

ON THE DIAMETER OF THE INTERSECTION GRAPH
OF A FINITE SIMPLE GROUP

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Abstract. Let G be a finite group. The intersection graph Δ_G of G is an undirected graph without loops and multiple edges defined as follows: the vertex set is the set of all proper nontrivial subgroups of G , and two distinct vertices X and Y are adjacent if $X \cap Y \neq 1$, where 1 denotes the trivial subgroup of order 1. A question was posed by Shen (2010) whether the diameters of intersection graphs of finite non-abelian simple groups have an upper bound. We answer the question and show that the diameters of intersection graphs of finite non-abelian simple groups have an upper bound 28. In particular, the intersection graph of a finite non-abelian simple group is connected.

Keywords: intersection graph; finite simple group; diameter

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1. INTRODUCTION

Csákéany and Pollák [3] introduced the intersection graph of nontrivial proper subgroups of a finite group. Let G be a finite group which is not a cyclic group of prime order. The *intersection graph* Δ_G of G is the undirected graph whose vertex set is the set of all proper nontrivial subgroups of G , and two distinct vertices X and Y are joined by an edge if $X \cap Y \neq 1$, where 1 denotes the trivial subgroup of order 1. This definition was inspired by the intersection graph of nontrivial proper subsemigroups of a semigroup [1]. In [3], the authors posed the problem to classify the finite groups whose intersection graphs are disconnected. In 2010, the problem was solved by Shen [10]. Zelinka [13] studied the intersection graphs of finite abelian groups, and conjectured that two finite abelian groups with isomorphic intersection

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graphs are isomorphic. Very recently, Kayacan and Yaraneri [6] investigated the conjecture and showed that it is almost true.

It was shown in [10] that the intersection graph of a finite non-abelian simple group is connected. At the end of the paper, the author put forward the problem: Do the diameters of intersection graphs of finite non-abelian simple groups have an upper bound? A graph related to the maximal subgroups of a group has been studied in [4]. It follows from [4], Theorem 1.1, that the diameters of intersection graphs of non-abelian simple groups have an upper bound 64. In this paper we prove the following theorem.

Theorem 1.1. *Let G be a finite non-abelian simple group. Then $\text{diam}(\Delta_G) \leq 28$. In particular, Δ_G is connected.*

All graphs considered in this paper are finite, simple, and undirected. Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ , respectively. If $\{x, y\}$ is an edge of Γ , then we say that x and y are adjacent in Γ , and we denote this by $x \sim y$. The *distance* of two vertices x and y , denoted by $d_\Gamma(x, y)$, is the length of the shortest path between x and y . We say $d_\Gamma(x, y) = \infty$ if x and y are not connected in Γ . A *component* of Γ is its maximal connected subgraph. If Γ is connected, the largest distance between a pair at vertices of Γ is called the *diameter* of Γ , and is denoted by $\text{diam}(\Gamma)$. For two integers a and b , we denote the greatest common divisor of a and b by (a, b) . Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of $|G|$, where $|G|$ is the order of G . The *prime graph* Γ_G of G is the graph whose vertex set is $\pi(G)$, and two distinct vertices p and q are adjacent if there is an element of order pq in G . For a positive integer i , π_i always denotes a component of Γ_G , and we say simply that π_i is a subset of $\pi(G)$.

2. THE RESULTS

Throughout this section, G denotes a finite non-abelian simple group.

Lemma 2.1 ([2], Corollary 2). *Let p be an odd prime divisor of $|G|$. Then $d_{\Gamma_G}(p, 2) = 1, 2$ or ∞ .*

Lemma 2.2. *Suppose that $H, K \in V(\Delta_G)$. Then*

- (1) *if $|H|$ and $|K|$ are even, then $d_{\Delta_G}(H, K) \leq 2$;*
- (2) *if $(|H|, |K|) \neq 1$, then $d_{\Delta_G}(H, K^g) \leq 2$ for some $g \in G$.*

Proof. (1) Take $a \in H$ and $b \in K$ such that $|a| = |b| = 2$. Since G is simple, we have that $\langle a, b \rangle$ is a proper subgroup of G . If $a \in K$ or $b \in H$, then H is adjacent

to K in Δ_G , and so $d_{\Delta_G}(H, K) = 1$. If $a \notin K$ and $b \notin H$, then $H \sim \langle a, b \rangle \sim K$, and hence $d_{\Delta_G}(H, K) \leq 2$.

