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1-COCYCLES ON THE GROUP OF CONTACTOMORPHISMS ON THE SUPERCIRCLE $S^{1|3}$ GENERALIZING THE SCHWARZIAN DERIVATIVE

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Abstract. The relative cohomology $\mathrm{H}^1_{\mathrm{diff}}(\mathbb{K}(1|3),\mathfrak{osp}(2,3);\mathcal{D}_{\lambda,\mu}(S^{1|3}))$ of the contact Lie superalgebra $\mathbb{K}(1|3)$ with coefficients in the space of differential operators $\mathcal{D}_{\lambda,\mu}(S^{1|3})$ acting on tensor densities on $S^{1|3}$, is calculated in N. Ben Fraj, I. Laraied, S. Omri (2013) and the generating 1-cocycles are expressed in terms of the infinitesimal super-Schwarzian derivative 1-cocycle $s(X_f)=D_1D_2D_3(f)\alpha_3^{1/2}$, $X_f\in\mathbb{K}(1|3)$ which is invariant with respect to the conformal subsuperalgebra $\mathfrak{osp}(2,3)$ of $\mathbb{K}(1|3)$.

In this work we study the supergroup case. We give an explicit construction of 1-cocycles of the group of contactomorphisms $\mathcal{K}(1|3)$ on the supercircle $S^{1|3}$ generating the relative cohomology $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|3),\ \mathrm{PC}(2,3);\ \mathcal{D}_{\lambda,\mu}(S^{1|3})$ with coefficients in $\mathcal{D}_{\lambda,\mu}(S^{1|3})$. We show that they possess properties similar to those of the super-Schwarzian derivative 1-cocycle $S_3(\Phi)=E_\Phi^{-1}(D_1(D_2),D_3)\alpha_3^{1/2}$, $\Phi\in\mathcal{K}(1|3)$ introduced by Radul which is invariant with respect to the conformal group $\mathrm{PC}(2,3)$ of $\mathcal{K}(1|3)$. These cocycles are expressed in terms of $S_3(\Phi)$ and possess its properties.

Keywords: contact vector field; cohomology of groups; group of contactomorphisms; super-Schwarzian derivative; invariant differential operator

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1. Introduction

The computation of the first cohomology space $\mathrm{H}^1(\mathrm{Diff}_+(S^1);\mathcal{F}_\lambda(S^1))$, where $\mathrm{Diff}_+(S^1)$ is the group of diffeomorphisms on the circle and $\mathcal{F}_\lambda(S^1)$ is the space of λ-densities, is due to Fuks (see [12]). It is generated by the nontrivial cocycles

$$1)\ f^{-1}\mapsto E_0(f)=\log(f'),$$
$$2)\ f^{-1}\mapsto A_0(f)=\frac{f''}{f'}\,\mathrm{d}x$$

and

$$3)\ f^{-1} \mapsto S_0(f) = \Big(\frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2\Big)\,\mathrm{d}x^2.$$

The cocycles 1), 2) and 3) are linked, respectively, to the well know geometries, viz. the Euclidean, affine and projective, and they are, respectively, invariant with respect to the groups $(\mathbb{R}, +)$, $(\mathrm{Aff}(1, \mathbb{R}), \circ)$ and $(\mathrm{PSL}(2, \mathbb{R}), \times)$.

The 1-cocycle $S_0\colon \mathrm{Diff}_+(S^1) \mapsto \mathcal{F}_2$, has the following proprieties: the cocycle condition $S_0(f \circ g) = g^* S_0(f) + S_0(g)$ for every $f, g \in \mathrm{Diff}_+(S^1)$, where $g^* S_0(f) = S_0(f) \circ g;\ (g')^2$ and the invariance with respect to the group $(\mathrm{PSL}(2, \mathbb{R}), \times)$, has been generalized to a 1-cocycle from $\mathrm{Diff}_+(S^1)$ with coefficients in $\mathcal{D}_{\lambda,\mu}(S^1)$, the space of linear differential operators acting from the space of λ-tenser densities $\mathcal{F}_\lambda(S^1)$ to $\mathcal{F}_\mu(S^1)$, and having the property of invariance with respect to $(\mathrm{PSL}(2, \mathbb{R}), \times)$ by Bouarroudj, Ovsienko, (see [8], [9], [7], [18]).

Recently Michel and Duval [17] have generalized Fuks constructions to the case of the supercircles $S^{1|1}$ and $S^{1|2}$ using a Cartan like formula [10]:

$$\begin{aligned} \mathcal{S}_1\colon\ \mathcal{K}(1|1) &\to \mathcal{Q}(S^{1|1}) = \mathcal{F}_2(S^{1|1}) \oplus \mathcal{F}_{3/2}(S^{1|1}) \\ \Phi^{-1} &\mapsto \mathcal{S}_1(\Phi) = \frac{1}{3}\alpha_1^2 D(S_{11}(\Phi)) + \alpha_1 \beta S_{11}(\Phi) \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_2\colon\ \mathcal{K}(1|2) &\to \mathcal{Q}(S^{1|2}) = \mathcal{F}_2(S^{1|2}) \oplus \mathcal{F}_{3/2}(S^{1|2}) \oplus \mathcal{F}_{3/2}(S^{1|2}) \oplus \mathcal{F}_1(S^{1|2}) \\ \Phi^{-1} &\mapsto \mathcal{S}_2(\Phi) = \frac{1}{6}\alpha_2^2 \Big(D_1 D_2 S_{12}(\Phi) + \frac{1}{2} S_{12}^2(\Phi)\Big) \\ &\qquad + \frac{1}{2}\alpha_2(\beta_1 D_2 + \beta_2 D_1) S_{12}(\Phi) + \beta_1 \beta_2 S_{12}(\Phi) \end{aligned}$$

where

$$S_{11}(\Phi) = 2(D^3 E_\Phi / E_\Phi - 3 D^2 E_\Phi D E_\Phi / (2 (E_\Phi)^2))$$

and

$$S_{12}(\Phi) = 2(D_2 D_1 E_\Phi / E_\Phi - 3 D_2 E_\Phi D_1 E_\Phi / (2 (E_\Phi)^2)).$$

Here α_1 and α_2 are the 1-forms defining, respectively, the contact structures on $S^{1|1}$ and $S^{1|2}$.

The projections of the above projective Schwarzian cocycles $\mathcal{S}$ on tensor densities, as defined by Michel and Duval (see [17]), give the super-Schwarzian derivatives on $\mathcal{K}(1|m)$, $m = 1, 2$, as introduced by Radul (see [20]):

$$\begin{cases} \Phi^{-1} \mapsto S_1(\Phi) = \Big(\dfrac{D^3 E_\Phi}{E_\Phi} - \dfrac{3}{2}\dfrac{D^2 E_\Phi D E_\Phi}{(E_\Phi)^2}\Big)\alpha_1^{3/2} & \text{for } m = 1, \\ \Phi^{-1} \mapsto S_2(\Phi) = \Big(\dfrac{D_2 D_1 E_\Phi}{E_\Phi} - \dfrac{3}{2}\dfrac{D_2 E_\Phi D_1 E_\Phi}{(E_\Phi)^2}\Big)\alpha_2^1 & \text{for } m = 2. \end{cases}$$

Recently Agrebaoui, Dammak and Mansour [1] have generalized the construction of Michel and Duval (see above) by building 1-cocycles with values in the space of linear differential operators acting on tensor densities on $S^{1|1}$ and $S^{1|2}$.

Our main aim in this paper is to extend the work of [1] to the case of the supercircle $S^{1|3}$.

2. Geometric structure on $S^{1|m}$

The purpose of this section is to give a down-to-earth description of the supercircle $S^{1|m}$ (see [6]). In particular, we give explicit descriptions of several classical objects of supergeometry in terms of the even affine coordinate x of S^1 and of m odd coordinates $\xi = (\xi_1, \ldots, \xi_m)$.

