

HILBERT-SCHMIDT HANKEL OPERATORS WITH  
ANTI-HOLOMORPHIC SYMBOLS ON COMPLETE  
PSEUDOCONVEX REINHARDT DOMAINS

MEHMET ÇELİK, Commerce, YUNUS E. ZEYTUNCU, Dearborn

Received September 2, 2015. First published February 24, 2017.

*Abstract.* On complete pseudoconvex Reinhardt domains in  $\mathbb{C}^2$ , we show that there is no nonzero Hankel operator with anti-holomorphic symbol that is Hilbert-Schmidt. In the proof, we explicitly use the pseudoconvexity property of the domain. We also present two examples of unbounded non-pseudoconvex domains in  $\mathbb{C}^2$  that admit nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols. In the first example the Bergman space is finite dimensional. However, in the second example the Bergman space is infinite dimensional and the Hankel operator  $H_{\bar{z}_1 \bar{z}_2}$  is Hilbert-Schmidt.

*Keywords:* canonical solution operator for  $\bar{\partial}$ -problem; Hankel operator; Hilbert-Schmidt operator

*MSC 2010:* 47B35, 32A36, 47B10

## 1. INTRODUCTION

**1.1. Setup and problem.** For a domain  $\Omega$  in  $\mathbb{C}^n$ , we denote the space of square integrable functions and the space of square integrable holomorphic functions on  $\Omega$  by  $L^2(\Omega)$  and  $A^2(\Omega)$  (the Bergman space of  $\Omega$ ), respectively. The Bergman projection operator,  $P$ , is the orthogonal projection from  $L^2(\Omega)$  onto  $A^2(\Omega)$ . It is an integral operator with the kernel called the Bergman kernel, which is denoted by  $B_\Omega(z, w)$ . Moreover, if  $\{e_n(z)\}_{n=0}^\infty$  is an orthonormal basis for  $A^2(\Omega)$  then the Bergman kernel can be represented as

$$B_\Omega(z, w) = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)}.$$

On complete Reinhardt domains the monomials  $\{z^\gamma\}_{\gamma \in \mathbb{N}^n}$  (or a subset of them) constitute an orthogonal basis for  $A^2(\Omega)$ .

For  $f \in A^2(\Omega)$ , the Hankel operator with the anti-holomorphic symbol  $\overline{f}$  is formally defined on  $A^2(\Omega)$  by

$$H_{\overline{f}}(g) = (I - P)(\overline{f}g).$$

Note that this (possibly unbounded) operator is densely defined on  $A^2(\Omega)$ .

For a multi-index  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ , we set

$$(1) \quad c_\gamma^2 = \int_{\Omega} |z^\gamma|^2 dV(z).$$

Then on complete Reinhardt domains the set  $\{z^\gamma/c_\gamma\}_{\gamma \in \mathbb{N}^n}$  gives a complete orthonormal basis for  $A^2(\Omega)$ . Each  $f \in A^2(\Omega)$  can be written in the form  $f(z) = \sum_{\gamma \in \mathbb{N}^n} f_\gamma z^\gamma/c_\gamma$  where the sum converges in  $A^2(\Omega)$ , but also uniformly on compact subsets of  $\Omega$ . For the coefficients  $f_\gamma$ , we have  $f_\gamma = \langle f(z), z^\gamma/c_\gamma \rangle_\Omega$ .

**Definition 1.** A linear bounded operator  $T$  on a Hilbert space  $H$  is called a *Hilbert-Schmidt operator* if there is an orthonormal basis  $\{\xi_j\}$  for  $H$  such that the sum  $\sum_{j=1}^{\infty} \|T(\xi_j)\|^2$  is finite.

The sum does not depend on the choice of the orthonormal basis  $\{\xi_j\}$ . For more on Hilbert-Schmidt operators see [10], Section X.

