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CONGRUENCES FOR WOLSTENHOLME PRIMES

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Abstract. A prime p is said to be a Wolstenholme prime if it satisfies the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$. For such a prime p, we establish an expression for $\binom{2p-1}{p-1} \pmod{p^8}$ given in terms of the sums $R_i := \sum_{k=1}^{p-1} 1/k^i$ $(i = 1, 2, 3, 4, 5, 6)$. Further, the expression in this congruence is reduced in terms of the sums R_i $(i = 1, 3, 4, 5)$. Using this congruence, we prove that for any Wolstenholme prime p we have

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^7}.$$

Moreover, using a recent result of the author, we prove that a prime p satisfying the above congruence must necessarily be a Wolstenholme prime.

Furthermore, applying a technique of Helou and Terjanian, the above congruence is given as an expression involving the Bernoulli numbers.

Keywords: congruence; prime power; Wolstenholme prime; Wolstenholme's theorem; Bernoulli number

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1. Introduction and statements of results

Wolstenholme's theorem (see, e.g., [23], [7]) asserts that if p is a prime greater than 3, then the binomial coefficient $\binom{2p-1}{p-1}$ satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

It is well known (see, e.g., [8]) that this theorem is equivalent to the assertion that for any prime $p \geqslant 5$ the numerator of the fraction

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1},$$

written in reduced form, is divisible by p^2. A. Granwille [7] established broader generalizations of Wolstenholme's theorem. As an application, it is obtained in [7] that for a prime $p \geqslant 5$ we have

$$\binom{2p-1}{p-1}\Big/\binom{2p}{p}^3 \equiv \binom{3}{2}\Big/\binom{2}{1}^3 \pmod{p^5}.$$

Notice that C. Helou and G. Terjanian [9] established many Wolstenholme type congruences modulo p^k with a prime p and $k \in \mathbb{N}$ such that $k \leqslant 6$. One of their main results ([9], Proposition 2, pages 488–489) is a congruence of the form $\binom{np}{mp} \equiv f(n,m,p)\binom{n}{m} \pmod{p}$, where $p \geqslant 3$ is a prime number, $m, n \in \mathbb{N}$ with $0 \leqslant m \leqslant n$, and f is a function on m, n and p involving the Bernoulli numbers B_k. As an application, ([9], Corollary 2 (2), page 493; also see Corollary 6 (2), page 495), for any prime $p \geqslant 5$ we have

$$\binom{2p-1}{p-1} \equiv 1 - p^3 B_{p^3-p^2-2} + \frac{1}{3}p^5 B_{p-3} - \frac{6}{5}p^5 B_{p-5} \pmod{p^6}.$$

A similar congruence modulo p^7 (Corollary 1.2) is obtained in this paper for Wolstenholme primes.

A prime p is said to be a *Wolstenholme prime* if it satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}.$$

The two known such primes are 16843 and 2124679, and in 2007 R. J. McIntosh and E. L. Roettger [17] reported that these primes are the only two Wolstenholme primes less than 10^9. However, using an argument based on the prime number theorem, McIntosh [16], page 387, conjectured that there are infinitely many Wolstenholme primes, and that no prime satisfies the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}$.

Wolstenholme primes form a subset of irregular primes. Indeed, Wolstenholme primes are those irregular primes p which divide the numerator of B_{p-3} (see, e.g., [16] or [19]). Recall that the irregular primes as well as Wieferich and related primes are connected with the first case of Fermat's last theorem; see [21], Lecture I, pages 9–12, and [21], Lecture VIII, pages 151–154, [2], [3], [12], [13], [22].

The following result is basic in our investigations.

Proposition 1.1. *Let p be a Wolstenholme prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} \equiv 1 &+ p\sum_{k=1}^{p-1}\frac{1}{k} - \frac{p^2}{2}\sum_{k=1}^{p-1}\frac{1}{k^2} + \frac{p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} - \frac{p^4}{4}\sum_{k=1}^{p-1}\frac{1}{k^4} \\ &+ \frac{p^5}{5}\sum_{k=1}^{p-1}\frac{1}{k^5} - \frac{p^6}{6}\sum_{k=1}^{p-1}\frac{1}{k^6} \pmod{p^8}.\end{aligned}$$

The above congruence can be simplified as follows.

Proposition 1.2. *Let p be a Wolstenholme prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}\sum_{k=1}^{p-1}\frac{1}{k} - \frac{p^2}{4}\sum_{k=1}^{p-1}\frac{1}{k^2} + \frac{7p^3}{12}\sum_{k=1}^{p-1}\frac{1}{k^3} + \frac{5p^5}{12}\sum_{k=1}^{p-1}\frac{1}{k^5} \pmod{p^8}.$$

Reducing the modulus in the previous congruence, we can obtain the following simpler congruences.

Corollary 1.1. *Let p be a Wolstenholme prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} &\equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} - 2p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \\ &\equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} + \frac{2p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} \pmod{p^7}.\end{aligned}$$

The *Bernoulli numbers* B_k ($k \in \mathbb{N}$) are defined by the generating function

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{\mathrm{e}^x - 1}.$$

It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for odd $n \geqslant 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geqslant 1$. These and many other properties can be found, for instance, in [10] or [4].

The second congruence from Corollary 1.1 can be given in terms of the Bernoulli numbers by the following result.