We denote by $\mathcal{C}^\infty(S^{1|m}) = \mathcal{C}^\infty(S^1) \otimes \bigwedge(\xi_1, \ldots, \xi_m)$ the $\mathbb{Z}/2\mathbb{Z}$-graded ring of superfunctions, where $\bigwedge(\xi_1, \ldots, \xi_m)$ is the Grassman algebra, i.e., the free algebra generated by $\xi_1, \ldots, \xi_m$ quotiented by the relations $\xi_i \xi_j = -\xi_j \xi_i$. It follows that a function on $S^{1|m}$ can be written in the form

$$F = f_0(x) + \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant m} \xi_{i_1} \ldots \xi_{i_k} f_{i_1, \ldots, i_k}(x), \quad \text{where } f_0, f_{i_1, \ldots, i_k} \in \mathcal{C}^\infty(S^1).$$

Let us denote by J the ideal of $\mathcal{C}^\infty(S^{1|m})$ generated by nilpotent functions ($f_0 \equiv 0$). Recalling that the quotient $\mathcal{C}^\infty(S^{1|m})/J$ is isomorphic to the algebra $\mathcal{C}^\infty(S^1)$ of smooth functions on S^1, we denote by $\pi\colon \mathcal{C}^\infty(S^{1|m}) \to \mathcal{C}^\infty(S^1)$ the canonical projection.

A diffeomorphism on $S^{1|m}$ is, by definition, a parity preserving algebra automorphism of the algebra of functions $\mathcal{C}^\infty(S^{1|m})$. We denote by $\mathrm{Diff}(S^{1|m})$ the group of diffeomorphisms on $S^{1|m}$. Let us write $\xi = (\xi_1, \ldots, \xi_m)$, which allows us to use the shorthand notation $\Phi(x, \xi)$ to denote a diffeomorphism of $S^{1|m}$ defined by a family $(\varphi(x, \xi), \psi_1(x, \xi), \ldots, \psi_m(x, \xi))$ where φ is an even function and ψ_i, where $i = 1, \ldots, m$, are odd functions.

Every $\Phi \in \mathrm{Diff}(S^{1|m})$ induces a diffeomorphism $\Pi(\Phi) \in \mathrm{Diff}(S^1)$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{C}^\infty(S^{1|m}) \arrow[r, "\pi"] \arrow[d, "\Phi"'] & \mathcal{C}^\infty(S^1) \arrow[d, "\Pi(\Phi)"] \\ \mathcal{C}^\infty(S^{1|m}) \arrow[r, "\pi"] & \mathcal{C}^\infty(S^1) \end{tikzcd} \tag{2.1}$$

Hence, every diffeomorphism on $S^{1|m}$ induces a diffeomorphism on S^1 and $\Pi\colon \mathrm{Diff}(S^{1|m}) \to \mathrm{Diff}(S^1)$ is a group homomorphism.

The space $\mathrm{Vect}(S^{1|m})$ of vector fields on $S^{1|m}$ is by definition the space of all graded derivations of the superalgebra $\mathcal{C}^\infty(S^{1|m})$. Namely $X(fg) = X(f)g + (-1)^{p(X)p(f)} fX(g)$, where $p(X)$, $p(f)$ are equal to $0, 1$ according to their parity. It is a Lie superalgebra where the bracket $[X, Y]$ of two vector fields X and Y is defined by their graded commutator:

$$[X, Y](f) = X(Y(f)) - (-1)^{p(X)p(Y)} Y(X(f)), \quad f \in \mathcal{C}^\infty(S^{1|m}).$$

The space $\mathrm{Vect}(S^{1|m})$ is a free left $\mathcal{C}^\infty(S^{1|m})$-module of rank $(1, m)$ with basis $(\partial_x, \partial_{\xi_1}, \ldots, \partial_{\xi_m})$. The parity of the vector fields is defined by $p(\partial_x) = 0$ and $p(\partial_{\xi_1}) = \ldots = p(\partial_{\xi_m}) = 1$.

The space of differential 1-forms $\Omega^1(S^{1|m})$ is a free right $\mathcal{C}^\infty(S^{1|m})$-module of rank $(1, m)$ and $(\mathrm{d}x, \mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_m)$ is a basis. The parity is given by $p(\mathrm{d}x) = 0$ and $p(\mathrm{d}\xi_i) = 1$. The duality between the generators of the space of vector fields and the generators of the space of 1-forms is determined by

$$\langle \partial_x, \mathrm{d}x \rangle = \langle \partial_{\xi_i}, \mathrm{d}\xi_i \rangle = 1 \quad \text{and} \quad \langle \partial_x, \mathrm{d}\xi_i \rangle = \langle \partial_{\xi_i}, \mathrm{d}x \rangle = \langle \partial_{\xi_i}, \mathrm{d}\xi_j \rangle = 0 \quad \text{where } i \neq j.$$

The differential $d\colon \mathcal{C}^\infty(S^{1|m}) \to \Omega^1(S^{1|m})$ is defined by

$$\langle X, \mathrm{d}f \rangle = X(f) \quad \text{for } X \in \mathrm{Vect}(S^{1|m}) \text{ and } f \in \mathcal{C}^\infty(S^{1|m}).$$

Let $\Omega^*(S^{1|m}) = \bigwedge^* \Omega^1(S^{1|m})$ be the space of exterior differential forms on $S^{1|m}$. Then d extends uniquely to a parity preserving derivation (called the exterior differential) $d\colon \Omega^*(S^{1|m}) \to \Omega^*(S^{1|m})$ by requiring that

$$d^2 = 0 \quad \text{and} \quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a \alpha \wedge d\beta \quad \text{for } \alpha \in \Omega^a(S^{1|m}).$$

For $\alpha \in \Omega^a(S^{1|m})$ and $\beta \in \Omega^b(S^{1|m})$ we will consider the following choice for the sign rule:

$$\alpha \wedge \beta = (-1)^{ab + p(\alpha)p(\beta)} \beta \wedge \alpha.$$

2.1. The Lie superalgebra of contact vector fields on $S^{1|3}$. The standard contact structure on the supercircle $S^{1|3}$ with local coordinates (x, ξ_1, ξ_2, ξ_3) is defined by the contact 1-form

$$\alpha_3 = \mathrm{d}x + \xi_1\, \mathrm{d}\xi_1 + \xi_2\, \mathrm{d}\xi_2 + \xi_3\, \mathrm{d}\xi_3.$$

Let $\Sigma = \mathrm{Ker}(\alpha_3) \subset \mathrm{T}S^{1|3}$ be the maximall non-integrable distribution. The odd vector fields $D_i = \partial_{\xi_i} + \xi_i \partial_x$, $i = 1, 2, 3$ form a basis of Σ as a $C^\infty(S^{1|3})$-module. They satisfy $D_i^2 = [D_i, D_i]/2 = \partial_x$.

A contact vector field on $S^{1|3}$ is a vector field $X \in \mathrm{Vect}(S^{1|3})$ such that $\mathcal{L}_X\alpha_3 = F_X\alpha_3$ for some function $F_X \in C^\infty(S^{1|3})$, where $\mathcal{L}_X\alpha_3$ is the Lie derivative of α_3 in the direction of X. We denote by $\mathbb{K}(1|3)$ the Lie superalgebra of all contact vector fields.

We have

$$\langle D_i, \mathcal{L}_X\alpha_3\rangle = \langle [D_i, X], \alpha_3\rangle$$

where the Lie derivative satisfies:

$$Y\langle X, \alpha_3\rangle = \langle \mathcal{L}_Y X, \alpha_3\rangle + \langle X, \mathcal{L}_Y\alpha_3\rangle.$$

These relations give an alternative definition of $\mathbb{K}(1|3)$:

$$\mathbb{K}(1|3) = \{X \in \mathrm{Vect}(S^{1|3})\colon\ [X, D_i] \in \ker(\alpha_3) \ \text{ for } i = 1, 2, 3\}.$$

One can check that for every $X \in \mathbb{K}(1|3)$ there exists a unique function $f \in \mathcal{C}^\infty(S^{1|3})$ called the *contact Hamiltonian* such that $X = X_f$, where

$$X_f = f\partial_x + \frac{(-1)^{p(f)}}{2}\sum_{i=1}^{3} D_i(f)D_i.$$

The multiplicative function is given by $F_{X_f} = -f'/2$.

