_{[t_i]}, F_g)$ for notational convenience. The map $\sigma: \mathbb{C}G \times C(G/H \times G) \rightarrow C(G/H \times G)$ given on the basis elements of $C(G/H \times G)$ as

$$\sigma(h \times (t_i, g)) = (ht_i, hgh^{-1})$$

for $h, g \in G$, can be linearly extended both to $\mathbb{C}G$ and $C(G/H \times G)$. One can show that σ defines an automorphic action of $\mathbb{C}G$ on $C(G/H \times G)$. Here and from now on, by $h(t_i, g)$ we always denote $\sigma(h \times (t_i, g))$.

Definition 3.1. The associative algebra generated by $\{(t_i, g), \alpha: g, \alpha \in G; i = 1, 2, \dots, k\}$ is called the crossed product of $C(G/H \times G)$ by $\mathbb{C}G$ with respect to σ , and we denote it by $C(G/H \times G) \rtimes_{\sigma} \mathbb{C}G$ or $C(G/H \times G) \rtimes \mathbb{C}G$ for convenience.

Using the linear basis elements $(t_i, g; \alpha)$ of $C(G/H \times G) \rtimes \mathbb{C}G$, the product is defined as

$$(t_i, g; \alpha)(t_j, h; \beta) = \delta_{[t_i], [\alpha t_j]} \delta_{g\alpha, \alpha h} (t_i, g; \alpha\beta),$$

$$\text{where } \delta_{\zeta, \eta} = \begin{cases} 1 & \text{if } \zeta = \eta, \\ 0 & \text{if } \zeta \neq \eta. \end{cases}$$

It is clear that the element $\sum_{g \in G} \sum_{i=1}^k (t_i, g; u)$ is the unit of $C(G/H \times G) \rtimes \mathbb{C}G$, where u is a unit of G . Moreover, if H is a normal subgroup of G , then $C(G/H \times G) \rtimes \mathbb{C}G$ is a coalgebra. In fact, the coproduct Δ and counit ε are defined on the basis elements as

$$\begin{aligned} \Delta(t_i, g; \alpha) &= \sum_{j=1}^k \sum_{h \in G} (t_j, h; \alpha) \otimes (t_j^{-1} t_i, h^{-1} g; \alpha), \\ \varepsilon(t_i, g; \alpha) &= \delta_{[t_i], [t_1]} \delta_{g, u} \end{aligned}$$

and are linearly extended to $C(G/H \times G) \rtimes \mathbb{C}G$. Since ε is not an algebra homomorphism, $C(G/H \times G) \rtimes \mathbb{C}G$ is not a bialgebra. Naturally, a question may emerge: in what case $C(G/H \times G) \rtimes \mathbb{C}G$ can be a bialgebra. However, this is unknown.

Lemma 3.2. *The elements $\sum_{i=1}^k (t_i, g; \alpha)$ and $\sum_{h \in G} (t_1, h; u)$ satisfy the following covariant relations:*

- (1) $\sum_{h \in G} (t_1, h; u) \sum_{i=1}^k (t_i, g; \alpha) \sum_{h \in G} (t_1, h; u) = \delta_{[u], [\alpha]} \sum_{i=1}^k (t_i, g; \alpha) \sum_{h \in G} (t_1, h; u);$
- (2) $\sum_{h \in G} (t_1, h; u)$ is an idempotent element in $C(G/H \times G) \rtimes \mathbb{C}G$, that is

$$\left(\sum_{h \in G} (t_1, h; u) \right)^2 = \sum_{h \in G} (t_1, h; u).$$

Proof. (1) Notice that

$$\begin{aligned} \sum_{h \in G} (t_1, h; u) \sum_{i=1}^k (t_i, g; \alpha) \sum_{h \in G} (t_1, h; u) \\ = \sum_{h \in G} (t_1, h; u) \sum_{i=1}^k \sum_{h \in G} \delta_{[t_i], [\alpha]} \delta_{g\alpha, \alpha h} (t_i, g; \alpha) \\ = \sum_{h \in G} (t_1, h; u) (\alpha, g; \alpha) = \sum_{h \in G} \delta_{[u], [\alpha]} \delta_{h, g} (t_1, g; \alpha) \\ = \delta_{[u], [\alpha]} (t_1, g; \alpha), \end{aligned}$$