Corollary 1.2. *Let p be a Wolstenholme prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 - p^3 B_{p^4-p^3-2} - \frac{3}{2}p^5 B_{p^2-p-4} + \frac{3}{10}p^6 B_{p-5} \pmod{p^7}.$$

The above congruence can be given by the following expression involving lower order Bernoulli numbers.

Corollary 1.3. *Let p be a Wolstenholme prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} \equiv 1 &- p^3\Big(\frac{8}{3}B_{p-3} - 3B_{2p-4} + \frac{8}{5}B_{3p-5} - \frac{1}{3}B_{4p-6}\Big)\\ &- p^4\Big(\frac{8}{9}B_{p-3} - \frac{3}{2}B_{2p-4} + \frac{24}{25}B_{3p-5} - \frac{2}{9}B_{4p-6}\Big)\\ &- p^5\Big(\frac{8}{27}B_{p-3} - \frac{3}{4}B_{2p-4} + \frac{72}{125}B_{3p-5} - \frac{4}{27}B_{4p-6} + \frac{12}{5}B_{p-5} - B_{2p-6}\Big)\\ &- \frac{2}{25}p^6B_{p-5} \pmod{p^7}.\end{aligned}$$

Combining the first congruence in Corollary 1.1 and a recent result of the author in [18], Theorem 1.1, we obtain a new characterization of Wolstenholme primes as follows.

Corollary 1.4 ([18], Remark 1.6)**.** *A prime p is a Wolstenholme prime if and only if*

$$\binom{2p-1}{p-1} \equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} - 2p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^7}.$$

Remark 1.1. A computation shows that no prime $p < 10^5$ satisfies the second congruence in Corollary 1.1, except the Wolstenholme prime 16843. Accordingly, an interesting question is as follows: *Is it true that the second congruence in Corollary* 1.1 *implies that a prime p is necessarily a Wolstenholme prime*? We conjecture that this is true.

A proof of Proposition 1.1 is given in the next section. Proofs of Proposition 1.2 and Corollaries 1.1–1.3 are presented in Section 3.

2. Proof of Proposition 1.1

For the proof of Proposition 1.1, we will need some auxiliary results.

Lemma 2.1. *For any prime $p \geqslant 7$, we have*

$$2\sum_{k=1}^{p-1}\frac{1}{k} \equiv -p\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^4}. \tag{2.1}$$

P r o o f. The above congruence is in fact the congruence (14) in ([25], Proof of Theorem 3.2). □

Lemma 2.2. *For any prime $p \geqslant 7$, we have*

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^5}, \tag{2.2}$$

and

$$\binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^5}. \tag{2.3}$$

P r o o f. Let $R_1(p) = \sum_{k=1}^{p-1} 1/k$. Following ([25], Definition 3.1) we define $w_p < p^2$ to be the unique nonnegative integer such that $w_p \equiv R_1(p)/p^2 \pmod{p^2}$. Then by ([25], Theorem 3.2), for all nonnegative integers n and r with $n \geqslant r$,

$$\binom{np}{rp} \Big/ \binom{n}{r} \equiv 1 + w_p nr(n-r)p^3 \pmod{p^5}. \tag{2.4}$$

Since $\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1}$, taking $n = 2$ and $r = 1$, (2.4) becomes

$$\binom{2p-1}{p-1} \equiv 1 + 2w_p p^3 \pmod{p^5},$$

which is actually (2.2). Now the congruence (2.3) follows immediately from (2.2) and (2.1) of Lemma 2.1. □

Lemma 2.3. *The following statements about a prime $p \geqslant 7$ are equivalent:*

(i) *p is a Wolstenholme prime;*
(ii) $\sum_{k=1}^{p-1} 1/k \equiv 0 \pmod{p^3}$;
(iii) $\sum_{k=1}^{p-1} 1/k^2 \equiv 0 \pmod{p^2}$;
(iv) *p divides the numerator of the Bernoulli number B_{p-3}.*

P r o o f. The equivalences (i)⇔(ii)⇔(iii) are immediate from Lemma 2.2 if we consider the congruences (2.2) and (2.3) modulo p^4. Further, by a special case of Glaisher's congruence ([5], page 21, [6], page 323; also see [16], Theorem 2), we have

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4},$$

which implies the equivalence (i)⇔(iv). This concludes the proof. □

For the proof of Proposition 1.1, we use the congruences (2.2) and (2.3) of Lemma 2.2 with $\pmod{p^4}$ instead of $\pmod{p^5}$. By a classical result of E. Lehmer [15]; (also see [24], Theorem 2.8), $\sum_{k=1}^{p-1} 1/k \equiv -\frac{1}{3}B_{p-3} \pmod{p^3}$. Substituting this into Glaisher's congruence given above, we obtain immediately (2.2) of Lemma 2.2, with $\pmod{p^4}$ instead of $\pmod{p^5}$.

Notice that the congruence (2.3) is also given in [16], page 385, but its proof is there omitted.

For a prime $p \geqslant 3$ and a positive integer $n \leqslant p-2$ we denote

$$R_n(p) := \sum_{i=1}^{p-1} \frac{1}{k^n} \quad \text{and} \quad H_n(p) := \sum_{1 \leqslant i_1 < i_2 < \ldots < i_n \leqslant p-1} \frac{1}{i_1 i_2 \ldots i_n}.$$

In the sequel we shall often write R_n and H_n instead of $R_n(p)$ and $H_n(p)$, respectively.