The Lie bracket of two contact vector fields X_f and X_g is given by $[X_f, X_g] = X_{\{f,g\}}$ where

$$\{f, g\} = fg' - f'g + \frac{(-1)^{p(f)}}{2}\sum_{i=1}^{3} D_i(f)D_i(g).$$

2.2. The group of contactomorphisms on $S^{1|3}$. As $\mathcal{C}^\infty(S^{1|3})$-modules, the family $(\partial_x, D_1, D_2, D_3)$ forms another local basis of the Lie superalgebra of vector fields $\mathrm{Vect}(S^{1|3})$ and the family $(\alpha_3, \mathrm{d}\xi_1, \mathrm{d}\xi_2, \mathrm{d}\xi_3)$ forms another local basis of $\Omega^1(S^{1|3})$. Moreover, we have:

$$\langle \partial_x, \alpha_3\rangle = \langle D_i, \mathrm{d}\xi_i\rangle = 1 \quad \text{and} \quad \langle \partial_x, \mathrm{d}\xi_i\rangle = \langle D_i, \alpha_3\rangle = \langle D_i, \mathrm{d}\xi_j\rangle = 0 \quad \text{where } i \neq j.$$

The local basis $(\partial_x, D_1, D_2, D_3)$ transforms, under the diffeomorphism

$$\Phi(x, \xi) = \begin{cases} y = \varphi(x, \xi), \\ \psi = (\psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi)) \end{cases}$$

of $S^{1|3}$, as follows:

$$
(2.2)\qquad \begin{pmatrix} \partial_x \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} \varphi' + \psi \cdot \psi' & \psi_1' \ \psi_2' \ \psi_3' \\ D_1\varphi - \psi \cdot D_1\psi & D_1\psi_1 \ D_1\psi_2 \ D_1\psi_3 \\ D_2\varphi - \psi \cdot D_2\psi & D_2\psi_1 \ D_2\psi_2 \ D_2\psi_3 \\ D_3\varphi - \psi \cdot D_3\psi & D_3\psi_1 \ D_3\psi_2 \ D_3\psi_3 \end{pmatrix} \begin{pmatrix} \partial_y \\ \widetilde{D}_1 \\ \widetilde{D}_2 \\ \widetilde{D}_3 \end{pmatrix}
$$

and the local basis $(\widetilde{\alpha_3}, \mathrm{d}\psi_1, \mathrm{d}\psi_2, \mathrm{d}\psi_3)$ transforms as

$$
(2.3)\qquad (\widetilde{\alpha_3}, \mathrm{d}\psi_1, \mathrm{d}\psi_2, \psi_3) = (\alpha_3, \mathrm{d}\xi_1, \mathrm{d}\xi_2, \mathrm{d}\xi_3) \begin{pmatrix} \varphi' + \psi \cdot \psi' & \psi_1' \ \psi_2' \ \psi_3' \\ D_1\varphi - \psi \cdot D_1\psi & D_1\psi_1 \ D_1\psi_2 \ D_1\psi_3 \\ D_2\varphi - \psi \cdot D_2\psi & D_2\psi_1 \ D_2\psi_2 \ D_2\psi_3 \\ D_3\varphi - \psi \cdot D_3\psi & D_3\psi_1 \ D_3\psi_2 \ D_3\psi_3 \end{pmatrix}
$$

where $\widetilde{D}_i = \partial_{\psi_i} + \psi_i \partial_y$, $i = 1, 2, 3$ and $\widetilde{\alpha_3} = \mathrm{d}y + \sum_{i=1}^{3} \psi_i \, \mathrm{d}\psi_i$.
Moreover, if $\psi = (\psi_1, \psi_2, \psi_3)$ and $\theta = (\theta_1, \theta_2, \theta_3)$, we have

$$
(2.4)\qquad \psi \cdot \theta = \psi_1\theta_1 + \psi_2\theta 2 + \psi_3\theta_3.
$$

According to the transformation rule (2.2), we have:

$$
\partial_x = (\varphi' + \psi \cdot \psi')\partial_y + \sum_{j=1}^{3} \psi_j' \widetilde{D}_j,
$$

while from (2.3) we can derive the relation

$$
\mathrm{d}\psi_i = \alpha_3 \psi_i' + \sum_{j=1}^{3} \mathrm{d}\xi_j D_j \psi_i \quad \text{for } i = 1, 2, 3.
$$

A diffeomorphism Φ satisfying $\Phi^*\alpha_3 = E_\Phi \alpha_3$ for some positive function E_Φ on $S^{1|3}$ is called a *contactomorphism* on $S^{1|3}$. Using the transformation (2.3), we deduce that Φ is a contactomorphism if and only if

$$
(2.5)\qquad D_i\varphi - \psi \cdot D_i\psi = 0 \quad \text{for } i = 1, 2, 3
$$

where $D_i\psi = (D_i\psi_1, D_i\psi_2, D_i\psi_3)$ and "$\cdot$" is as in (2.4). Therefore we have

$$
(2.6)\qquad E_\Phi = \varphi' + \psi \cdot \psi' = D_i\psi \cdot D_i\psi \quad \text{for all } i.
$$

The set of all contactomorphisms is a subgroup of $\mathrm{Diff}(S^{1|3})$ and we shall denote it by $\mathcal{K}(1|3)$.

If Φ is a contactomorphism, we have from transformation (2.2) the relations

$$D_i = \sum_{j=1}^{3} D_i\psi_j \widetilde{D}_j \quad \text{for } i = 1, 2, 3,$$

as well as the following properties (see [20]):

1) $D_i\psi \cdot D_j\psi = E_\Phi \delta_{ij}$.

2) $\mathrm{Ber}(y, \psi) = E_\Phi^{-1/2}\mathrm{Ber}(x, \xi)$ where $\mathrm{Ber}(x, \xi)$ is the volume form on $S^{1|3}$ and the coefficient $E_\Phi^{-1/2}$ is the Berizinian of the matrix of equation (2.3). Recall that the Berizinian of a matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is

$$\mathrm{Ber}(M) = \det(A - BD^{-1}C)\det(D^{-1}).$$

Let us write $\Phi = (\varphi(x, \xi), \psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi)) \in \mathcal{K}(3)$ where

$$\varphi(x, \xi) = \varphi_0(x) + \sum_{i=1}^{3} \xi_i \varphi_i(x) + \dots$$

and

$$\psi_j(x, \xi) = \psi_{j0}(x) + \sum_{i=1}^{3} \xi_i \psi_{ji}(x) + \dots, \quad j = 1, 2, 3.$$

Using equations (2.6) and the diagram (2.1), we deduce that the map Π transforms a $\Phi \in \mathcal{K}(3)$ to an orientation-preserving diffeomorphism $\Pi(\Phi) \in \mathrm{Diff}_+(S^1)$.

2.3. The conformal embedding in $\mathcal{K}(1|m)$ and $\mathbb{K}(1|m)$. The consideration of conformal embedding, as well as projective embedding in the classical case, is natural and indispensable for various points of view. For instance, one can cite the conformally equivariant symbol calculus (see [13], [25]) and the relative cohomology (see [14], [3], [11] among others). In this section, we consider the question of the relative cohomology.

Let $V^t = (p, q, \xi_1, \dots, \xi_m) \in \mathbb{R}^{2|m}$. A general linear transformation is given by a matrix

$$M = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & e \end{pmatrix} \tag{2.7}$$

where the entries a, b, c, d are even elements, α, β form an odd row vector of size m, γ, δ form an odd column vector of size m, and e is an even matrix of size $m \times m$.

The matrix M acts by the conformal projective transformation on $\mathbb{R}^{1|m}$

$$\Phi_M(x,\xi) = \Big(\frac{ax+b+\alpha\xi}{cx+d+\beta\xi}, \frac{\gamma x+\delta+e\xi}{cx+d+\beta\xi}\Big), \tag{2.8}$$

where $x = q/p$, $\xi_i = \theta_i/p$; $p \neq 0$ are the affine coordinates in a chart U of $S^{1|m}$. The above diffeomorphism preserves the contact form α if and only if

$$\begin{aligned} ad - bc - \gamma^t\delta &= 1, \\ e^t e + 2\alpha^t\beta &= 1, \\ \gamma^t e - a\beta + c\alpha &= 0, \\ \delta^t e - b\beta + d\alpha &= 0, \end{aligned} \tag{2.9}$$

where the notation x^t denotes the transposition of the row or column vector x.