which together with $\sum_{i=1}^k (t_i, g; \alpha) \sum_{h \in G} (t_1, h; u) = (\alpha, g; \alpha)$ completes the proof.

$$\begin{aligned} (2) \left(\sum_{h \in G} (t_1, h; u) \right)^2 &= \sum_{h \in G} \sum_{g \in G} (t_1, h; u) (t_1, g; u) = \sum_{h \in G} \sum_{g \in G} \delta_{g, h} (t_1, h; u) = \\ &= \sum_{h \in G} (t_1, h; u). \end{aligned}$$

□

Now we give the main theorem of this paper.

Theorem 3.3. *There is an isomorphism of algebras*

$$\langle D(G), e \rangle \cong C(G/H \times G) \rtimes \mathbb{C}G.$$

Proof. Let $\pi: \langle D(G), e \rangle \rightarrow C(G/H \times G) \rtimes \mathbb{C}G$ be a map with

$$\begin{aligned} (g, \alpha) &\mapsto \sum_{i=1}^k (t_i, g; \alpha) \\ e &\mapsto \sum_{g \in G} (t_1, g; u). \end{aligned}$$

First, π is a well-defined map and can be linearly extended to $\langle D(G), e \rangle$ preserving algebraic structure. Indeed, by Lemma 2.6 and Lemma 3.2 we obtain that

$$\pi(e(g, \alpha)e) = \pi(e)\pi(g, \alpha)\pi(e) = \pi(E(g, \alpha)e).$$

And we have

$$\begin{aligned} \pi(g, \alpha)\pi(h, \beta) &= \sum_{i=1}^k (t_i, g; \alpha) \sum_{j=1}^k (t_j, h; \beta) = \sum_{i=1}^k \sum_{j=1}^k \delta_{[t_i], [\alpha t_j]} \delta_{g\alpha, \alpha h} (t_i, g; \alpha\beta) \\ &= \delta_{g\alpha, \alpha h} \sum_{i=1}^k (t_i, g, \alpha\beta) = \pi(\delta_{g\alpha, \alpha h}(g, \alpha\beta)) \\ &= \pi((g, \alpha)(h, \beta)). \end{aligned}$$

Thus, π is an algebra homomorphism.

In order to complete the proof of the theorem we need to show that π is bijective. Indeed, for any $(t_i, g; \alpha) \in C(G/H \times G) \rtimes \mathbb{C}G$, choose $(g, t_i)e(t_i^{-1}gt_i, t_i^{-1}\alpha) \in \langle D(G), e \rangle$ such that

$$\pi((g, t_i)e(t_i^{-1}gt_i, t_i^{-1}\alpha)) = (t_i, g; \alpha),$$

which implies that π is surjective, and then $\dim \langle D(G), e \rangle \geq k|G|^2$. From the following relations, we know that every element $(g, h)e(\alpha, \beta)$ in $\langle D(G), e \rangle$ can be linearly expressed by $\{(g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h): g, h \in G; i = 1, 2, \dots, k\}$. For $(m, n) \in D(G)$,

$$\begin{aligned} (g, h)e(\alpha, \beta)(m, n) &= (g, h)e(\delta_{\alpha\beta, \beta m}(\alpha, \beta n)) \\ &= \delta_{[\beta n], [u]} \delta_{\alpha\beta, \beta m}(g, h)(\alpha, \beta n) \\ &= \delta_{[\beta n], [u]} \delta_{gh, h\alpha} \delta_{\alpha\beta, \beta m}(g, h\beta n), \end{aligned}$$

which yields induce that

$$(g, h)e(h^{-1}gh, \beta)(m, n) = \delta_{[\beta n], [u]} \delta_{h^{-1}gh, \beta m\beta^{-1}}(g, h\beta n).$$