Lemma 2.4 ([1], Theorem 3; also see [24], Remark 2.3)**.** *For any prime $p \geqslant 3$ and a positive integer $n \leqslant p-3$, we have*

$$R_n(p) \equiv 0 \pmod{p^2} \quad \textit{if } n \textit{ is odd}, \quad \textit{and} \quad R_n(p) \equiv 0 \pmod{p} \quad \textit{if } n \textit{ is even}.$$

Lemma 2.5 (Newton's formula, see, e.g., [11])**.** *Let m and s be positive integers such that $m \leqslant s$. Define the symmetric polynomials*

$$P_m(s) = P_m(s; x_1, x_2, \ldots, x_s) = x_1^m + x_2^m + \ldots + x_s^m,$$

and

$$A_m(s) = A_m(s; x_1, x_2, \ldots, x_s) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant s} x_{i_1} x_{i_2} \ldots x_{i_m}.$$

Then for $n = 1, 2, \ldots, s$, we have

$$P_n(s) - A_1(s)P_{n-1}(s) + A_2(s)P_{n-2}(s) + \ldots + (-1)^{n-1}A_{n-1}(s)P_1(s) + (-1)^n n A_n(s) = 0.$$

Lemma 2.6 (see [20], Lemma 2.2, the case $l = 1$)**.** *For any prime $p \geqslant 5$ and a positive integer $n \leqslant p-3$, we have*

$$H_n(p) \equiv 0 \pmod{p^2} \quad \textit{if } n \textit{ is odd} \quad \textit{and} \quad H_n(p) \equiv 0 \pmod{p} \quad \textit{if } n \textit{ is even}.$$

Lemma 2.6 is an immediate consequence of a result of X. Zhou and T. Cai [26], Lemma 2; (also see [24], Theorem 2.14).

Lemma 2.7. *For any Wolstenholme prime p, we have*

$$R_2(p) \equiv -2H_2(p) \pmod{p^6}, \quad R_3(p) \equiv 3H_3(p) \pmod{p^5},$$
$$R_4(p) \equiv -4H_4(p) \pmod{p^4}, \quad R_5(p) \equiv 5H_5(p) \pmod{p^4}$$
$$\text{and} \qquad R_6(p) \equiv -6H_6(p) \pmod{p^3}.$$

P r o o f. By Newton's formula (see Lemma 2.5), for $n = 2, 3, 4, 5, 6$ we have

$$R_n + (-1)^n n H_n = H_1 R_{n-1} - H_2 R_{n-2} + \ldots + (-1)^n H_{n-1} R_1. \tag{2.5}$$

First note that by Lemma 2.3, $R_1 = H_1 \equiv 0 \pmod{p^3}$ and $R_2 \equiv 0 \pmod{p^2}$. Therefore, (2.5) implies $R_2 + 2H_2 = H_1 R_1 \equiv 0 \pmod{p^6}$, so that $R_2 \equiv -2H_2 \pmod{p^6}$. From this and Lemma 2.3 we conclude that $H_2 \equiv R_2 \equiv 0 \pmod{p^2}$.

Further, by Lemma 2.4 and Lemma 2.6, $R_3 \equiv H_3 \equiv R_5 \equiv H_5 \equiv 0 \pmod{p^2}$ and $R_4 \equiv H_4 \equiv 0 \pmod{p}$. Substituting the previous congruences for H_i and R_i $(i = 1, 2, 3, 4, 5)$ into (2.5) with $n = 3, 4, 5, 6$, we get

$$R_3 - 3H_3 = H_1 R_2 - H_2 R_1 \equiv 0 \pmod{p^5},$$
$$R_4 + 4H_4 = H_1 R_3 - H_2 R_2 + H_3 R_1 \equiv 0 \pmod{p^4},$$
$$R_5 - 5H_5 = H_1 R_4 - H_2 R_3 + H_3 R_2 - H_4 R_1 \equiv 0 \pmod{p^4},$$
$$R_6 + 6H_6 = H_1 R_5 - H_2 R_4 + H_3 R_3 - H_4 R_2 + H_5 R_1 \equiv 0 \pmod{p^3}.$$

This completes the proof. □

P r o o f of Proposition 1.1. For any prime $p \geqslant 7$, we have

$$\begin{aligned}\binom{2p-1}{p-1} &= \frac{(p+1)(p+2)\ldots(p+k)\ldots(p+(p-1))}{1\cdot 2\ldots k\ldots p-1} \\ &= \Big(\frac{p}{1}+1\Big)\Big(\frac{p}{2}+1\Big)\ldots\Big(\frac{p}{k}+1\Big)\ldots\Big(\frac{p}{p-1}+1\Big) \\ &= 1 + \sum_{i=1}^{p-1}\frac{p}{i} + \sum_{1\leqslant i_1<i_2\leqslant p-1}\frac{p^2}{i_1 i_2} + \ldots + \sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant p-1}\frac{p^k}{i_1 i_2\ldots i_k} \\ &\quad + \ldots + \frac{p^{p-1}}{(p-1)!} = 1 + \sum_{k=1}^{p-1} p^k H_k = 1 + \sum_{k=1}^{6} p^k H_k + \sum_{k=7}^{p-1} p^k H_k.\end{aligned}$$