In this paper, we investigate the following problem. On a given Reinhardt domain in  $\mathbb{C}^n$ , characterize the symbols for which the corresponding Hankel operators are Hilbert-Schmidt. This question was first studied in  $\mathbb{C}$  on the unit disc in [2]. The problem was studied on higher dimensional domains in [13], Theorem at page 2, where the author showed that when  $n \geq 2$ , on an  $n$ -dimensional complex ball there are no nonzero Hilbert-Schmidt Hankel operators (with anti-holomorphic symbols) on the Bergman space. The result was revisited in [11] with a more robust approach. On more general domains in higher dimensions, the problem was explored in [6], Theorem 1.1, where the authors extended the result [13], Theorem at page 2, to bounded pseudoconvex domains of finite type in  $\mathbb{C}^2$  with smooth boundary. Moreover, the authors of the current article studied the same problem on complex ellipsoids [3], in  $\mathbb{C}^2$  with not necessarily smooth boundary.

The same question was investigated on Cartan domains of tube type in [1], Section 2, and on strongly pseudoconvex domains in [8], [9]. Arazy studied the natural generalization of Hankel operators on Cartan domains (circular, convex, irreducible bounded symmetric domains in  $\mathbb{C}^n$ ) of tube type and rank  $r > 1$  in  $\mathbb{C}^n$  for which  $n/r$  is an integer. He showed that there is no non-trivial Hilbert-Schmidt Hankel

operator with anti-holomorphic symbols on those type of domains. Li and Peloso, independently, obtained the same result on strongly pseudoconvex domains with smooth boundary.

## 1.2. Results. Let

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2: z_1 \in \mathbb{D} \text{ and } |z_2| < e^{-\varphi(z_1)}\}$$

( $\varphi(z_1) = \varphi(|z_1|)$ ) be a complete pseudoconvex Reinhardt domain where monomials  $\{z^\alpha\}$  (or a subset of monomials) form a complete system for  $A^2(\Omega)$ . In this paper, we show that on complete pseudoconvex Reinhardt domains in  $\mathbb{C}^2$  there are no nonzero Hilbert-Schmidt Hankel operator with anti-holomorphic symbols. Moreover, we also present examples of unbounded non-pseudoconvex domains on which there are nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols.

**Theorem 1.** *Let  $\Omega$  be as above and  $f \in A^2(\Omega)$ . If the Hankel operator  $H_{\bar{f}}$  is Hilbert-Schmidt on  $A^2(\Omega)$  then  $f$  is constant.*

**Remark 1.** Theorem 1 generalizes Zhu's result on the unit ball in  $\mathbb{C}^n$ , see [13], Schnider's result on the unit ball in  $\mathbb{C}^n$  and its variations, see [11]. Theorem 1 also generalizes the result in [6], Theorem 1.1, by dropping the finite type condition on complete pseudoconvex Reinhardt domains.

**Remark 2.** The new ingredient in the proof of Theorem 1 is the explicit use of the pseudoconvexity property of the domain  $\Omega$ , see the assumption made at (6) and how it is used at (10). Additionally, we employ the key estimate (4) proven in [3].

**Remark 3.** After completing this note, the authors have learned that by using the estimate (4), Le obtained the same result on bounded complete Reinhardt domains without the pseudoconvexity assumption, see [7]. Although our statement requires pseudoconvexity, it also works on unbounded domains. The complex function theory on unbounded domains (and its relation to pseudoconvexity) has been investigated recently in [4], [5] and new phenomenas have been observed.

Wiegerinck in [12] constructed Reinhardt domains (unbounded but with finite volume) in  $\mathbb{C}^2$  for which the Bergman spaces are  $k$ -dimensional. In fact, for these domains the Bergman spaces are spanned by monomials of the form  $\{(z_1 z_2)^j\}_{j=1}^{k-1}$ . Therefore, Hankel operators with nontrivial anti-holomorphic symbols are Hilbert-Schmidt. We revisit these and similar domains in the last section to present examples of domains that admit nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols.