Let us consider the 1-form $\omega = (pdq - qdp + \theta_i d\theta_i)/2$ on $\mathbb{R}^{2|m}$. The orthosymplectic group $\mathrm{OSp}(2,m)$ is the sub-supergroup of $\mathrm{GL}(2,m)$ preserving the symplectic form $d\omega$ (see [16]). The system of equations (2.9) implies that $M \in \mathrm{OSp}(2,m)$.

The assignment $M \mapsto \Phi_M$ is a morphism from $\mathrm{OSp}(2,m)$ to $\mathcal{K}(m)$ whose image we denote by $\mathrm{PC}(2,m)$; it is known as the subgroup of conformal projective transformations of $\mathcal{K}(1|m)$.

The associated Lie sub-superalgebra of contact vector fields consists of X_f, $f \in \mathbb{R}[x,\xi]$ and $\deg(f) \leqslant 2$ and is isomorphic to the orthosymplectic Lie algebra $\mathfrak{osp}(2,m)$.

2.4. $\mathcal{K}(1|m)$-module structures on differential operators.

2.4.1. The module of tensor densities. Let $\mathcal{F}_\lambda(S^{1|m})$ be the space of λ-*densities* (or the space of tensor densities of degree λ) associated with the contact structure on $S^{1|m}$. By definition, we have

$$\mathcal{F}_\lambda(S^{1|m}) = \{f(x,\xi)\,\alpha_m^\lambda\colon\ f(x,\xi) \in C^\infty(S^{1|m})\},$$

where $\lambda \in \mathbb{C}$. The action of $\mathcal{K}(m)$ on the space $\mathcal{F}_\lambda(S^{1|m})$ is given by

$$\Phi_\lambda f\alpha_m^\lambda = ((E_{\Phi^{-1}})^\lambda f \circ \Phi^{-1})\alpha_m^\lambda, \quad f \in C^\infty(S^{1|m}). \tag{2.10}$$

We can deduce from (2.2) that the Lie superalgebra of contact vector fields $\mathbb{K}(1|m)$ is isomorphic to $\mathcal{F}_{-1}(S^{1|m})$ for the adjoint action. We have also the isomorphisms $\mathcal{F}_\lambda(S^{1|m})^* \cong \mathcal{F}_{(2-m)/2-\lambda}(S^{1|m})$, in particular $\mathbb{K}(1|m)^* \cong \mathcal{F}_{(4-m)/2}(S^{1|m})$.

The action of $\mathbb{K}(1|m)$ on the space $\mathcal{F}_\lambda(S^{1|m})$ is given by

$$L^\lambda_{X_f}(g(x,\xi)\,\alpha_m^\lambda) = ((X_f(g) + \lambda\, f'\, g)(x,\xi))\alpha_m^\lambda, \tag{2.11}$$

which is nothing but the Lie derivative along the vector field X_f.

2.4.2. Module of differential operators on tensor densities. Denote $|\boldsymbol{i}| = i_1 + \ldots + i_m$, the differential operator A is of order at most $k \in \mathbb{N}$ if $\boldsymbol{i} \leqslant k$. The operator A on $S^{1|m}$ can be expressed in the form

$$A = \sum_{\boldsymbol{i} \geqslant 0} a_{\boldsymbol{i}}(x,\xi) D^{\boldsymbol{i}}$$

where $\boldsymbol{i} = (i_1, \ldots, i_m)$ is a multi-index and $D^{\boldsymbol{i}} = D_1^{i_1} \ldots D_m^{i_m}$. A differential operator A acts from the tensor densities $\mathcal{F}_\lambda(S^{1|m})$ of degree λ to $\mathcal{F}_\mu(S^{1|m})$ as

$$A(f(x,\xi)\alpha_m^\lambda) = A(f(x,\xi))\alpha_m^\mu.$$

We shall denote by $\mathcal{D}_{\lambda,\mu}(S^{1|m})$ the space of differential operators $A\colon \mathcal{F}_\lambda(S^{1|m}) \to \mathcal{F}_\mu(S^{1|m})$. We denote by $\mathcal{D}^k_{\lambda,\mu}(S^{1|m})$ the space of linear differential operators of order k. In particular $\mathcal{D}^0_{\lambda,\mu}(S^{1|m}) \cong \mathcal{F}_{\mu-\lambda}(S^{1|m})$ and $\mathcal{D}^1_{\lambda,\lambda}(S^{1|m}) \cong \mathrm{Vect}(S^{1|m}) \oplus C^\infty(S^{1|m})$. One has a filtration

$$\mathcal{D}^0_{\lambda,\mu}(S^{1|m}) \subset \mathcal{D}^1_{\lambda,\mu}(S^{1|m}) \subset \ldots \subset \mathcal{D}^k_{\lambda,\mu}(S^{1|m}) \subset \ldots$$

We also consider the *finer filtration* by the spaces $\mathcal{D}^{p/2}_{\lambda,\mu}(S^{1|m})$ where p is an integer consisting of operators $\sum\limits_{|\boldsymbol{i}|=0}^{|\boldsymbol{i}|=p} a_{\boldsymbol{i}}(x,\xi) D^{\boldsymbol{i}}$ where $|\boldsymbol{i}| = \sum\limits_{s=1}^{m} i_s$.

Definition 2.1. We define a two parameter family of actions of $\mathcal{K}(1|m)$ on $\mathcal{D}_{\lambda,\mu}(S^{1|m})$ by

$$\Phi_{\lambda,\mu}(A) = \Phi_\mu \circ A \circ \Phi_\lambda^{-1}. \tag{2.12}$$

The action of the contact Lie superalgebra $\mathbb{K}(1|m)$ on the space $\mathcal{D}_{\lambda,\mu}(S^{1|m})$ is given by

$$\mathcal{L}^{\lambda,\mu}_f(A) = L^\mu_{X_f} \circ A - (-1)^{p(f)p(A)} A \circ L^\lambda_{X_f}. \tag{2.13}$$

The space of symbols of $\mathcal{D}_{\lambda,\mu}(S^{1|m})$ is defined by

$$\mathcal{SB}^\bullet_\delta(S^{1|m}) := \bigoplus_{s \in 1/2\mathbb{N}} \frac{\mathcal{D}^s_{\lambda,\mu}(S^{1|m})}{\mathcal{D}^{s-1/2}_{\lambda,\mu}(S^{1|m})}$$

where we let $\mathcal{D}^{-1/2}_{\lambda,\mu}(S^{1|m}) = \{0\}$ and $\delta = \mu - \lambda$. For $m = 1$, we have the isomorphism of $\mathcal{K}(1|1)$-modules

$$\mathcal{SB}^s_\delta(S^{1|1}) := \frac{\mathcal{D}^s_{\lambda,\mu}(S^{1|1})}{\mathcal{D}^{s-1/2}_{\lambda,\mu}(S^{1|1})} \simeq \mathcal{F}_{\delta-s}(S^{1|1}).$$

3. Schwarzian derivatives on supercircles as 1-cocycles

The first cohomology space $\mathrm{H}^1(\mathbb{K}(1|m), \mathbb{K}(1|m)^*)$ is nontrivial if and only if $m = 0, 1, 2, 3$. It is one dimensional and is generated by the 1-cocycle $s\colon \mathbb{K}(m) \to \mathbb{K}(m)^*$ equal to, respectively (see [2], Proposition 5.4):

$$\begin{array}{ccccc} & m=0 & m=1 & m=2 & m=3 \\ s(X_f) = & f'''\,\mathrm{d}x^2 & D(f'')\alpha_1^{3/2} & D_1D_2(f')\alpha_2^1 & D_1D_2D_3(f)\alpha_3^{1/2}. \end{array} \tag{3.1}$$

These 1-cocycles are $\mathfrak{osp}(2, m)$ (or conformally)-invariant.