Since by Lemma 2.6, $p^9 \,\Big|\, \sum\limits_{k=7}^{p-1} p^k H_k$ for any prime $p \geqslant 11$, the above identity yields

$$\binom{2p-1}{p-1} \equiv 1 + pH_1 + p^2 H_2 + p^3 H_3 + p^4 H_4 + p^5 H_5 + p^6 H_6 \pmod{p^8}.$$

Now by Lemma 2.7, for $n = 2, 3, 4, 5, 6$, we have

$$H_n \equiv (-1)^{n-1}\frac{R_n}{n} \pmod{p^{e_n}} \quad \text{for } e_2 = 6,\ e_3 = 5,\ e_4 = 4,\ e_5 = 4 \text{ and } e_6 = 3.$$

Substituting the above congruences into the previous one, and setting $H_1 = R_1$, we obtain

$$\binom{2p-1}{p-1} \equiv 1 + pR_1 - \frac{p^2}{2}R_2 + \frac{p^3}{3}R_3 - \frac{p^4}{4}R_4 + \frac{p^5}{5}R_5 - \frac{p^6}{6}R_6 \pmod{p^8}.$$

This is the desired congruence from Proposition 1.1. □

3. Proofs of Proposition 1.2 and Corollaries 1.1–1.3

In order to prove Proposition 1.2 and Corollaries 1.1–1.3, we need some auxiliary results.

Lemma 3.1. *Let p be a prime, and let m be any even positive integer. Then the denominator d_m of the Bernoulli number B_m, written in reduced form, is given by*

$$d_m = \prod_{p-1|m} p,$$

where the product is taken over all primes p such that $p-1$ divides m.

Proof. The assertion is an immediate consequence of the von Staudt-Clausen theorem (see, e.g., [10], page 233, Theorem 3) which asserts that $B_m + \sum_{p-1|m} 1/p$ is an integer for all even m, where the summation is over all primes p such that $p-1$ divides m. □

Recall that for a prime p and a positive integer n, we denote

$$R_n(p) = R_n = \sum_{k=1}^{p-1}\frac{1}{k^n} \quad \text{and} \quad P_n(p) = \sum_{k=1}^{p-1} k^n.$$

Lemma 3.2 ([9], page 8)**.** *Let p be a prime greater than 5, and let n, r be positive integers. Then*

$$P_n(p) \equiv \sum_{s-\mathrm{ord}_p(s)\leqslant r} \frac{1}{s}\binom{n}{s-1} p^s B_{n+1-s} \pmod{p^r}, \tag{3.1}$$

where $\mathrm{ord}_p(s)$ is the largest power of p dividing s, and the summation is taken over all integers $1 \leqslant s \leqslant n+1$ such that $s - \mathrm{ord}_p(s) \leqslant r$.

The following result is well known as the Kummer congruences.

Lemma 3.3 ([10], page 239)**.** *Suppose that $p \geqslant 3$ is a prime and m, n, r are positive integers such that m and n are even, $r \leqslant n-1 \leqslant m-1$, and $m \not\equiv 0 \pmod{p-1}$. If $n \equiv m \pmod{\varphi(p^r)}$, where $\varphi(p^r) = p^{r-1}(p-1)$ is Euler's totient function, then*

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \pmod{p^r}. \tag{3.2}$$

The following congruences are also due to Kummer.

Lemma 3.4 ([14]; also see [9], page 20)**.** *Let $p \geqslant 3$ be a prime and let m, r be positive integers such that m is even, $r \leqslant m-1$ and $m \not\equiv 0 \pmod{p-1}$. Then*

$$\sum_{k=0}^{r} (-1)^k \binom{m}{k} \frac{B_{m+k(p-1)}}{m+k(p-1)} \equiv 0 \pmod{p^r}. \tag{3.3}$$

Lemma 3.5. *For any prime $p \geqslant 11$, we have*

(i) $R_1(p) \equiv -\frac{1}{2}p^2 B_{p^4-p^3-2} - \frac{1}{4}p^4 B_{p^2-p-4} + \frac{1}{6}p^5 B_{p-3} + \frac{1}{20}p^5 B_{p-5} \pmod{p^6}$,
(ii) $R_3(p) \equiv -\frac{3}{2}p^2 B_{p^4-p^3-4} \pmod{p^4}$,
(iii) $R_4(p) \equiv pB_{p^4-p^3-4} \pmod{p^3}$,
(iv) $pR_6(p) \equiv -\frac{2}{5}R_5(p) \pmod{p^4}$.