## 2. AN IDENTITY AND AN ESTIMATE ON REINHARDT DOMAINS

The set  $\{z^\gamma/c_\gamma\}_{\gamma \in \mathbb{N}^n}$  is an orthonormal basis for  $A^2(\Omega)$ . In order to prove Theorem 1, we will look at the sum

$$(2) \quad \sum_{\gamma} \left\| H_{\bar{f}} \left( \frac{z^\gamma}{c_\gamma} \right) \right\|^2 = \sum_{\alpha} |f_\alpha|^2 \sum_{\gamma} \left( \frac{c_{\alpha+\gamma}^2}{c_\gamma^2} - \frac{c_\gamma^2}{c_{\gamma-\alpha}^2} \right)$$

for  $f \in A^2(\Omega)$ . For detailed computation of (2) and of the later estimate (4) we refer to [3].

The term  $\sum_{\gamma} (c_{\gamma+\alpha}^2/c_\gamma^2 - c_\gamma^2/c_{\gamma-\alpha}^2)$  in the identity (2) plays an essential role in the rest of the proof, and we label it as,

$$(3) \quad S_\alpha := \sum_{\gamma} \left( \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} - \frac{c_\gamma^2}{c_{\gamma-\alpha}^2} \right).$$

Note that the Cauchy-Schwarz inequality guarantees that  $c_{\gamma+\alpha}^2/c_\gamma^2 - c_\gamma^2/c_{\gamma-\alpha}^2 \geq 0$  for all  $\alpha$  and  $\gamma$ .

The computations above hold on any domains where the monomials (or a subset of monomials) form an orthonormal basis for the Bergman space.

Now, we estimate the term  $S_\alpha$  on complete pseudoconvex Reinhardt domains. Our goal is to show that  $S_\alpha$  diverges for all nonzero  $\alpha$  on these domains. By (2), this will be sufficient to conclude Theorem 1.

In earlier results,  $S_\alpha$ 's were computed explicitly to obtain the divergence. Here we obtain the divergence by using the estimate (4):

For any sufficiently large  $N$ , we have

$$(4) \quad S_\alpha \geq \sum_{|\gamma|=N} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2}$$

for any nonzero  $\alpha$ , see [3].

## 3. COMPUTATIONS ON COMPLETE PSEUDOCONVEX REINHARDT DOMAINS, PROOF OF THEOREM 1

Let  $\varphi(r) \in C^2([0, 1))$ , define the complete Reinhardt domain

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 \in \mathbb{D} \quad \text{and} \quad |z_2| < e^{-\varphi(z_1)}\}.$$

Note that  $\varphi(z_1) = \varphi(|z_1|)$ .

If  $\limsup_{r \rightarrow 1^-} \varphi(r)$  is finite then there exists  $c > 0$  such that for any  $z_1 \in \mathbb{D}$  the fiber in the  $z_2$  direction contains a disc of radius  $c$ . Hence,  $\Omega$  contains a polydisc  $\mathbb{D} \times c\mathbb{D}$ . This indicates that there are no nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on  $\Omega$ . This also indicates that there are no compact Hankel operators with anti-holomorphic symbols.

Therefore, from this point we assume

$$\limsup_{r \rightarrow 1^-} \varphi(r) = \infty.$$

In fact, the later assumption (6) made on the domain forces  $\varphi(r)$  not to oscillate, so we can assume

$$(5) \quad \lim_{r \rightarrow 1^-} \varphi(r) = \infty.$$

On the other hand,  $\Omega$  is pseudoconvex if and only if  $z_1 \rightarrow \varphi(|z_1|)$  is a subharmonic function on  $\mathbb{D}$ . A simple calculation gives  $\Delta\varphi(z_1) = \varphi''(r) + \varphi'(r)/r$ . We assume  $\Omega$  is pseudoconvex hence we have

$$(6) \quad \varphi''(r) + \frac{1}{r}\varphi'(r) \geq 0 \quad \text{on } (0, 1).$$

Our goal is to show that the sum  $\sum_{|\gamma|=N} c_{\gamma+\alpha}^2 / c_\gamma^2$  diverges for any nonzero  $\alpha$  on a complete pseudoconvex Reinhardt domain  $\Omega$ . We start with computing  $c_\gamma$ 's.