The Schwarzian derivative on $S^{1|m}$ is the map

$$S_m\colon\ \mathcal{K}(1|m) \to \mathbb{K}(1|m)^*$$

having the following characteristic properties:

1) S is invariant under the transformation $\Phi_M \in \mathrm{PC}(2, m)$ given by (2.8).
2) S satisfies the cocycle condition: for every $\Phi, \Psi \in \mathcal{K}(1|m)$

$$S_m(\Psi \circ \Phi) = E_\Phi^{(4-m)/2} S_m(\Psi) \circ \Phi + S_m(\Phi).$$

The super-Schwarzian derivatives (see [20], [17]) are given for $\Phi \in \mathcal{K}(m)$, $m = 0, 1, 2, 3$ by

$$\begin{cases} S_0(\Phi) = \Big(\dfrac{\Phi'''}{\Phi'} - \dfrac{3}{2}\Big(\dfrac{\Phi''}{\Phi'}\Big)^2\Big)\mathrm{d}x^2 & \text{for } m=0, \\ S_1(\Phi) = \Big(\dfrac{D^3E_\Phi}{E_\Phi} - \dfrac{3}{2}\dfrac{D^2E_\Phi DE_\Phi}{(E_\Phi)^2}\Big)\alpha_1^{3/2} & \text{for } m=1, \\ S_2(\Phi) = \Big(\dfrac{D_2D_1E_\Phi}{E_\Phi} - \dfrac{3}{2}\dfrac{D_2E_\Phi D_1E_\Phi}{(E_\Phi)^2}\Big)\alpha_2^1 & \text{for } m=2, \\ S_3(\Phi) = E_\Phi^{-1}(D_1(D_2), D_3)\alpha_3^{1/2} & \text{for } m=3. \end{cases} \tag{3.2}$$

Remark 3.1 ([20]). For a vector fields v_i of the form

$$v_i = \sum_{p=1}^n a_{ip}\widetilde{D}_p$$

we have that $v_i(v_j) = \sum\limits_{k=1}^{n} v_i(a_{jk})\widetilde{D}_k$ is the coefficients-wise application of v_i to v_j. The superskewsymmetric scalar product $(.,.)$ is determined by the formula

$$(v_i, v_j) = \left(\sum_{p=1}^{n} a_{ip}\widetilde{D}_p, \sum_{k=1}^{n} b_{jk}\widetilde{D}_k\right) = \sum_{k=1}^{n}(-1)^{\widetilde{b}_{ik}} a_{ik}b_{ik}.$$

Using (2.2) we obtain the explicit expression of S_3:

$$S_3(\varphi) = E_{\varphi}^{-1}(D_1D_2\psi_1D_3\psi_1 + D_1D_2\psi_2D_3\psi_2 + D_1D_2\psi_3D_3\psi_3)\alpha_3^{1/2}.$$

Since the diffeomorphisms in $\mathrm{Diff}(S^{1|m})$ preserve parity, we can define the flow of $X \in \mathrm{Vect}(S^{1|m})$ as $\Phi_\varepsilon = \mathrm{Id} + \varepsilon X + O(\varepsilon^2)$, where $p(\varepsilon X) = 0$.

The infinitesimal versions of the Schwarzian derivatives are easy to compute we substitute Φ_ε in the expressions (3.2) of $S(\Phi)$ for $m = 0, 1, 2$ and 3 and differentiate with respect to ε at $\varepsilon = 0$; we get the conformal invariant 1-cocycle $s\colon \mathbb{K}(1|m) \to \mathbb{K}(1|m)^*$ given by (3.1).

4. The first cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|1), \mathrm{PC}(2,1); \mathcal{D}_{\lambda,\mu}(S^{1|1}))$

4.1. Differentiable cohomology. Let G be a Lie or Fréchet-Lie group (for example the Fréchet-Lie group of diffeomorphisms on a manifold M). Let V be a G-module and φ a representation of G on V. A 1-cocycle of G with coefficients in V is a smooth map $C\colon\ G \to V$, given by

$$C(gh) = \varphi(g)C(h) + C(g), \quad \text{where } g, h \in G.$$

Each vector $a \in V$ determines a 1-cocycle given by

$$B_a(g) = \varphi(g)a - a, \quad g \in G.$$

Such a 1-cocycle is called a 1-coboundary. Denote the sets of 1-cocycles and 1-coboundaries respectively by $Z^1(G, V)$ and $B^1(G, V)$. Then the first cohomology space, denoted by $H^1(G, V)$, is defined as the quotient space:

$$H^1(G, V) = \frac{Z^1(G, V)}{B^1(G, V)}.$$

Similarly, a 1 cocycle for a Lie algebra $\mathfrak{g}$ is a linear map $F\colon\ \mathfrak{g} \to V$, given by the equation

$$c([x, y]) = \varrho(x)c(y) - \varrho(y)c(x), \quad x, y \in \mathfrak{g}.$$

Also, each vector $a \in V$ determines a 1-cocycle b_a given by the equation

$$b_a(x) = \varrho(x)a, \quad x \in \mathfrak{g}.$$

Such a 1-cocycle is called a 1-coboundary. The first cohomology space, denoted by $H^1(\mathfrak{g}, V)$, is the quotient group

$$H^1(\mathfrak{g}, V) = \frac{Z^1(\mathfrak{g}, V)}{B^1(\mathfrak{g}, V)}$$

where $Z^1(\mathfrak{g}, V)$ and $B^1(\mathfrak{g}, V)$ are the sets of 1-cocycles and 1-coboundaries, respectively.

Suppose that G is a connected group. In the case of the group of contactomorphisms $\mathcal{K}(1|m)$, we can define the flow of $X \in \mathbb{K}(1|m)$, namely $\Phi_t = \mathrm{Id} + tX + O(t^2)$ as $\exp(tX)$, only if $p(tX) = 0$. For odd vector fields X, the parameter t must therefore be odd.

Let $C\colon\ G \to V$ be a differentiable 1-cocycle and set

$$(4.1) \quad c(X) = \frac{\mathrm{d}}{\mathrm{d}t} C(\exp(tX))_{|t=0}, \quad X \in \mathfrak{g} \text{ and } \exp\colon\ \mathfrak{g} \to G \text{ is the exponential map.}$$

We have

$$\begin{aligned} c([X, Y]) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} C(\exp(tX)\exp(sY)\exp(-tX))_{|s=0,t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\{(\exp(tX)\exp(sY))C(\exp(-tX)) \\ &\quad + \exp(tX)C(\exp(sY)) + C(\exp(tX))\}_{|s=0,t=0} \\ &= -\,\varrho(Y)c(X) + \varrho(X)c(Y). \end{aligned}$$

Then c is a 1-cocycle on $\mathfrak{g}$ (see [19] page 70 and Section 8.4). So the map α: $Z^1(G, V) \to Z^1(\mathfrak{g}, V)$ defined by (4.1) is injective.

If $B \in B^1(G, V)$, then it has the functional form, $B(g) = \varphi(g)a - a$ for all $g \in G$, for some $a \in V$. The differential of B_a defined by $b(X) = (\mathrm{d}/\mathrm{d}t)B_a(\exp(tX))_{|t=0} = \varrho(X)a$ is a 1 coboundary of $\mathfrak{g}$.

Therefore we have the induced map

$$\alpha_\sharp\colon\ H^1(G, V) \to H^1(\mathfrak{g}, V).$$

Let us prove that if G is connected, $\alpha_\sharp$ is injective. Let $C \in Z^1(G, V)$ be such that $\alpha(C) \in B^1(\mathfrak{g}, V)$, then there exists $a \in V$ such that

$$\alpha(C)(X) = \varrho(X)a, \quad X \in \mathfrak{g}.$$

Now, define the function $C_a\colon\ G \to V$, by the relation

$$C_a(g) = C(g) - \varphi(g)a + a.$$

Hence $C_a \in Z^1(G,V)$.

Now $\alpha(C_a)(X) = \alpha(C)(X) - \varrho(X)a = 0$ for all $X \in \mathfrak{g}$. Since G is connected, C_a is identically zero and then $C(g) = \varphi(g)a - a$, i.e. $C \in B^1(G,V)$, and so $\alpha_\sharp$ is injective.

