Czechoslovak Mathematical Journal, 65 (140) (2015), 375–379

A SPECTRAL BOUND FOR GRAPH IRREGULARITY

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(Received November 3, 2014)

Abstract. The imbalance of an edge $e=\{u,v\}$ in a graph is defined as $i(e)=|d(u)-d(v)|$, where $d(\cdot)$ is the vertex degree. The irregularity $I(G)$ of G is then defined as the sum of imbalances over all edges of G. This concept was introduced by Albertson who proved that $I(G)\leqslant 4n^3/27$ (where $n=|V(G)|$) and obtained stronger bounds for bipartite and triangle-free graphs. Since then a number of additional bounds were given by various authors. In this paper we prove a new upper bound, which improves a bound found by Zhou and Luo in 2008. Our bound involves the Laplacian spectral radius λ.

Keywords: irregularity; Laplacian matrix; degree; Laplacian index

MSC 2010: 05C35, 05C50, 05C07

1. Introduction

Albertson [2] has defined the *irregularity* of a graph G as:

$$I(G)=\sum_{(u,v)\in E(G)}|d(u)-d(v)|,$$

where $d(u)$ is the degree of vertex u. Clearly $I(G)$ is zero if and only if G is regular and for non-regular graphs $I(G)$ is a measure of the defect of regularity. Albertson proved the following upper bound:

$$I(G)\leqslant\frac{4n^3}{27}. \tag{1.1}$$

Abdo, Cohen and Dimitrov [1] improved Albertson's bound:

$$I(G)\leqslant\Big\lfloor\frac{n}{3}\Big\rfloor\Big\lceil\frac{2n}{3}\Big\rceil\Big(\Big\lceil\frac{2n}{3}\Big\rceil-1\Big). \tag{1.2}$$

Additional upper bounds on $I(G)$ have been given by various authors: Hansen and Mélot [5], Henning and Rautenbach [6], Zhou and Luo [10], and Fath-Tabar [3]. These bounds are, strictly speaking, noncomparable but the bound of Zhou and Luo seems to be much sharper than the others for most graphs. We obtain here a new upper bound which is always less than the Zhou-Luo bound or equal to it.

To state the results, let us define the quantity $Z_G = \sum_{u \in V(G)} d(u)^2$. It is sometimes called the *first Zagreb index* of G (cf. [3]).

Theorem 1.1 ([10], Theorem 1)**.** *Let G be a graph on n vertices and with m edges. Then:*

$$I(G) \leqslant \sqrt{m(nZ_G - 4m^2)}.$$

Let us now recall the definition of the Laplacian matrix L of the graph $G = (V, E)$ whose vertices are labelled $\{1, 2, \ldots, n\}$:

$$L_{ij} = \begin{cases} -1 & \text{if } (i,j) \in E, \\ 0 & \text{if } (i,j) \notin E \text{ and } i \neq j, \\ -\sum_{k \neq i} L_{ik} & \text{if } i = j. \end{cases}$$

It is obvious from the definition that L is a positive semidefinite matrix. Surveys of its variegated and fascinating properties can be found in [8], [9]. One simple fact will be germane to us here: the largest eigenvalue $\lambda_{\max}$ of L satisfies $\lambda_{\max} \leqslant n$.

We can now state our new result which is clearly an improvement upon the Zhou-Luo bound:

Theorem 1.2. *Let G be a graph on n vertices and with m edges. Then:*

$$I(G) \leqslant \sqrt{m(nZ_G - 4m^2)(\lambda_{\max}/n)}. \tag{1.3}$$

2. Proof of the main result

The quadratic form defined by L has the following useful expression (where we identify the vector $x \in \mathbb{R}^n$ with a function $x\colon V(G) \to \mathbb{R}$):

$$x^T L x = \sum_{(u,v) \in E(G)} (x(u) - x(v))^2. \tag{2.1}$$

We also need Fiedler's [4] well-known characterization of $\lambda_{\max}$:

Lemma 2.1.

$$\lambda_{\max} = 2n \max_x \frac{\sum\limits_{(u,v)\in E(G)} (x(u)-x(v))^2}{\sum\limits_{u\in V(G)} \sum\limits_{v\in V(G)} (x(u)-x(v))^2},$$

where x is a nonconstant vector.