P r o o f. If s is a positive integer such that $\operatorname{ord}_p(s) = e \geqslant 1$, then for $p \geqslant 11$ we have $s - e \geqslant p^e - e \geqslant 10$. This shows that the condition $s - \operatorname{ord}_p(s) \leqslant 6$ implies that $\operatorname{ord}_p(s) = 0$, and thus, $s \leqslant 6$ must hold for such an s. Therefore, by Lemma 3.2,

$$P_n(p) \equiv \sum_{s=1}^{6} \frac{1}{s} \binom{n}{s-1} p^s B_{n+1-s} \pmod{p^6} \quad \text{for } n = 1, 2, \ldots. \tag{3.4}$$

By Euler's theorem, for $1 \leqslant k \leqslant p-1$ and positive integers n, e we have $1/k^{\varphi(p^e)-n} \equiv k^n \pmod{p^e}$, where $\varphi(p^e) = p^{e-1}(p-1)$ is Euler's totient function. Hence, $R_{\varphi(p^e)-n}(p) \equiv P_n(p) \pmod{p^e}$. In particular, if $n = \varphi(p^6)-1 = p^5(p-1)-1$, then by Lemma 3.1, $p^6 \mid p^6 B_{p^5(p-1)-6}$ for each prime $p \geqslant 11$. Therefore, using the fact that $B_{p^5(p-1)-1} = B_{p^5(p-1)-3} = B_{p^5(p-1)-5} = 0$, (3.4) yields

$$\begin{aligned} R_1(p) \equiv P_{p^5(p-1)-1}(p) &\equiv \frac{1}{2}(p^5(p-1)-1)p^2 B_{p^5(p-1)-2} \\ &\quad + \frac{1}{4}\frac{(p^5(p-1)-1)(p^5(p-1)-2)(p^5(p-1)-3)}{6} p^4 B_{p^5(p-1)-4} \pmod{p^6}, \end{aligned}$$

whence we have

$$R_1(p) \equiv -\frac{p^2}{2}B_{p^6-p^5-2} - \frac{p^4}{4}B_{p^6-p^5-4} \pmod{p^6}. \tag{3.5}$$

By the Kummer congruences (3.2) from Lemma 3.3, we have

$$B_{p^6-p^5-2} \equiv \frac{p^6-p^5-2}{p^4-p^3-2}B_{p^4-p^3-2} \equiv \frac{2B_{p^4-p^3-2}}{p^3+2} \equiv \Big(1-\frac{p^3}{2}\Big)B_{p^4-p^3-2} \pmod{p^4}.$$

Substituting this into (3.5), we obtain

$$R_1(p) \equiv -\frac{p^2}{2}B_{p^4-p^3-2} + \frac{p^5}{4}B_{p^4-p^3-2} - \frac{p^4}{4}B_{p^6-p^5-4} \pmod{p^6}. \tag{3.6}$$

Similarly, we have

$$B_{p^4-p^3-2} \equiv \frac{p^4-p^3-2}{p-3}B_{p-3} \equiv \frac{2}{3}B_{p-3} \pmod{p}$$

and

$$B_{p^6-p^5-4} \equiv \frac{p^6-p^5-4}{p^2-p-4}B_{p^2-p-4} \equiv \frac{4B_{p^2-p-4}}{p+4} \equiv \Big(1-\frac{p}{4}\Big)B_{p^2-p-4} \pmod{p^2}.$$

Substituting the above two congruences into (3.6), we get

$$R_1(p) \equiv -\frac{p^2}{2}B_{p^4-p^3-2} + \frac{p^5}{6}B_{p-3} - \frac{p^4}{4}B_{p^2-p-4} + \frac{p^5}{16}B_{p^2-p-4} \pmod{p^6}. \tag{3.7}$$

Finally, since

$$B_{p^2-p-4} \equiv \frac{p^2-p-4}{p-5}B_{p-5} \equiv \frac{4}{5}B_{p-5} \pmod{p},$$

the substitution of the above congruence into (3.7) immediately gives the congruence (i).

To prove the congruences (ii) and (iii), note that if $n-3 \not\equiv 0 \pmod{p-1}$, then by Lemma 3.1, $p^4 \mid p^4B_{n-3}$ for odd $n \geqslant 5$, while $B_{n-3} = 0$ for even $n \geqslant 6$. Therefore, reducing the modulus in (3.4) to p^4, for all odd $n \geqslant 3$ with $n-3 \not\equiv 0 \pmod{p-1}$ and for all even $n \geqslant 2$ we have

$$P_n(p) \equiv pB_n + \frac{p^2}{2}nB_{n-1} + \frac{p^3}{6}n(n-1)B_{n-2} \pmod{p^4}. \tag{3.8}$$

In particular, for $n = p^4-p^3-3$ we have $B_{p^4-p^3-3} = B_{p^4-p^3-5} = 0$, and thus (3.8) yields

$$R_3(p) \equiv P_{p^4-p^3-3}(p) \equiv \frac{p^2(p^4-p^3-3)}{2}B_{p^4-p^3-4} \equiv -\frac{3p^2}{2}B_{p^4-p^3-4} \pmod{p^4}.$$

Similarly, if $n = p^4 - p^3 - 4$, then since $p^4 - p^3 - 6 \not\equiv 0 \pmod{p-1}$, by Lemma 3.1 we have $p^3 \mid p^3 B_{p^4-p^3-6}$ for each prime $p \geqslant 11$. Using this and the fact that $B_{p^4-p^3-5} = 0$, from (3.8) modulo p^3 we find that

$$R_4(p) \equiv P_{p^4-p^3-4}(p) \equiv pB_{p^4-p^3-4} \pmod{p^3}.$$

It remains to show (iv). If n is odd such that $n-3 \not\equiv 0 \pmod{p-1}$, then by (3.8) and Lemma 3.1, $P_n(p) \equiv (n/2)p^2 B_{n-1} \pmod{p^4}$ and $P_{n-1}(p) \equiv pB_{n-1} \pmod{p^3}$. Thus, for such an n we have

$$P_n(p) \equiv \frac{n}{2} p P_{n-1} \pmod{p^4}.$$

In particular, for $n = p^4 - p^3 - 5$, from the above we get

$$\begin{aligned} R_5(p) \equiv P_{p^4-p^3-5}(p) &\equiv \frac{(p^4-p^3-5)p}{2} P_{p^4-p^3-6}(p) \\ &\equiv -\frac{5}{2} p P_{p^4-p^3-6}(p) \equiv -\frac{5}{2} p R_6(p) \pmod{p^4}. \end{aligned}$$