Le H be a Lie subgroup of G. We suppose that H and G are connected. Let $\mathfrak{h}$ and $\mathfrak{g}$ be their Lie algebras. If we denote $\alpha\colon\ Z^1(G,H,V) \to Z^1(\mathfrak{g},\mathfrak{h},V)$ then the induced cohomological map $\alpha_\sharp\colon\ H^1(G,H,V) \to H^1(\mathfrak{g},\mathfrak{h},V)$ will be injective.

4.2. The case of $S^{1|1}$. We consider the first relative differentiable cohomology space of $\mathcal{K}(1|1)$ under $\mathrm{PC}(2,1)$ with coefficients in the space $\mathcal{D}_{\lambda,\mu}(S^{1|1})$ of differential operators acting from $\mathcal{F}_\lambda(S^{1|1})$ to $\mathcal{F}_\mu(S^{1|1})$. In other words, we will consider only the cohomology class of local 1-cocycles vanishing on $\mathrm{PC}(2,1)$. We will denote this space by $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|1),\mathrm{PC}(2,1),\mathcal{D}_{\lambda,\mu}(S^{1|1}))$. A 1-cocycle $C\colon\ \mathcal{K}(1|1) \to \mathcal{D}_{\lambda,\mu}(S^{1|1})$ satisfies the conditions

$$\begin{aligned} C(\Phi\circ\Psi) &= \Phi_{\lambda,\mu} S(\Psi) + C(\Phi),\\ C(\Phi) &= 0, \quad \Phi \in \mathrm{PC}(2,m). \end{aligned}$$

This space has the following structure:

Theorem 4.1.

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|1),\mathrm{PC}(2,1);\mathcal{D}_{\lambda,\mu}(S^{1|1})) = \begin{cases} \mathbb{R} & \textit{if } \mu-\lambda = \frac{3}{2} \textit{ and } \lambda \neq -\frac{1}{2},\\ \mathbb{R} & \textit{if } \mu-\lambda = 2 \textit{ for all } \lambda,\\ \mathbb{R} & \textit{if } \mu-\lambda = \frac{5}{2} \textit{ and } \lambda \neq -1,\\ \mathbb{R} & \textit{if } \mu-\lambda = 3 \ \textit{ and } \lambda \in \{0,\ -\frac{5}{2}\},\\ \mathbb{R} & \textit{if } \mu-\lambda = 4 \ \textit{ and } \lambda = \frac{1}{3}(-7 \pm \sqrt{33}),\\ 0 & \textit{otherwise.} \end{cases}$$

The corresponding generating 1*-cocycles* $\mathcal{S}_{\lambda,\mu}\colon\ \mathcal{K}(1|1) \to \mathcal{D}_{\lambda,\mu}(S^{1|1})$ *are respectively given by*

$$\begin{aligned} &\mathcal{S}_{\lambda,\lambda+3/2}(\Phi^{-1}) = S_1(\Phi), \\ &\mathcal{S}_{\lambda,\lambda+5/2}(\Phi^{-1}) = 3S_1(\Phi)D^2 + D(S_1(\Phi))D - 2\lambda D^2(S_1(\Phi)),\\ &\mathcal{S}_{\lambda,\lambda+2}(\Phi^{-1}) = S_1(\Phi)D + \frac{2}{3}\lambda D(S_1(\Phi)), \end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathcal{S}_{\lambda,\lambda+3}(\Phi^{-1}) &= S_1(\Phi)D^3 + \frac{2\lambda+1}{3}(D(S_1(\Phi))D^2 - D^2(S_1(\Phi))D)\\ &\quad - \frac{\lambda(2\lambda+1)}{6}D^3(S_1(\Phi)),\\ \mathcal{S}_{\lambda,\lambda+4}(\Phi^{-1}) &= S_1(\Phi)D^5 + \frac{2(\lambda+1)}{3}(D(S_1(\Phi))D^4 - 2D^2(S_1(\Phi))D^3)\\ &\quad - \frac{(\lambda+1)(2\lambda+1)}{3}D^3(S_1(\Phi))D^2\\ &\quad + \frac{2\lambda+1}{12}(2(\lambda+1)D^4(S_1(\Phi)) - 3(\lambda+3)S_1(\Phi)D(S_1(\Phi)))D\\ &\quad + \frac{\lambda(\lambda+1)(2\lambda+1)}{15}D^5(S_1(\Phi))\\ &\quad + \frac{\lambda(\lambda+3)(2\lambda-3)}{10}S_1(\Phi)D^2(S_1(\Phi))\\ &\quad - \frac{2\lambda(\lambda+1)(\lambda+3)}{15}D(S_1(\Phi))^2.\end{aligned}$$

The first group of differentiable cohomology $\mathrm{H}^1_{\mathrm{diff}}(\mathbb{K}(1|1), \mathfrak{osp}(2,1); \mathcal{D}_{\lambda,\mu}(S^{1|1}))$ (we mean the cohomology with cocycles given by multi-differential operators) of the Lie algebra of contact vector fields $\mathbb{K}(1|1)$ on $S^{1|1}$, vanishing on the projective embedding of $\mathfrak{osp}(2,1)$ in $\mathbb{K}(1|1)$ and with coefficients in $\mathcal{D}_{\lambda,\mu}(S^{1|1})$, has been calculated recently by Conley [11] and Ben Ammar et al. [3]. Their announced result is the following theorem:

Theorem 4.2 ([3], [11]).

$$\mathrm{H}^1_{\mathrm{diff}}(\mathbb{K}(1|1), \mathfrak{osp}(2,1); \mathcal{D}_{\lambda,\mu}(S^{1|1})) = \begin{cases} \mathbb{R} & \text{if } \mu-\lambda = \frac{3}{2} \text{ and } \lambda \neq -\frac{1}{2},\\ \mathbb{R} & \text{if } \mu-\lambda = 2 \text{ for all } \lambda,\\ \mathbb{R} & \text{if } \mu-\lambda = \frac{5}{2} \text{ and } \lambda \neq -1,\\ \mathbb{R} & \text{if } \mu-\lambda = 3 \text{ and } \lambda \in \left\{0, -\frac{5}{2}\right\},\\ \mathbb{R} & \text{if } \mu-\lambda = 4 \text{ and } \lambda = \frac{1}{4}(-7 \pm \sqrt{33}),\\ 0 & \text{otherwise.}\end{cases}$$

The corresponding generating 1*-cocycles* $s_{\lambda,\mu}\colon \mathbb{K}(1|1) \to \mathcal{D}_{\lambda,\mu}(S^{1|1})$ *are respectively given by*

$$\begin{aligned} &(4.3)\quad s_{\lambda,\lambda+3/2}(X_f) = s(X_f) \quad \text{for } \lambda \neq -\frac{1}{2},\\ &s_{\lambda,\lambda+5/2}(X_f) = 3s(X_f)D^2 + (-1)^{p(f)}D(s(X_f))D - 2\lambda\frac{\partial}{\partial x}(s(X_f)) \quad \text{for } \lambda \neq -1,\\ &s_{\lambda,\lambda+2}(X_f) = (-1)^{p(f)}s(X_f)D - \frac{2}{3}\lambda D(s(X_f)) \quad \text{for all } \lambda,\end{aligned}$$

$$
\begin{aligned}
s_{\lambda,\lambda+3}(X_f) = {}& (-1)^{p(f)}s(X_f)D^3\\
&- \frac{2\lambda+1}{3}\Big(D(s(X_f))D^2 + (-1)^{p(f)}\frac{\partial}{\partial x}(s(X_f))D\Big)\\
&+ \frac{\lambda(2\lambda+1)}{6}D^{(3)}(s(X_f)) \quad \text{for } \lambda = 0, -\frac{5}{2},\\
s_{\lambda,\lambda+4}(X_f) = {}& (-1)^{p(f)}(s(X_f))D^5\\
&- \frac{2(\lambda+1)}{3}\Big(D(s(X_f))D^4 + 2(-1)^{p(f)}\frac{\partial}{\partial x}(s(X_f))D^3\Big)\\
&+ \frac{(\lambda+1)(2\lambda+1)}{6}\Big(2D^3(s(X_f))D^2 + (-1)^{p(f)}\frac{\partial}{\partial x}^2(s(X_f))D\Big)\\
&- \frac{\lambda(\lambda+1)(2\lambda+1)}{15}D\Big(\frac{\partial}{\partial x}^2(s(X_f))\Big) \quad \text{for } \lambda = \frac{-7\pm\sqrt{33}}{4}.
\end{aligned}
$$