And for $h \in H$, there exists t_i such that $h \in t_i H$, that is $t_i^{-1}h \in H$. Then

$$\begin{aligned} (g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h\beta)(m, n) &= \delta_{t_i^{-1}gh\beta, t_i^{-1}h\beta m} (g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h\beta n) \\ &= \delta_{h^{-1}gh, \beta m\beta^{-1}}(g, t_i)E(t_i^{-1}gt_i, t_i^{-1}h\beta n) \\ &= \delta_{[\beta n], [u]} \delta_{h^{-1}gh, \beta m\beta^{-1}}(g, t_i)(t_i^{-1}gt_i, t_i^{-1}h\beta n) \\ &= \delta_{[\beta n], [u]} \delta_{h^{-1}gh, \beta m\beta^{-1}}(g, h\beta n). \end{aligned}$$

From the above, π is surjective and $\{(g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h): g, h \in G; i = 1, 2, \dots, k\}$ is a linear basis of $\langle D(G), e \rangle$, which mean π is bijective. \square

Remark 3.4. The basic constructions do not depend on the choice of conditional expectations (see Proposition 2.10.11 in [16]), which, together with Theorem 3.3, implies that the crossed product does not depend on the choice of a conditional expectation.

Example 3.5. (1) Consider the basic construction from the conditional expectation $\Phi: \mathbb{C}G \rightarrow \mathbb{C}H$ defined by

$$\Phi\left(\sum_{g \in G} \lambda_g g\right) = \sum_{h \in H} \lambda_h h$$

where $\lambda_g \in \mathbb{C}$ for $g \in G$. Let $\tau: \mathbb{C}G \rightarrow \text{Aut } C(G/H)$ be the action induced by translation from the left, where $\text{Aut } C(G/H)$ stands for the collection of all automorphisms on $C(G/H)$. There is an inclusion $i: \mathbb{C}G \rightarrow D(G)$ given by $i(g) = \sum_{f \in G} (f, g)$, which together with Theorem 3.3 yields that $\langle \mathbb{C}G, \varphi \rangle$ is algebra isomorphic to $C(G/H) \rtimes_{\tau} \mathbb{C}G$, where we view Φ as an idempotent operator on $\mathbb{C}G$, denoted by φ .

(2) If G is an Abelian finite group, then $D(G)$ reduces to a symmetry group $\widehat{G} \times G$, where \widehat{G} denotes the Pontryagin dual of G (the group of characters of G). Let $E: \widehat{G} \times G \rightarrow \widehat{G} \times H$ be a conditional expectation such that

$$E\left(\sum_{g, h \in G} \lambda_{g, h}(g, h)\right) = \sum_{g \in G, h \in H} \lambda_{g, h}(g, h).$$

Set $\varsigma: \mathbb{C}G \times C(G/H \times G) \rightarrow C(G/H \times G)$ given on the basis elements of $C(G/H \times G)$ as

$$\sigma(h \times (t_i, g)) = (ht_i, g) \quad \text{for } h, g \in G.$$

One can prove that ς defines an automorphic action of $\mathbb{C}G$ on $C(G/H \times G)$. Then, $\langle \widehat{G} \times G, e \rangle$ is algebra isomorphic to $C(G/H \times G) \rtimes_{\varsigma} \mathbb{C}G$. Moreover, there is an algebra isomorphism between $\langle \widehat{G} \times G, e \rangle$ and $\widehat{G} \times (C(G/H) \rtimes_{\tau} \mathbb{C}G)$, where τ is defined as above.

Finally, we consider the conditional expectation from $\langle D(G), e \rangle$ onto $D(G)$.