This implies (iv) and the proof is complete. □

Lemma 3.6. *For any prime p and any positive integer r, we have*

$$2R_1 \equiv -\sum_{i=1}^{r} p^i R_{i+1} \pmod{p^{r+1}}. \tag{3.9}$$

P r o o f. Multiplying the identity

$$1 + \frac{p}{i} + \ldots + \frac{p^{r-1}}{i^{r-1}} = \frac{p^r - i^r}{i^{r-1}(p-i)}$$

by $-p/i^2$, $1 \leqslant i \leqslant p-1$, we obtain

$$-\frac{p}{i^2}\Big(1 + \frac{p}{i} + \ldots + \frac{p^{r-1}}{i^{r-1}}\Big) = \frac{-p^{r+1} + pi^r}{i^{r+1}(p-i)} \equiv \frac{p}{i(p-i)} \pmod{p^{r+1}}.$$

Therefore,

$$\Big(\frac{1}{i} + \frac{1}{p-i}\Big) \equiv -\Big(\frac{p}{i^2} + \frac{p^2}{i^3} + \ldots + \frac{p^r}{i^{r+1}}\Big) \pmod{p^{r+1}},$$

from which we immediately obtain (3.9) after summing over all i from 1 to $p-1$. □

P r o o f of Proposition 1.2. We begin with the congruence from Proposition 1.1:

$$\binom{2p-1}{p-1} \equiv 1 + pR_1 - \frac{p^2}{2}R_2 + \frac{p^3}{3}R_3 - \frac{p^4}{4}R_4 + \frac{p^5}{5}R_5 - \frac{p^6}{6}R_6 \pmod{p^8}. \tag{3.10}$$

As by Lemma 2.4 we have $p^2 \mid R_7$, Lemma 3.6 with $r = 7$ yields

$$2R_1 \equiv -pR_2 - p^2R_3 - p^3R_4 - p^4R_5 - p^5R_6 \pmod{p^8}, \tag{3.11}$$

and after multiplying by $p/4$ it follows that

$$-\frac{p^4}{4}R_4 \equiv \frac{p}{2}R_1 + \frac{1}{4}(p^2R_2 + p^3R_3 + p^5R_5 + p^6R_6) \pmod{p^8}.$$

Substituting this into the congruence (3.10), we obtain

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}R_1 - \frac{p^2}{4}R_2 + \frac{7p^3}{12}R_3 + \frac{9p^5}{20}R_5 + \frac{p^6}{12}R_6 \pmod{p^8}.$$

Further, from (iv) of Lemma 3.5 we see that

$$p^6R_6 \equiv -\frac{2}{5}p^5R_5 \pmod{p^8}.$$

The substitution of this into the previous congruence immediately gives

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}R_1 - \frac{p^2}{4}R_2 + \frac{7p^3}{12}R_3 + \frac{5p^5}{12}R_5 \pmod{p^8},$$

as desired. □

Remark 3.1. Proceeding in the same way as in the previous proof and using (3.11), we can eliminate R_2 to obtain

$$\binom{2p-1}{p-1} \equiv 1 + 2pR_1 + \frac{5p^3}{6}R_3 + \frac{p^4}{4}R_4 + \frac{17p^5}{30}R_5 \pmod{p^8}.$$

Remark 3.2. If we suppose that there exists a prime p such that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}$, then by Lemma 2.2, for such a p we must have $R_1 \equiv 0 \pmod{p^4}$ and $R_2 \equiv 0 \pmod{p^3}$. Starting with these two congruences, in the same manner as in the proof of Lemma 2.7, it can be deduced that for $n = 2, 3, 4, 5, 6, 7, 8$,

$$H_n \equiv (-1)^{n-1}\frac{R_n}{n} \pmod{p^{e_n}},$$

where $e_2 = 8$, $e_3 = 7$, $e_4 = 6$, $e_5 = 5$, $e_6 = 4$, $e_7 = 3$ and $e_8 = 2$. Since as in the proof of Proposition 1.1 we have

$$\binom{2p-1}{p-1} \equiv 1 + pH_1 + p^2H_2 + p^3H_3 + p^4H_4 + p^5H_5 + p^6H_6 + p^7H_7 + p^8H_8 \pmod{p^{10}},$$

then substituting the previous congruences into the right hand side of the above congruence and setting $H_1 = R_1$, we obtain

$$\binom{2p-1}{p-1} \equiv 1 + pR_1 - \frac{p^2}{2}R_2 + \frac{p^3}{3}R_3 - \frac{p^4}{4}R_4 + \frac{p^5}{5}R_5 - \frac{p^6}{6}R_6 + \frac{p^7}{7}R_7 - \frac{p^8}{8}R_8 \pmod{p^{10}}.$$