(2) Take a prime p in $\pi(H) \cap \pi(K)$. Choose a in H and b in K such that $|a| = |b| = p$. Let P be a Sylow p -subgroup of G such that $a \in P$. Then $b \in P^{g^{-1}}$ for some $g \in G$. It follows that $b^g \in P$. Note that $b^g \in K^g$. Hence $H \sim P \sim K^g$, as desired. \square

Lemma 2.3. *Let J_4 be Janko's large simple sporadic group. Then Δ_{J_4} is connected and $\text{diam}(\Delta_{J_4}) \leq 4$.*

Proof. By the main result of [7], we get all conjugacy classes of maximal subgroups of J_4 . Observe that every maximal subgroup of J_4 is of even order. Let H and K be two distinct vertices of Δ_{J_4} . Then there exist two maximal subgroups M_1 and M_2 such that $H \subseteq M_1$ and $K \subseteq M_2$. By (1) of Lemma 2.2, one has $d_{\Delta_{J_4}}(M_1, M_2) \leq 2$. This implies that $d_{\Delta_{J_4}}(H, K) \leq 4$. It follows that Δ_{J_4} is connected and $\text{diam}(\Delta_{J_4}) \leq 4$. \square

A subgroup H of a group K is called *isolated* if $H \cap H^k = 1$ or $H \cap H^k = H$ for each $k \in K$, and for each $h \in H \setminus \{1\}$, $C_K(h) \subseteq H$. A group K is called *Frobenius* if there exists a nontrivial proper subgroup H of K such that $N_K(H) = H$ and $C_K(h) \subseteq H$ for all nontrivial $h \in H$.

Lemma 2.4 ([11], Theorem 3). *Let π_i be a component of Γ_G not containing 2. Then G has a nilpotent Hall π_i -subgroup which is isolated in G .*

Lemma 2.5. *Let P and H be two distinct subgroups of G such that $|P| = p$ and $2 \mid |H|$, where p is an odd prime. If p and 2 are connected in Γ_G , then $d_{\Delta_G}(H, P) \leq 6$.*

Proof. Note that p and 2 are connected. By Lemma 2.1, one has that $d_{\Gamma_G}(p, 2) = 1$ or 2.

Suppose that $d_{\Gamma_G}(p, 2) = 1$. Then G has an element x of order $2p$. Note that $(|x|, |P|) = p$. By (2) of Lemma 2.2, we have $d_{\Delta_G}(\langle x \rangle^g, P) \leq 2$ for some $g \in G$. Furthermore, $d_{\Delta_G}(\langle x \rangle^g, H) \leq 2$ by (1) of Lemma 2.2. It follows that $d_{\Delta_G}(H, P) \leq 4$.

Suppose that $d_{\Gamma_G}(p, 2) = 2$. Let $2 \sim q \sim p$ be a path from 2 to p in Γ_G . Then there exist $y, z \in G$ such that $|y| = 2q$ and $|z| = pq$. It follows from Lemma 2.2 that

$$d_{\Delta_G}(\langle z \rangle^{g_1}, P) \leq 2, \quad d_{\Delta_G}(\langle z \rangle^{g_1}, \langle y \rangle^{g_2}) \leq 2, \quad d_{\Delta_G}(\langle y \rangle^{g_2}, H) \leq 2$$

for some $g_1, g_2 \in G$. This forces that $d_{\Delta_G}(H, P) \leq 6$, as desired. \square

Lemma 2.6. *Suppose that Γ_G has at most five components, and P and H are two subgroups of G such that $|P| = p$ and $2 \mid |H|$, where p is an odd prime. If p is not connected to 2 in Γ_G , then $d_{\Delta_G}(H, P) \leq 14$.*

Proof. Suppose that π_1 and π_2 are two components of Γ_G such that $2 \in \pi_1$ and $p \in \pi_2$. Since 2 and p are not connected, G has a nilpotent Hall π_2 -subgroup A which is isolated in G by Lemma 2.4. Let Q be a Sylow q -subgroup of A , where q is a prime. Suppose that Q is non-abelian. By the main theorem of [2], Q contains a nontrivial element which commutes with an involution. Hence G has an element of order $2q$, and so 2 is adjacent to q in Γ_G . It follows that $2 \in \pi_2$, a contradiction. This implies that Q is abelian, and so is A . By [9], Theorem 1, one of the following holds: $G \cong \text{PSL}(2, q^t)$ for some positive integer t , $N_G(Q)$ contains an involution, or Q is cyclic.