We have

$$\begin{aligned} c_\gamma^2 &= \int_{\Omega} |z^\gamma|^2 dV(z) = \int_{\mathbb{D}} |z_1|^{2\gamma_1} \int_{|z_2| < e^{-\varphi(|z_1|)}} |z_2|^{2\gamma_2} dA(z_2) dA(z_1) \\ &= \int_{\mathbb{D}} \left\{ |z_1|^{2\gamma_1} \frac{2\pi}{2\gamma_2 + 2} e^{-(2\gamma_2+2)\varphi(|z_1|)} \right\} dA(z_1) = \frac{2\pi^2}{\gamma_2 + 1} \int_0^1 r^{2\gamma_1+1} e^{-(2\gamma_2+2)\varphi(r)} dr. \end{aligned}$$

For sufficiently large  $x$  and  $y$ , consider the ratio

$$(7) \quad R_{x,y} := \frac{\int_0^1 r^{x+2\alpha_1} e^{-(y+2\alpha_2)\varphi(r)} dr}{\int_0^1 r^x e^{-y\varphi(r)} dr},$$

and define

$$\Phi_{x,y}(r) := \frac{r^x e^{-y\varphi(r)}}{\int_0^1 r^x e^{-y\varphi(r)} dr}.$$

Note that  $\Phi_{x,y}(0) = 0$ ,  $\Phi_{x,y}(1) = 0$ , and  $\int_0^1 \Phi_{x,y}(r) dr = 1$ .

Also, define

$$(8) \quad g_\alpha(r) = r^{2\alpha_1} e^{-2\alpha_2 \varphi(r)}.$$

Note that  $g_\alpha(r)$  does not vanish inside the interval  $(0, 1)$ , but may vanish at  $r = 0$  and  $r = 1$  depending on  $\alpha$ . Now, we can rewrite the ratio  $R_{x,y}$  as

$$(9) \quad R_{x,y} = \int_0^1 \Phi_{x,y}(r) r^{2\alpha_1} e^{-2\alpha_2 \varphi(r)} dr = \int_0^1 \Phi_{x,y}(r) g_\alpha(r) dr.$$

Our goal is to find a sub-interval  $(a, b) \subset\subset (0, 1)$  such that for sufficiently large  $x$  and  $y$

$$\int_a^b \Phi_{x,y}(r) dr \geq \frac{1}{2}.$$

For this purpose, we analyse  $\Phi_{x,y}(r)$  further on  $(0, 1)$  and locate the local maximum of  $\Phi_{x,y}(r)$ . We have

$$\frac{d}{dr} \Phi_{x,y}(r) = (x - y\varphi'(r)r)(r^{x-1} e^{-y\varphi(r)}) \left( \int_0^1 r^x e^{-y\varphi(r)} dr \right)^{-1}.$$

Therefore,

$$\frac{d}{dr} \Phi_{x,y}(r) = 0 \quad \text{on } (0, 1) \text{ when } x - y\varphi'(r)r = 0.$$

We label  $f_{x,y}(r) := x - y\varphi'(r)r$ . Note that  $f_{x,y}(r)$  controls the sign of  $\frac{d}{dr} \Phi_{x,y}(r)$ , since the rest of the terms in  $\frac{d}{dr} \Phi_{x,y}(r)$  is positive. Furthermore,

$$f_{x,y}(0) = x > 0$$

and

$$(10) \quad \frac{d}{dr} f_{x,y}(r) = -y(\varphi'(r) + r\varphi''(r)) < 0 \quad (\text{by the assumption (6)}).$$