5. The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|2), \mathrm{PC}(2,2), \mathcal{D}_{\lambda,\mu}(S^{1|2}))$

Theorem 5.1. *The first cohomology space of $\mathcal{K}(1|2)$ with coefficients in $\mathcal{D}_{\lambda,\mu}(S^{1|2})$ has ben calculated by Agrebaoui, Dammak and Mansour (see [1]). This space has the following structure:*

$$
\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|2), \mathrm{PC}(2,2); \mathcal{D}_{\lambda,\mu}(S^{1|2})g) = \begin{cases} \mathbb{R} & \text{if } \mu-\lambda = 1 \text{ and } \lambda \neq -\frac{1}{2},\\ \mathbb{R}^2 & \text{if } \mu-\lambda = 2 \text{ and } \lambda \neq -1,\\ \mathbb{R} & \text{if } \mu-\lambda = 2 \text{ for } \lambda = -1,\\ 0 & \text{otherwise.} \end{cases}
$$

It is respectively generated by the following 1-cocycles $\mathcal{C}_{\lambda,\mu}\colon \mathcal{K}(1|2) \to \mathcal{D}_{\lambda,\mu}(S^{1|2})$:

$$
\begin{aligned}
\mathcal{C}_{\lambda,\lambda+1}(\Phi^{-1}) &= S_2(\Phi), \lambda \neq -\frac{1}{2},\\
\mathcal{C}_{\lambda,\lambda+2}(\Phi^{-1}) &= 2S_2(\Phi)\partial_x - D_2(S_2(\Phi))D_2 - D_1(S_2(\Phi))D_1 - 2\lambda\frac{\partial}{\partial x}(S_2(\Phi)),\ \lambda \neq -1,\\
\widetilde{\mathcal{C}}_{\lambda,\lambda+2}(\Phi^{-1}) &= 2S_2(\Phi)D_1D_2 + (2\lambda+1)(D_2(S_2(\Phi))D_1 - D_1(S_2(\Phi))D_2)\\
&\quad + \frac{2\lambda}{3}((2\lambda+1)D_1D_2(S_2(\Phi)) + (\lambda+2)S_2(\Phi)^2) \quad \text{for all } \lambda,
\end{aligned}
$$

where $S_2(\Phi)$ is the super-Schwarzian derivative on $S^{1|2}$ which is given by the expression

$$
S_2(\Phi) = \Big(\frac{D_2D_1E_\Phi}{E_\Phi} - \frac{3}{2}\frac{D_2E_\Phi D_1E_\Phi}{(E_\Phi)^2}\Big)\alpha_2^{1}
$$

and $D_i = \partial_{\xi_i} + \xi_i\,\partial_x$, $i = 1, 2$.

Consider the first relative cohomology on the contact Lie algebra $\mathbb{K}(1|2)$ with respect to the sub super-algebra $\mathfrak{osp}(2,2)$ and with coefficients in $\mathcal{D}_{\lambda,\mu}(S^{1|2})$. The following theorem is a consequence of a general result proved by Ben Fraj (see [4]).

Theorem 5.2.

$$\mathrm{H}^1_{\mathrm{diff}}(\mathbb{K}(1|2),\mathfrak{osp}(2,2);\mathcal{D}_{\lambda,\mu}(S^{1|2}))=\begin{cases}\mathbb{R} & \text{if } \mu-\lambda=1 \text{ and } \lambda\neq-\frac{1}{2},\\ \mathbb{R}^2 & \text{if } \mu-\lambda=2 \text{ and } \lambda\neq-1,\\ \mathbb{R} & \text{if } \mu-\lambda=2 \text{ and } \lambda=-1,\\ 0 & \text{otherwise.}\end{cases}$$

The corresponding generators are respectively given by the following 1*-cocycles* $c_{\lambda,\mu}:\mathbb{K}(1|2)\to\mathcal{D}_{\lambda,\mu}(S^{1|2})$:

$$\begin{aligned}c_{\lambda,\lambda+1}(X_f)&=s(X_f)\quad \text{for}\ \ \lambda\neq-\frac{1}{2},\\ c_{\lambda,\lambda+2}(X_f)&=2s(X_f)\partial_x-(-1)^{p(f)}(D_2(s(X_f))D_2\\ &\qquad+D_1(s(X_f))D_1)-2\lambda\frac{\partial}{\partial x}(s(X_f))\quad \text{for}\ \lambda\neq-1,\\ \widetilde{c}_{\lambda,\lambda+2}(X_f)&=2s(X_f)D_1D_2+(2\lambda+1)((-1)^{p(f)}(D_1(s(X_f))D_2\\ &\qquad-D_2(s(X_f))D_1)+\frac{2\lambda}{3}D_1D_2(s(X_f)))\quad \text{for all}\ \lambda.\end{aligned}\tag{5.1}$$

6. The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|3),\mathrm{PC}(2,3),\mathcal{D}_{\lambda,\mu}(S^{1|3}))$

In this section we will compute the first relative cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|3),\mathrm{PC}(2,3),\mathcal{D}_{\lambda,\mu}(S^{1|3}))$.

6.1. The case of $S^{1|3}$. The main theorem is

Theorem 6.1. *The first cohomology space of* $\mathcal{K}(1|3)$ *with coefficients in* $\mathcal{D}_{\lambda,\mu}(S^{1|3})$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1|3),\mathrm{PC}(2,3);\mathcal{D}_{\lambda,\mu}(S^{1|3}))=\begin{cases}\mathbb{R} & \text{if } \mu-\lambda=\frac{1}{2},\\ \mathbb{R} & \text{if } \mu-\lambda=\frac{3}{2} \text{ and } \lambda\neq-1,\\ 0 & \text{otherwise.}\end{cases}$$

It is respectively generated by the following two 1*-cocycles* $\mathcal{C}_{\lambda,\mu}\colon\ \mathcal{K}(1|3)\to\mathcal{D}_{\lambda,\mu}(S^{1|3})$:

$$
\begin{aligned}
(6.1)\qquad & \mathcal{C}_{\lambda,\lambda+1/2}(\Phi^{-1}) = S_3(\Phi),\\
& \mathcal{C}_{\lambda,\lambda+3/2}(\Phi^{-1}) = S_3(\Phi)\partial_x + D_3(S_3(\Phi))D_3 + D_2(S_3(\Phi))D_2\\
& \qquad\qquad + D_1(S_3(\Phi))D_1 - 2\lambda\frac{\partial}{\partial x}(S_3(\Phi)),\quad \lambda\neq -1,
\end{aligned}
$$

where $S_3(\Phi)$ *is the super-Schwarzian derivative on* $S^{1|3}$ *which is given by the expression*

$$S_3(\Phi) = E_\Phi^{-1}(D_1(D_2), D_3)\alpha_3^{1/2}$$

and $D_i = \partial_{\xi_i} + \xi_i\,\partial_x$, $i = 1,2,3$.

The proof of Theorem 6.1 involves the relative cohomology on the contact Lie algebra $\mathbb{K}(1|3)$ with respect to the sub super-algebra $\mathfrak{osp}(2,3)$ and with coefficients in $\mathcal{D}_{\lambda,\mu}(S^{1|3})$. The following theorem is a consequence of a general result proved by Ben Fraj, Laraiedh and Omri (see [5]).