Theorem 3.6. *The map $\widetilde{E}: \langle D(G), e \rangle \rightarrow D(G)$ defined on the basis elements of $\langle D(G), e \rangle$ by*

$$\widetilde{E}((g, t_i)e(t_i^{-1}gt_i, t_i^{-1}h)) = \frac{1}{k}(g, h)$$

for $g, h \in G$ and $i = 1, 2, \dots, k$, linearly extended to $\langle D(G), e \rangle$, is a conditional expectation of index-finite type, and we call it the dual conditional expectation of $E: D(G) \rightarrow D(G; H)$. More precisely, let $w_i = \sqrt{k} \sum_{\alpha, \beta \in G} (\alpha, t_i)e(\beta, u)$ and $z_i =$

$\sqrt{k} \sum_{\eta, \xi \in G} (\eta, u) e(\xi, t_i^{-1})$, then $\{(w_i, z_i) : i = 1, 2, \dots, k\}$ is a quasi-basis for \tilde{E} and $\text{Index } \tilde{E} = \text{Index } E$.

Proof. Since $\sum_{g \in G} \sum_{i=1}^k (g, t_i) e(t_i^{-1} g t_i, t_i^{-1})$ and I are the units of $\langle D(G), e \rangle$ and $D(G)$, respectively, we have

$$\tilde{E} \left(\sum_{g \in G} \sum_{i=1}^k (g, t_i) e(t_i^{-1} g t_i, t_i^{-1}) \right) = \frac{1}{k} \sum_{g \in G} \sum_{i=1}^k (g, u) = \sum_{g \in G} (g, u) = I.$$

Since $\text{Index } E = kI$ is in the center of $D(G)$, \tilde{E} is a $D(G)$ -bimodule homomorphism.

For $g, h \in G$ and $t_j, j = 1, 2, \dots, k$, we have that

$$\begin{aligned} \sum_{i=1}^k w_i \tilde{E}(z_i(g, t_j) e(t_j^{-1} g t_j, t_j h)) \\ &= \sqrt{k} \sum_{i=1}^k w_i \tilde{E} \left(\sum_{\eta, \xi \in G} (\eta, u) e(\xi, t_i^{-1})(g, t_j) e(t_j^{-1} g t_j, t_j h) \right) \\ &= \sqrt{k} \sum_{i=1}^k w_i \tilde{E} \left(\sum_{\eta \in G} (\eta, u) E(t_i^{-1} g t_i, t_i^{-1} t_j) e(t_j^{-1} g t_j, t_j h) \right) \\ &= \frac{1}{\sqrt{k}} \sum_{i=1}^k w_i E(t_i^{-1} g t_i, t_i^{-1} t_j) (t_j^{-1} g t_j, t_j h) \\ &= \frac{1}{\sqrt{k}} w_j (t_j^{-1} g t_j, u) (t_j^{-1} g t_j, t_j h) \\ &= \sum_{\alpha \in G} (\alpha, t_j) e(t_j^{-1} g t_j, u) (t_j^{-1} g t_j, t_j h) \\ &= \sum_{\alpha \in G} (\alpha, t_j) (t_j^{-1} g t_j, u) e(t_j^{-1} g t_j, t_j h) \\ &= (g, t_j) e(t_j^{-1} g t_j, t_j h) \end{aligned}$$

and similarly

$$\sum_{i=1}^k \tilde{E}((g, t_j) e(t_j^{-1} g t_j, t_j h) w_i) z_i = (g, t_j) e(t_j^{-1} g t_j, t_j h).$$