Suppose that $G \cong \text{PSL}(2, q^t)$. By [5], (Chapter 2, Section 8), there exists an element u in $N_G(Q)$ such that $N_G(\langle u \rangle)$ is dihedral. Thus we have a path $Q \sim N_G(Q) \sim N_G(\langle u \rangle)$ in Δ_G . Note that $d_{\Delta_G}(N_G(\langle u \rangle), H) \leq 2$. Then $d_{\Delta_G}(Q, H) \leq 4$. If $p = q$, then it is not hard to see that $d_{\Delta_G}(P, H) \leq 6$, as required. Assume that $q \neq p$. Since A is abelian, p and q are adjacent in Γ_G . In other words, there exists a proper subgroup of G such that its order is divisible by p and q . This implies that $d_{\Delta_G}(P, Q^g) \leq 4$ for some $g \in G$ by Lemma 2.2. Consequently $d_{\Delta_G}(P, H) \leq 8$, as desired.

Suppose that $N_G(Q)$ contains an involution. Note that $N_G(Q)$ is a proper subgroup of G . We get $d_{\Delta_G}(Q, H) \leq 3$. It means that if $p = q$, then $d_{\Delta_G}(P, H) \leq 5$; if $p \neq q$, then $d_{\Delta_G}(P, H) \leq 7$, as desired.

Now assume that Q is cyclic. Then A is cyclic. If $N_G(A) = A$, then G is Frobenius, and so G has a nontrivial normal subgroup which is the Frobenius kernel, a contradiction. It follows that there exists a nontrivial element $x \in N_G(A) \setminus A$. Note that A is a Hall subgroup. We may assume that $|x|$ is a prime p_3 and $p_3 \notin \pi_2$. Since $A\langle x \rangle$ is a proper subgroup of G , it follows that $d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, P) \leq 2$ for some $g_1 \in G$. Suppose that $p_3 \in \pi_1$. Then according to Lemma 2.5, we have $d_{\Delta_G}(\langle x \rangle^{g_1}, H) \leq 6$. Since $d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, P) \leq 2$ and $\langle x \rangle^{g_1} \subseteq A^{g_1}\langle x \rangle^{g_1}$, one has $d_{\Delta_G}(H, P) \leq 8$, as desired.

Now suppose that $p_3 \notin \pi_1$. Let π_3 be the component of Γ_G containing p_3 and r_2 be the minimal prime of π_2 . Take $a \in A$ with $|a| = r_2$. Note that A is cyclic. Since $x \in N_G(A) \setminus A$, one gets that $\langle a \rangle \langle x \rangle$ is a proper subgroup of G . If $p_3 > r_2$, then $\langle a \rangle \langle x \rangle$ is cyclic and since A is isolated in G , one has $x \in A$, a contradiction. Note that $p_3 \neq r_2$. It follows that $p_3 < r_2$. Moreover, by Lemma 2.4, there exists a nilpotent Hall π_3 -subgroup B in G such that it is isolated in G . Similarly, for any Sylow r -subgroup R of B , $G \cong \text{PSL}(2, r^t)$ for some positive integer t , $N_G(R)$ contains an involution, or R is cyclic. Note that $d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, P) \leq 2$ and $d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, B^{g_0}) \leq 2$ for some g_0 in G . If $G \cong \text{PSL}(2, r^t)$, then $R^{g_0} \sim N_G(R^{g_0}) \sim N_G(\langle v \rangle)$ for some $v \in N_G(R^{g_0})$, where $N_G(\langle v \rangle)$ is dihedral, so $d_{\Delta_G}(H, P) \leq 8$, as desired. If $N_G(R)$ contains an involution, then $d_{\Delta_G}(H, P) \leq 7$, as desired. Thus, we get that R is cyclic, and so is B . Then there exists an element $y \in N_G(B) \setminus B$ such that $|y|$ is a prime

p_4 and $p_4 \notin \pi_3$. Since $d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, P) \leq 2$ and $d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, B^{g_2}\langle y \rangle^{g_2}) \leq 2$ for some $g_2 \in G$, one has $d_{\Delta_G}(B^{g_2}\langle y \rangle^{g_2}, P) \leq 4$. If $p_4 \in \pi_1$, by Lemma 2.5, one gets $d_{\Delta_G}(H, P) \leq 10$, as required.