Theorem 6.2.

$$
\mathrm{H}^1_{\mathrm{diff}}(\mathbb{K}(1|3),\mathfrak{osp}(2,3);\mathcal{D}_{\lambda,\mu}(S^{1|2})) = \begin{cases} \mathbb{R} & \text{if } \mu-\lambda=\frac{1}{2},\\ \mathbb{R} & \text{if } \mu-\lambda=\frac{3}{2} \text{ and } \lambda\neq -1,\\ 0 & \text{otherwise.}\end{cases}
$$

The corresponding generators are respectively given by the following 1*-cocycles* $c_{\lambda,\mu}\colon$ $\mathbb{K}(1|3)\to\mathcal{D}_{\lambda,\mu}(S^{1|3})$:

$$
\begin{aligned}
(6.2)\qquad & c_{\lambda,\lambda+1/2}(X_f) = s(X_f),\\
& c_{\lambda,\lambda+3/2}(X_f) = s(X_f)\partial_x - (-1)^{p(f)}(D_3(s(X_f))D_3 + D_2(s(X_f))D_2\\
& \qquad\qquad + D_1(s(X_f))D_1) - 2\lambda\frac{\partial}{\partial x}(s(X_f)) \text{ for } \lambda\neq -1.
\end{aligned}
$$

It is well known (see [12]) that the dimension of the first cohomology space of a Lie group is bounded by the dimension of the first cohomology space of its Lie algebra. If we compare Theorem 6.1 and Theorem 6.2, then we are left with the task of proving that the cocycles in Theorem 6.1 are nontrivial.

Consider a 1-cocycle of $\mathcal{K}(1|3)$ with values in the space of differential operators

$$C\colon\ \mathcal{K}(1|3)\to\mathcal{D}_{\lambda,\mu}(S^{1|3}).$$

The cocycle relation reads

$$C(\Phi \circ \Psi) = \Phi_\mu \circ C(\Psi) \circ \Phi_\lambda^{-1} + C(\Phi). \tag{6.3}$$

The first summand on the right hand side of (6.3) is the $\mathcal{K}(1|3)$-action on $\mathcal{D}_{\lambda,\mu}(S^{1|3})$.

Let us check first that the maps (6.1) satisfy the 1-cocycle condition (6.3).

6.2. Proof of existence. The first map $C_{\lambda,\lambda+1/2}\colon \mathcal{K}(1|3) \to \mathcal{D}_{\lambda,\lambda+1/2}(S^{1|3})$ of Theorem 6.1 is the zero order operator of multiplication by the Schwarzian derivative, namely, for $\Phi \in \mathcal{K}(1|3)$, set

$$C_{\lambda,\lambda+1/2}(\Phi^{-1})\colon\ F\alpha_3^\lambda \mapsto S(\Phi)F\alpha_3^{\lambda+1/2}.$$

The Schwarzian derivative

$$\Phi^{-1} \mapsto S_3(\Phi) = E_\Phi^{-}1(D_1(D_2), D_3)\alpha_3^{\lambda+1/2}$$

is a 1-cocycle, which implies the 1-cocycle relation (6.3) for the map $C_{\lambda,\lambda+1/2}$.

The cocycle condition (6.3) for the map $\mathcal{S}_{\lambda,\lambda+3/2}$ of Theorem 6.1 reads

$$\mathcal{C}_{\lambda,\lambda+3/2}((\Phi_1 \circ \Phi)^{-1}) \circ \Phi_\lambda^{-1} = \Phi_{\lambda+3/2}^{-1} \circ \mathcal{C}_{\lambda,\lambda+3/2}(\Phi_1^{-1}) + \mathcal{C}_{\lambda,\lambda+3/2}(\Phi^{-1}) \circ \Phi_\lambda^{-1}. \tag{6.4}$$

To check the equality (6.4), we replace the identities in the second summand on the right hand side:

$$\begin{aligned}
\Phi_\lambda^{-1} \cdot F\alpha^\lambda &= (E_\Phi^\lambda F \circ \Phi)\alpha^\lambda,\\
D_i(\Phi_\lambda^{-1} \cdot F\alpha^\lambda) &= \lambda E_\Phi^{\lambda-1} D_i(E_\Phi) F \circ \Phi\\
&\quad + E_\Phi^\lambda \left(\sum_{j=1}^3 D_1\psi_j \widetilde{D}_j(F) \circ \Phi\right)\alpha^\lambda \quad \text{for } i = 1,2,3,\\
(\Phi_\lambda^{-1} \cdot F\alpha^\lambda)' &= \lambda E_\Phi^{\lambda-1} E'_\Phi F \circ \Phi + E_\Phi^{\lambda+1} F \circ \Phi + E_\Phi^\lambda \left(\sum_{j=1}^3 \psi'_j \widetilde{D}_j(F) \circ \Phi\right)\alpha^\lambda.
\end{aligned}$$

We develop the right hand side using further the identities

$$\begin{aligned}
S(\Phi_1 \circ \Phi) &= E_\Phi^{1/2} S(\Phi_1) \circ \Phi + S(\Phi),\\
D_i(S(\Phi_1 \circ \Phi)) &= \frac{1}{2} E_\Phi^{-1/2} D_i(E_\Phi) S(\Phi_1) \circ \Phi\\
&\quad + E_\Phi^{1/2} \left(\sum_{j=1}^3 D_i\psi_j \widetilde{D}_j(S(\Phi_1)) \circ \Phi + D_i(S(\Phi)\right) \quad \text{for } \ i = 1,2,3,\\
(S(\Phi_1 \circ \Phi)) &= \frac{1}{2} E_\Phi^{1/2} E'_\Phi S(\Phi_1) \circ \Phi + E_\Phi^{1/2} (S(\Phi_1))' \circ \Phi\\
&\quad + E_\Phi^{1/2} \left(\sum_{j=1}^3 \psi'_j \widetilde{D}_j(S(\Phi_1)) \circ \Phi\right) + (S(\Phi))'.
\end{aligned}$$

We prove the equality (6.4) by idetifying its sides.

6.3. Trivialization of the cocycle $\mathcal{C}_{\lambda,\lambda+3/2}$. Let $C\colon \mathcal{K}(1|3) \longrightarrow \mathcal{D}_{\lambda,\mu}(S^{1|3})$ be a 1-cocycle vanishing on $\mathrm{PC}(2,3)$. Assume that C is a coboundary, then there exists $B \in \mathcal{D}_{\lambda,\mu}$ such that

$$C(\Phi) = \delta(B)(\Phi) = \Phi_{\lambda,\mu}(B) - B, \quad \Phi \in \mathcal{K}(1|3).$$

In particular, for $\Phi \in \mathrm{PC}(2,3)$ we obtain that $\Phi_{\lambda,\mu}(B) = B$ is a $\mathrm{PC}(2,3)$-invariant differential operator. The $\mathrm{PC}(2,3)$-invariant differential operators acting on tensor densities are classified by the following proposition:

Proposition 6.3. *The unique* $\mathrm{PC}(2,3)$*-invariant differential operators (up to a multiplicative constant) are given by*

$$\begin{aligned} D_1D_2D_3\partial_k\colon\ \mathcal{F}_{-(2+k)/2}(S^{1|3}) &\longrightarrow \mathcal{F}_{(1+k)/2}(S^{1|3}),\\ F\alpha_3^{-(2+k)/2} &\longmapsto D_1D_2D_3(F^{(k)})\alpha_3^{(1+k)/2} \end{aligned} \tag{6.5}$$

for all positive integers k. If $(\lambda,\mu) \neq (-(2+k)/2, (1+k)/2)$, then there is no nontrivial $\mathrm{PC}(2,3)$*-invariant differential operator from $\mathcal{F}_\lambda(S^{1|3})$ to $\mathcal{F}_\mu(S^{1|3})$.*

To complete the proof of Theorem 6.1, we verify that for $\lambda = -1$ we have

$$\mathcal{C}_{-1,1/2}(\Phi^{-1}) = -\delta(D_3D_2D_1)(\Phi).$$

Hence, the 1-cocycle $\mathcal{C}_{-1,1/2}$ is a coboundary. For the generic values of λ, the cocycles described in Theorem 6.1 are nontrivial due to Proposition 6.3.

Remark 6.4. For $m = 3$, there exists no projection from $\mathcal{D}_{\lambda,\mu}(S^{1|3})$ to $\mathcal{F}_{1/2} = \mathbb{K}(1|3)$ intertwining the $\mathcal{K}(1|3)$ action. Moreover Michel and Duval provide a clear cut explanation of the fact that $S(\Phi)$ cannot be derived as a quadratic differential by the Cartan formula for $m \geqslant 3$, and thus help us understand why the Radul expression for $m = 3$ involves pseudo-differential operators.

In particular, there exists a pseudodifferential operator such that

$$S_3(\varphi) = \delta(D_3D_2D_1\partial^{-1})(\Phi).$$

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