P r o o f of Theorem 1.2. The first step is to apply the Cauchy-Schwarz inequality:

$$I(G) = \sum_{(u,v)\in E(G)} |d(u)-d(v)| \leqslant \sqrt{m} \sqrt{\sum_{(u,v)\in E(G)} (d(u)-d(v))^2}. \tag{2.2}$$

In light of (2.1) we have:

$$I(G) \leqslant \sqrt{m}\sqrt{d^T Ld}.$$

We now turn to estimate $d^T Ld$ using Lemma 2.1 and Lagrange's identity:

$$\begin{aligned} d^T Ld &\leqslant \frac{\lambda_{\max}}{2n} \sum_{u\in V(G)} \sum_{v\in V(G)} (d(u)-d(v))^2 \\ &= \frac{\lambda_{\max}}{n}\left[n \sum_{v\in V(G)} d(u)^2 - \Big(\sum_{v\in V(G)} d(v)\Big)^2\right]. \end{aligned}$$

Clearly, the latter expression is equal to

$$\frac{\lambda_{\max}}{n}(nZ_G - 4m^2).$$

□

3. An example

Consider the *yoke* graph $G = Y_{n_1,n_2}$ which consists of two cycles of lengths n_1 and n_2 ($n_1 + n_2 = n$), connected by an edge. It is not hard to see that in this case $I(G) = 4$.

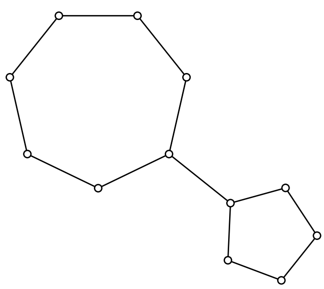

Figure 1. The graph $Y_{7,5}$.

In order to compare the bounds given by Theorems 1.1 and 1.2 we observe that:

$$m = n + 1, \quad Z_G = 4n + 10.$$

Therefore, the bound given by Theorem 1.1 is:

$$I(G) \leqslant \sqrt{2(n+1)(n-2)} = \Theta(n). \tag{3.1}$$

To estimate $\lambda_{\max}$ we can use a result due to Merris [7]. To state it, we define $m(v)$ to be the average degree of the neighbours of a vertex v.

Lemma 3.1 ([7])**.** $\lambda_{\max} \leqslant \max\{d(v) + m(v);\ v \in V(G)\}$.

Using this lemma we obtain that $\lambda_{\max} \leqslant 16/3$ for any yoke graph and therefore the bound of Theorem 1.2 is at least as good as:

$$I(G) \leqslant \sqrt{\frac{32}{3}\frac{(n+1)(n-2)}{n}} = \Theta(\sqrt{n}). \tag{3.2}$$

Obviously, (3.2) is much nearer to the true value of $I(G)$ than (3.1), although it still leaves something to be desired.

4. Another example—trees

We wish now to compare Theorem 1.2 to a specialized result of Zhou and Luo which gives a diffferent and very interesting bound for trees.

Theorem 4.1 ([10], Theorem 4)**.** *Let T be a tree with p pendant vertices. Then $I(G) \leqslant p(p-1)$.*

Before reporting a comparison between the two bounds, we wish to point out that by an observation of Albertson ([2], Corollary 5) $I(G)$ must be an even integer. Therefore, any upper bound on $I(G)$ can be replaced by the largest even integer not exceeding it.

We have computed the bounds of Theorem 1.2 (truncated to an even integer, as explained above) and of Theorem 4.1 (which always produces even integers) for the 106 non-isomorphic trees on ten vertices. The summary of the results is:

- ▷ For 46 trees Theorem 1.2 is better than Theorem 4.1.
- ▷ For 18 trees both bounds agree.
- ▷ For 42 trees Theorem 4.1 is better than Theorem 1.2.

To take some specific examples, consider first the tree in Figure 2. It is easy to compute that $I(T_{15}) = 22$. Since the graph has 7 pendant vertices, Theorem 4.1 yields the estimate $I(T_{15}) \leqslant 42$. On the other hand, the right-hand side of (1.3) is 27.8614 and so Theorem 1.2 gives us the estimate $I(G) \leqslant 26$.

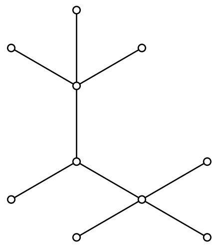

Figure 2. The tree T_{15}.

On the other hand, if we take the path P_n on n vertices, then $I(G) = 2$ and this value coincides with the bound of Theorem 4.1 since $p = 2$ in this case. However, Theorem 1.2 gives a poor estimate in this case—we have $m = n - 1$, $Z_G = 4n - 6$, and $\lambda_{\max} = 2(1 + \cos(\pi/n))$. Since $\lambda_{\max}$ tends to 4 as n grows, we can calculate that the right-hand side of (1.3) tends to $\sqrt{8(n-1)(n-2)/n} = \Theta(\sqrt{n})$.

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