Hence,  $f_{x,y}(r)$  decreases on  $(0, 1)$  and can vanish at a point. We will show that by choosing  $x, y$  appropriately we can guarantee that  $f_{x,y}(r)$  vanishes on  $(0, 1)$ . All we need is a point  $s \in (0, 1)$  such that

$$s\varphi'(s) > 0.$$

However, this is possible by the assumption (5). If there were no such point  $s \in (0, 1)$ , then  $\varphi(r)$  would not grow up to infinity. Moreover, if there exists  $s \in (0, 1)$  such that  $s\varphi'(s) > 0$  then since  $r\varphi'(r) > 0$  is an increasing function we have

$$r\varphi'(r) > 0, \quad r \in [s, 1).$$

Therefore, there exists a relatively compact subinterval  $(a, b)$  of  $(0, 1)$  such that

$$a\varphi'(a) > 0$$

and hence  $r\varphi'(r) > 0$  on  $(a, b)$ . Moreover, by choosing  $x$  and  $y$  appropriately we can make

$$f_{x,y}(a) > 0 \quad \text{and} \quad f_{x,y}(b) < 0.$$

That is,

$$x - ya\varphi'(a) > 0 \quad \text{and} \quad x - yb\varphi'(b) < 0.$$

Equivalently,

$$a\varphi'(a) < \frac{x}{y} \quad \text{and} \quad \frac{x}{y} < b\varphi'(b).$$

Therefore, as long as we keep

$$(11) \quad a\varphi'(a) < \frac{x}{y} < b\varphi'(b)$$

there exist a solution to  $x - yr\varphi'(r) = 0$  on the interval  $(a, b) \subset (0, 1)$ , and so we guarantee that the function  $\Phi_{x,y}(r)$  assumes its maximum somewhere inside  $(a, b)$ . Let us take the point  $\varrho_{xy} \in (a, b)$  where  $\Phi_{x,y}(r)$  takes its maximum value. We have

$$\int_0^{a/2} \Phi_{x,y}(r) \, dr \leq \int_{a/2}^{\varrho_{xy}} \Phi_{x,y}(r) \, dr \quad \text{and} \quad \int_{(1+b)/2}^1 \Phi_{x,y}(r) \, dr \leq \int_{\varrho_{xy}}^{(1+b)/2} \Phi_{x,y}(r) \, dr.$$

Hence, we deduce that

$$(12) \quad \int_{a/2}^{(1+b)/2} \Phi_{x,y}(r) \, dr \geq \int_0^1 \Phi_{x,y}(r) \, dr \geq \frac{1}{2}$$

as long as  $a\varphi'(a) < x/y < b\varphi'(b)$ . The inequality at (12) is the crucial step for the rest of the proof. It guarantees that the integral of  $\Phi_{x,y}(r)$  is located somewhere in the middle, i.e. does not lean towards any of the end points.

For a multi-index  $\gamma = (\gamma_1, \gamma_2)$ , let us write  $\Phi_\gamma(r) = \Phi_{\gamma_1, \gamma_2}(r)$ . Then

$$(13) \quad \begin{aligned} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} &= \frac{\gamma_2 + 1}{\gamma_2 + \alpha_2 + 1} \frac{\int_0^1 r^{2\gamma_1+2\alpha_1+1} e^{-(2\gamma_2+2+2\alpha_2)\varphi(r)} \, dr}{\int_0^1 r^{2\gamma_1+1} e^{-(2\gamma_2+2)\varphi(r)} \, dr} \\ &= \frac{\gamma_2 + 1}{\gamma_2 + \alpha_2 + 1} \int_0^1 \Phi_{2\gamma_1+1, 2\gamma_2+2}(r) g_\alpha(r) \, dr. \end{aligned}$$

Then

$$\begin{aligned}
 (14) \quad S_\alpha &\geq \sum_{|\gamma|=N} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} = \sum_{k=0}^N \frac{c_{\alpha+(k,N-k)}^2}{c_{(k,N-k)}^2} = \sum_{k=0}^N \frac{c_{(k+\alpha_1, N-k+\alpha_2)}^2}{c_{(k,N-k)}^2} \\
 &= \sum_{k=0}^N \frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1, 2(N-k)+2}(r) g_\alpha(r) \, dr.
 \end{aligned}$$