Let p_4 lie in the component π_4 of Γ_G . We may assume that $\pi_4 \neq \pi_1$. Let r_3 be the minimal prime of π_3 . Similarly, we get $p_4 < r_3$. It means that $p_4 < r_3 < r_2$. Thus $\pi_4 \neq \pi_i$ for each $i = 1, 2, 3$. It follows from Lemma 2.4 that there exists a nilpotent Hall π_4 -subgroup C such that it is isolated in G . Similarly, C is abelian and there is a nontrivial element $z \in N_G(C) \setminus C$ such that $|z|$ is a prime $p_5 \notin \pi_4$. Let R be a Sylow r -subgroup of C . Since the prime graph $\Gamma_{\text{PSL}(2, r^t)}$ has precisely three components (see [8]), $N_G(R)$ contains an involution or R is cyclic. If $N_G(R)$ contains an involution, then $d_{\Delta_G}(H, P) \leq 9$, as required. Thus, we may assume that C is cyclic. Let r_4 be the minimal prime of π_4 . Similarly $p_5 < r_4$. Namely $p_5 < r_4 \leq p_4 < r_3 < r_2$.

Suppose that $p_5 \in \pi_1$. Note that

$$d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, P) \leq 2, \quad d_{\Delta_G}(A^{g_1}\langle x \rangle^{g_1}, B^{g_2}\langle y \rangle^{g_2}) \leq 2$$

and

$$d_{\Delta_G}(C^{g_3}\langle z \rangle^{g_3}, B^{g_2}\langle y \rangle^{g_2}) \leq 2$$

for some $g_3 \in G$. By Lemma 2.5 one has $d_{\Delta_G}(\langle z \rangle^{h_3}, H) \leq 6$. This implies that $d_{\Delta_G}(H, P) \leq 12$, as desired.

Let π_5 be the component of Γ_G containing p_5 . We now may assume that $\pi_5 \neq p_i$ for each $i = 1, \dots, 4$. Similarly, G has a cyclic Hall π_5 -subgroup D , and there exists an element $w \in N_G(D) \setminus D$ such that $|w|$ is a prime $p_6 \notin \pi_5$. Let r_5 be the minimal prime of π_5 . We conclude that $p_6 < r_5 < r_4 < r_3 < r_2$. Since Γ_G has at most five components, we have $p_6 \in \pi_1$. Consequently $d_{\Delta_G}(D^{g_4}\langle w \rangle^{g_4}, P) \leq 8$ for some $g_4 \in G$, and by Lemma 2.5, $d_{\Delta_G}(\langle w \rangle^{g_4}, H) \leq 6$. This implies that $d_{\Delta_G}(H, P) \leq 14$. \square

Proof of Theorem 1.1. Note that if $G \not\cong J_4$, then Γ_G has at most five components by [8], [11]. If $G \cong J_4$, then by Lemma 2.3 one has $\text{diam}(\Delta_G) \leq 4$. Thus we assume that Γ_G has at most five components. Let L and K be any two distinct subgroups of G . Then G has two prime-order subgroups P and Q such that $P \subseteq L$ and $Q \subseteq K$. Take a proper subgroup H of even order in G . By Lemmas 2.2, 2.5 and 2.6, one has that $d_{\Delta_G}(H, P) \leq 14$ and $d_{\Delta_G}(H, Q) \leq 14$. This means that $d_{\Delta_G}(L, K) \leq 28$. Consequently, Δ_G is connected and $\text{diam}(\Delta_G) \leq 28$. \square

It follows from [12] that Γ_G has precisely five components if and only if $G \cong E_8(q)$, where $q \equiv 0, \pm 1 \pmod{5}$. The proof of the following corollary is straightforward.

Corollary 2.7. *Suppose that $G \not\cong E_8(q)$, where $q \equiv 0, \pm 1 \pmod{5}$. Then Δ_G is connected and $\text{diam}(\Delta_G) \leq 24$.*

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