Since by Lemma 2.4, $p^2 \mid R_7$ and $p \mid R_8$, from the above we get

$$\binom{2p-1}{p-1} \equiv 1 + pR_1 - \frac{p^2}{2}R_2 + \frac{p^3}{3}R_3 - \frac{p^4}{4}R_4 + \frac{p^5}{5}R_5 - \frac{p^6}{6}R_6 \pmod{p^9}.$$

Then as in the above proof, using (3.11) and the fact that by (iv) of Lemma 3.5, $p^6R_6(p) \equiv -(2/5)p^5R_5(p) \pmod{p^9}$, we can find that

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}R_1 - \frac{p^2}{4}R_2 + \frac{7p^3}{12}R_3 + \frac{5p^5}{12}R_5 \pmod{p^9}.$$

P r o o f of Corollary 1.1. In view of the fact that by Lemma 2.4, $p^2 \mid R_5$, the congruence from Proposition 1.2 immediately yields

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}R_1 - \frac{p^2}{4}R_2 + \frac{7p^3}{12}R_3 \pmod{p^7}. \tag{3.12}$$

Lemma 3.6 with $r = 5$ and the fact that by Lemma 2.4, $p^2 \mid R_5$ and $p \mid R_6$ imply

$$2R_1 \equiv -pR_2 - p^2R_3 - p^3R_4 \pmod{p^6}.$$

From (ii) and (iii) of Lemma 3.5 we see that $pR_4 \equiv -\frac{2}{3}R_3 \pmod{p^4}$, so that $p^3R_4 \equiv -\frac{2}{3}p^2R_3 \pmod{p^6}$. Substituting this into the previous congruence, we obtain

$$2R_1 + pR_2 \equiv -\frac{1}{3}p^2R_3 \pmod{p^6},$$

whence we have

$$p^3R_3 \equiv -6pR_1 - 3p^2R_2 \pmod{p^7}. \tag{3.13}$$

Substituting this into (3.12), we get

$$\binom{2p-1}{p-1} \equiv 1 - 2pR_1 - 2p^2R_2 \pmod{p^7},$$

which is actually the first congruence from Corollary 1.1. Finally, from (3.13) we have

$$p^2R_2 \equiv -2pR_1 - \frac{1}{3}p^3R_3 \pmod{p^7},$$

and substituting this into (3.12) gives

$$\binom{2p-1}{p-1} \equiv 1 + 2pR_1 + \frac{2}{3}p^3R_3 \pmod{p^7}. \tag{3.14}$$

This completes the proof. □

P r o o f of Corollary 1.2. By (ii) of Lemma 3.5, we have

$$p^3R_3(p) \equiv -(3/2)p^5B_{p^4-p^3-4} \pmod{p^7}.$$

Substituting this into (3.14), we obtain

$$\binom{2p-1}{p-1} \equiv 1 + 2pR_1 - p^5B_{p^4-p^3-4} \pmod{p^7}. \tag{3.15}$$

By Lemma 2.3, $p \mid B_{p-3}$ so that $p^6 \mid (p^5/6)B_{p-3}$, and hence from (i) of Lemma 3.5 we obtain

$$2pR_1(p) \equiv -p^3B_{p^4-p^3-2} - \frac{p^5}{2}B_{p^2-p-4} + \frac{p^6}{10}B_{p-5} \pmod{p^7}.$$

Furthermore, by the Kummer congruences (3.2), since $p^4 - p^3 - 2 \not\equiv 0 \pmod{p-1}$ and $p^4 - p^3 - 2 \equiv p^2 - p - 2 \pmod{\varphi(p^2)}$, we have

$$B_{p^4-p^3-4} \equiv \frac{p^4-p^3-4}{p^2-p-4}B_{p^2-p-4} \equiv \frac{4}{p+4}B_{p^2-p-4} \equiv \Big(1 - \frac{p}{4}\Big)B_{p^2-p-4} \pmod{p^2}.$$

The substitution of the above two congruences into (3.15) immediately gives

$$\binom{2p-1}{p-1} \equiv 1 - p^3B_{p^4-p^3-2} - \frac{3p^5}{2}B_{p^2-p-4} + \frac{p^6}{10}B_{p-5} + \frac{p^6}{4}B_{p^2-p-4} \pmod{p^7}. \tag{3.16}$$

Finally, since by the Kummer congruences (3.2),

$$B_{p^2-p-4} \equiv \frac{p^2-p-4}{p-5}B_{p-5} \equiv \frac{4}{5}B_{p-5} \pmod{p},$$

after substitution of this into (3.16) we obtain

$$\binom{2p-1}{p-1} \equiv 1 - p^3B_{p^4-p^3-2} - \frac{3p^5}{2}B_{p^2-p-4} + \frac{3p^6}{10}B_{p-5} \pmod{p^7}. \tag{3.17}$$

This is the required congruence. □

P r o o f of Corollary 1.3. As noticed in [9], congruence (3) on page 494, combining the Kummer congruences (3.2) and (3.3) for $m = \varphi(p^n) - s$, $n, s \in \mathbb{N}$ with $s \not\equiv 0 \pmod{p-1}$, we obtain

$$\frac{B_{p^n-p^{n-1}-s}}{p^n-p^{n-1}-s} \equiv \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{B_{k(p-1)-s}}{k(p-1)-s} \pmod{p^n}. \tag{3.18}$$