We want to keep

$$\frac{2k+1}{2N-2k+2} \in (a\varphi'(a), b\varphi'(b)),$$

see (11). This is equivalent to asking  $k$  to be in the interval

$$\frac{2a\varphi'(a)}{2a\varphi'(a)+2}N + \frac{2a\varphi'(a)-1}{2a\varphi'(a)+2} < k < \frac{2b\varphi'(b)}{2b\varphi'(b)+2}N + \frac{2b\varphi'(b)-1}{2b\varphi'(b)+2}.$$

We further restrict  $k$  to the interval

$$I_N := \left( \frac{2a\varphi'(a)}{2a\varphi'(a)+2}N + \frac{2a\varphi'(a)-1}{2a\varphi'(a)+2}, \frac{2b\varphi'(b)}{2b\varphi'(b)+2}N + \frac{2b\varphi'(b)-1}{2b\varphi'(b)+2} \right) \cap (0, N).$$

Therefore, the estimate (14) can be rewritten as

$$(15) \quad S_\alpha \geq \sum_{k \in I_N} \frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1, 2(N-k)+2}(r) g_\alpha(r) \, dr.$$

When  $k \in I_N$  we have

$$\begin{aligned}
 &\frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1, 2(N-k)+2}(r) g_\alpha(r) \, dr \\
 &\geq \frac{1}{1+\alpha_2} \int_{a/2}^{(1+b)/2} \Phi_{2k+1, 2(N-k)+2}(r) g_\alpha(r) \, dr \\
 &\geq \frac{1}{1+\alpha_2} \left( \min_{a/2 \leq r \leq (1+b)/2} \{g_\alpha(r)\} \right) \int_{a/2}^{(1+b)/2} \Phi_{2k+1, 2(N-k)+2}(r) \, dr \\
 \text{by (12)} \quad &\geq \frac{1}{1+\alpha_2} \left( \min_{a/2 \leq r \leq (1+b)/2} \{g_\alpha(r)\} \right) \frac{1}{2}.
 \end{aligned}$$

Let  $\lambda_\alpha := \left( \min_{a/2 \leq r \leq (1+b)/2} \{g_\alpha(r)\} \right) / (2(1+\alpha_2))$ . Note that  $\lambda_\alpha > 0$  since  $g_\alpha(r)$  is strictly positive on  $(a/2, (1+b)/2)$ , see (8). This gives us

$$\begin{aligned}
 S_\alpha &\geq \sum_{k \in I_N} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} \geq \sum_{k \in I_N} \frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1, 2(N-k)+2}(r) g_\alpha(r) \, dr \\
 &\geq \sum_{k \in I_N} \lambda_\alpha = |I_N| \lambda_\alpha.
 \end{aligned}$$



Note that the number of integers in  $I_N$  is comparable to  $N$ . Therefore,  $S_\alpha \gtrsim N$  and this suffices to conclude  $S_\alpha$  diverges for nonzero  $\alpha$ .

#### 4. EXAMPLES OF UNBOUNDED NON-PSEUDOCONVEXS DOMAIN WITH NONZERO HILBERT-SCHMIDT HANKEL OPERATORS

In this section, we present two examples of domains that admit nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols. In the first example, the Bergman space is finite dimensional and the claim holds for trivial reasons. In the second example, the Bergman space is infinite dimensional; however, some of the terms  $S_\alpha$ 's are bounded.

We start with defining the following domains from [12]:

$$\begin{aligned} X_1 &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| > e, |z_2| < \frac{1}{|z_1| \log |z_1|} \right\}, \\ X_2 &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_2| > e, |z_1| < \frac{1}{|z_2| \log |z_2|} \right\}, \\ X_3 &= \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq e, |z_2| \leq e \}, \\ \Omega_0 &= X_1 \cup X_2 \cup X_3, \\ B_m &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|, |z_2| > 1, ||z_1| - |z_2|| < \frac{1}{(|z_1| + |z_2|)^m} \right\}, \\ \Omega_k &= \Omega_0 \cup B_{4k}. \end{aligned}$$

Note that  $\Omega_0$  and  $\Omega_k$  are unbounded non-pseudoconvex complete Reinhardt domains with finite volume. The following proposition is also from [12].