This shows that $\{(w_i, z_i) : i = 1, 2, \dots, k\}$ is a quasi-basis for \tilde{E} . Hence \tilde{E} is of index-finite type and

$$\begin{aligned} \text{Index } \tilde{E} &= \sum_{i=1}^k w_i z_i = k \sum_{i=1}^k \sum_{\alpha, \beta, \xi, \eta \in G} (\alpha, t_i) e(\beta, u) (\eta, u) e(\xi, t_i^{-1}) \\ &= k \sum_{i=1}^k \sum_{\alpha, \beta, \xi \in G} (\alpha, t_i) E(\beta, u) e(\xi, t_i^{-1}) \\ &= k \sum_{i=1}^k \sum_{\alpha, \xi \in G} (\alpha, t_i) e(\xi, t_i^{-1}) = k \sum_{\gamma \in G} (\gamma, u) \\ &= \text{Index } E. \end{aligned}$$













Here we use the properties that $\left\{ \left(\sum_{\alpha \in G} (\alpha, t_i), \sum_{\xi \in G} (\xi, t_i^{-1}) \right) : i = 1, 2, \dots, k \right\}$ is a quasi-basis for E , and the unit $I = \sum_{\gamma \in G} (\gamma, u)$ of $D(G)$ can be expressed as

$$\sum_{i=1}^k \sum_{\alpha, \xi \in G} (\alpha, t_i) e(\xi, t_i^{-1}). \quad \square$$

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References

- [1] *P. Bántay*: Orbifolds and Hopf algebras. Phys. Lett., B 245 (1990), 477–479. MR
- [2] *O. Bratteli, D. W. Robinson*: Operator Algebras and Quantum Statistical Mechanics. 1. C^* - and W^* -Algebras, Symmetry Groups, Decomposition of States. Texts and Monographs in Physics, Springer, New York, 1987. zbl MR
- [3] *K. A. Dancer, P. S. Isaac, J. Links*: Representations of the quantum doubles of finite group algebras and spectral parameter dependent solutions of the Yang-Baxter equation. J. Math. Phys. 47 (2006), 103511, 18 pages. zbl MR
- [4] *L. Jiang*: Towards a quantum Galois theory for quantum double algebras of finite groups. Proc. Am. Math. Soc. 138 (2010), 2793–2801. zbl MR
- [5] *L. Jiang*: C^* -index of observable algebras in G -spin model. Sci. China, Ser. A 48 (2005), 57–66. zbl MR
- [6] *L. Jiang, G. Zhu*: C^* -index in double algebra of finite group. Trans. Beijing Inst. Technol. 23 (2003), 147–148. (In Chinese.) zbl MR
- [7] *V. F. R. Jones*: Subfactors and Knots. Expository lectures from the CBMS regional conference, Annapolis, USA, 1988. Regional Conference Series in Mathematics 80, AMS, Providence, 1991. zbl MR
- [8] *V. F. R. Jones*: Index for subfactors. Invent. Math. 72 (1983), 1–25. zbl MR
- [9] *C. Kassel*: Quantum Groups. Graduate Texts in Mathematics 155, Springer, Berlin, 1995. zbl MR
- [10] *Y. Kawahigashi, R. Longo*: Classification of local conformal nets. Case $c < 1$. Ann. Math. 160 (2004), 493–522. zbl MR

- [11] *H. Kosaki*: Extension of Jones' theory on index to arbitrary factors. *J. Funct. Anal.* *66* (1986), 123–140.  
- [12] *R. Longo*: Index of subfactors and statistics of quantum fields. II. Correspondences, braid group statistics and Jones polynomial. *Commun. Math. Phys.* *130* (1990), 285–309.  
- [13] *R. Longo*: Index of subfactors and statistics of quantum fields. I. *Commun. Math. Phys.* *126* (1989), 217–247.  
- [14] *D. E. Radford*: Minimal quasitriangular Hopf algebras. *J. Algebra* *157* (1993), 285–315.  
- [15] *M. E. Sweedler*: Hopf Algebras. Mathematics Lecture Note Series, W. A. Benjamin, New York, 1969.  
- [16] *Y. Watatani*: Index for C^* -subalgebras. *Mem. Am. Math. Soc.* *83* (1990).  

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