Now (3.18) with $n = 2$ and $s = 4$ gives

$$\frac{B_{p^2-p-4}}{p^2-p-4} \equiv \frac{2B_{p-5}}{p-5} - \frac{B_{2p-6}}{2p-6} \pmod{p^2},$$

or equivalently,

$$B_{p^2-p-4} \equiv -\frac{2(p+4)}{p-5}B_{p-5} + \frac{p+4}{2(p-3)}B_{2p-6} \pmod{p^2}.$$

Substituting $1/(p-5) \equiv -(5+p)/25 \pmod{p^2}$ and $1/(p-3) \equiv -(3+p)/9 \pmod{p^2}$, the above congruence becomes

$$B_{p^2-p-4} \equiv \frac{18p+40}{25}B_{p-5} - \frac{7p+12}{18}B_{2p-6} \pmod{p^2}. \tag{3.19}$$

Similarly, (3.18) with $n = 4$ and $s = 2$ yields

$$\frac{B_{p^4-p^3-2}}{p^4-p^3-2} \equiv \sum_{k=1}^{4}(-1)^{k+1}\binom{4}{k}\frac{B_{k(p-1)-2}}{k(p-1)-2} \pmod{p^4},$$

whence, multiplying by p^3+2, we get

$$\begin{aligned} -B_{p^4-p^3-2} &\equiv (p^3+2)\Big(\frac{4B_{p-3}}{p-3} - \frac{6B_{2p-4}}{2p-4} + \frac{4B_{3p-5}}{3p-5} - \frac{B_{4p-6}}{4p-6}\Big) \\ &\equiv p^3\Big(\frac{4B_{p-3}}{-3} - \frac{6B_{2p-4}}{-4} + \frac{4B_{3p-5}}{-5} - \frac{B_{4p-6}}{-6}\Big) \\ &\quad + 2\Big(\frac{4B_{p-3}}{p-3} - \frac{6B_{2p-4}}{2p-4} + \frac{4B_{3p-5}}{3p-5} - \frac{B_{4p-6}}{4p-6}\Big) \pmod{p^4}. \end{aligned} \tag{3.20}$$

As by the Kummer congruences (3.2),

$$\frac{B_{4p-6}}{4p-6} \equiv \frac{B_{3p-5}}{3p-5} \equiv \frac{B_{2p-4}}{2p-4} \equiv \frac{B_{p-3}}{p-3} \pmod{p},$$

we have

$$B_{4p-6} \equiv 2B_{p-3} \pmod{p}, \quad B_{3p-5} \equiv \frac{5}{3}B_{p-3} \pmod{p}, \quad B_{2p-4} \equiv \frac{4}{3}B_{p-3} \pmod{p}.$$

Substituting this into the first term on the right-hand side in the congruence (3.20), we obtain

$$p^3\Big(\frac{4B_{p-3}}{-3}-\frac{6B_{2p-4}}{-4}+\frac{4B_{3p-5}}{-5}-\frac{B_{4p-6}}{-6}\Big)\equiv-\frac{p^3}{3}B_{p-3}\equiv 0 \pmod{p^4},$$

where we have used the fact that by Lemma 2.3, p divides the numerator of B_{p-3}.

Further, as for all integers a, b, n such that $b\not\equiv 0 \pmod p$ we have

$$\frac{1}{ap-b}\equiv-\frac{1}{b}\sum_{k=0}^{3}\frac{a^kp^k}{b^k} \pmod{p^4},$$

applying this to $1/(p-3)$, $1/(2p-4)$ and $1/(3p-5)$, the second term on the right-hand side in the congruence (3.20) becomes

$$\begin{aligned}-B_{p^4-p^3-2}\equiv 2\Big(&-\frac{4}{3}\Big(1+\frac{p}{3}+\frac{p^2}{9}\Big)B_{p-3}+\frac{3}{2}\Big(1+\frac{p}{2}+\frac{p^2}{4}\Big)B_{2p-4}\\&-\frac{4}{5}\Big(1+\frac{3p}{5}+\frac{9p^2}{25}\Big)B_{3p-5}+\frac{1}{6}\Big(1+\frac{2p}{3}+\frac{4p^2}{9}\Big)B_{4p-6}\Big) \pmod{p^4}.\end{aligned}$$

Muptiplying by p^3, the above congruence becomes

$$\begin{aligned}-p^3B_{p^4-p^3-2}\equiv\ &-\frac{8}{3}\Big(p^3+\frac{p^4}{3}+\frac{p^5}{9}\Big)B_{p-3}+3\Big(p^3+\frac{p^4}{2}+\frac{p^5}{4}\Big)B_{2p-4}\\&-\frac{8}{5}\Big(p^3+\frac{3p^4}{5}+\frac{9p^5}{25}\Big)B_{3p-5}+\frac{1}{3}\Big(p^3+\frac{2p^4}{3}+\frac{4p^5}{9}\Big)B_{4p-6} \pmod{p^7}.\end{aligned}$$

Finally, substituting this and the congruence (3.19) into (3.17), we obtain the congruence from Corollary 1.3. □

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