**Proposition 1.** *Let  $k$  be a positive integer.*

- (i) *The Bergman space  $A^2(\Omega_k)$  is spanned by the monomials  $\{(z_1 z_2)^j\}_{j=0}^k$ .*
- (ii) *The Bergman space  $A^2(\Omega_0)$  is spanned by the monomials  $\{(z_1 z_2)^j\}_{j=0}^\infty$ .*

Next, we look at the Hankel operators on the Bergman spaces of  $\Omega_0$  and  $\Omega_k$ .

**Example 1.** We start with  $\Omega_k$ . Since  $A^2(\Omega_k)$  is finite dimensional, for any multi-index of the form  $(j, j)$  for  $j = 1, \dots, k$ , the term  $S_{(j,j)}$  is a finite sum and consequently finite when restricted to the subspace of  $A^2(\Omega_k)$  where the multiplication operator with the symbol  $\bar{f}$  is bounded. Hence, for any  $f \in A^2(\Omega_k)$ , the Hankel operator with the symbol  $\bar{f}$  is Hilbert-Schmidt on the subspace of  $A^2(\Omega_k)$  where the operator is bounded.

**Example 2.** Next, we look at  $\Omega_0$  and observe that the terms  $S_\alpha$  take a simpler form. Namely, for a multi-index  $(j, j)$ ,

$$S_{(j,j)} = \sum_{k=0}^{\infty} \left( \frac{c_{(k+j,k+j)}^2}{c_{(k,k)}^2} - \frac{c_{(k,k)}^2}{c_{(k-j,k-j)}^2} \right),$$

where

$$c_{(k,k)}^2 = \int_{\Omega_0} |z_1 z_2|^{2k} dV(z_1, z_2).$$

We will particularly compute  $S_{(1,1)}$ . A simple integration indicates

$$c_{(k,k)}^2 = 4\pi^2 \left( \frac{2}{2k+1} + \frac{e^{4k+4}}{(2k+2)^2} \right)$$

and by simple algebra we obtain

$$\frac{c_{(k+1,k+1)}^2}{c_{(k,k)}^2} - \frac{c_{(k,k)}^2}{c_{(k-1,k-1)}^2} = \frac{e^{8k+8} \frac{(2k+2)^4 - (2k+4)^2 (2k)^2}{(2k+4)^2 (2k)^2 (2k+2)^4} + e^{4k} \frac{p_1(k)}{p_2(k)} + \frac{p_3(k)}{p_4(k)}}{e^{8k+8} \frac{1}{(2k)^2 (2k+2)^2} + e^{4k} \frac{p_5(k)}{p_6(k)} + \frac{p_7(k)}{p_8(k)}}$$

where  $p_1(k), \dots, p_8(k)$  are polynomials in  $k$ . For large values of  $k$ , the first terms at the numerator and the denominator dominate and we obtain

$$\frac{c_{(k+1,k+1)}^2}{c_{(k,k)}^2} - \frac{c_{(k,k)}^2}{c_{(k-1,k-1)}^2} \approx \frac{\frac{(2k+2)^4 - (2k+4)^2 (2k)^2}{(2k+4)^2 (2k)^2 (2k+2)^4}}{\frac{1}{(2k)^2 (2k+2)^2}} \approx \frac{1}{k^2}.$$

Therefore,  $S_{(1,1)}$  is finite and the Hankel operator  $H_{\overline{z_1 z_2}}$  is Hilbert-Schmidt on  $A^2(\Omega_0)$ .

## 5. REMARKS

**5.1. Canonical solution operator for  $\bar{\partial}$ -problem:.** The canonical solution operator for  $\bar{\partial}$ -problem restricted to  $(0, 1)$ -forms with holomorphic coefficients is not a Hilbert-Schmidt operator on complete pseudoconvex Reinhardt domains because the canonical solution operator for  $\bar{\partial}$ -problem restricted to  $(0, 1)$ -forms with holomorphic coefficients is a sum of Hankel operators with  $\{\bar{z}_j\}_{j=1}^n$  as symbols (by Theorem 1 such Hankel operators are not Hilbert-Schmidt):

$$\bar{\partial}^* N_1(g) = \bar{\partial}^* N_1 \left( \sum_{j=1}^n g_j d\bar{z}_j \right) = \sum_{j=1}^n H_{\bar{z}_j}(g_j)$$

for any  $(0, 1)$ -form  $g$  with holomorphic coefficients.

**Acknowledgement.** We would like to thank Trieu Le and the anonymous referee for valuable comments on an earlier version of this manuscript.

### References

- [1] *J. Arazy*: Boundedness and compactness of generalized Hankel operators on bounded symmetric domains. *J. Funct. Anal.* **137** (1996), 97–151. [zbl](#) [MR](#) [doi](#)
- [2] *J. Arazy, S. D. Fisher, J. Peetre*: Hankel operators on weighted Bergman spaces. *Am. J. Math.* **110** (1988), 989–1053. [zbl](#) [MR](#) [doi](#)
- [3] *M. Çelik, Y. E. Zeytuncu*: Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on complex ellipsoids. *Integral Equations Oper. Theory* **76** (2013), 589–599. [zbl](#) [MR](#) [doi](#)
- [4] *P. Harrington, A. Raich*: Defining functions for unbounded  $C^m$  domains. *Rev. Mat. Iberoam.* **29** (2013), 1405–1420. [zbl](#) [MR](#) [doi](#)
- [5] *P. S. Harrington, A. Raich*: Sobolev spaces and elliptic theory on unbounded domains in  $\mathbb{R}^n$ . *Adv. Diff. Equ.* **19** (2014), 635–692. [zbl](#) [MR](#)
- [6] *S. G. Krantz, S.-Y. Li, R. Rochberg*: The effect of boundary geometry on Hankel operators belonging to the trace ideals of Bergman spaces. *Integral Equations Oper. Theory* **28** (1997), 196–213. [zbl](#) [MR](#) [doi](#)
- [7] *T. Le*: Hilbert-Schmidt Hankel operators over complete Reinhardt domains. *Integral Equations Oper. Theory* **78** (2014), 515–522. [zbl](#) [MR](#) [doi](#)
- [8] *H. Li*: Schatten class Hankel operators on the Bergman spaces of strongly pseudoconvex domains. *Proc. Am. Math. Soc.* **119** (1993), 1211–1221. [zbl](#) [MR](#) [doi](#)
- [9] *M. M. Peloso*: Hankel operators on weighted Bergman spaces on strongly pseudoconvex domains. *Ill. J. Math.* **38** (1994), 223–249. [zbl](#) [MR](#)
- [10] *J. R. Retherford*: Hilbert space: Compact operators and the trace theorem. London Mathematical Society Student Texts 27, Cambridge University Press, Cambridge, 1993. [zbl](#) [MR](#)
- [11] *G. Schneider*: A different proof for the non-existence of Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on the Bergman space. *Aust. J. Math. Anal. Appl.* (electronic only) **4** (2007), Artical No. 1, pages 7. [zbl](#) [MR](#)
- [12] *J. J. O. O. Wiegerinck*: Domains with finite-dimensional Bergman space. *Math. Z.* **187** (1984), 559–562. [zbl](#) [MR](#) [doi](#)
- [13] *K. H. Zhu*: Hilbert-Schmidt Hankel operators on the Bergman space. *Proc. Am. Math. Soc.* **109** (1990), 721–730. [zbl](#) [MR](#) [doi](#)

*Authors' addresses:* Mehmet Çelik, Texas A&M University-Commerce, Department of Mathematics, 1600 Education Dr., Binnion Hall Room 303A, Commerce, Texas, 75429-3011, USA, e-mail: [mehmet.celik@tamuc.edu](mailto:mehmet.celik@tamuc.edu); Yunus E. Zeytuncu, University of Michigan-Dearborn, Department of Mathematics and Statistics, 4901 Evergreen Road, 2014 CASL Building, Dearborn, Michigan, 48128, USA, e-mail: [zeytuncu@umich.edu](mailto:zeytuncu@umich.edu).