ON PRINCIPAL CONNECTION LIKE BUNDLES

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Abstract. Let $\mathcal{PB}_m$ be the category of all principal fibred bundles with m-dimensional bases and their principal bundle homomorphisms covering embeddings. We introduce the concept of the so called (r,m)-systems and describe all gauge bundle functors on $\mathcal{PB}_m$ of order r by means of the (r,m)-systems. Next we present several interesting examples of fiber product preserving gauge bundle functors on $\mathcal{PB}_m$ of order r. Finally, we introduce the concept of product preserving (r,m)-systems and describe all fiber product preserving gauge bundle functors on $\mathcal{PB}_m$ of order r by means of the product preserving (r,m)-systems.

Keywords: principal bundle; principal connection; gauge bundle functor; natural transformation

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Introduction

Let $\mathcal{M}f$ be the category of all manifolds and maps, $\mathcal{M}f_m$ the category of m-dimensional manifolds and their embeddings, $\mathcal{FM}$ the category of all fibred manifolds and their fibred maps, $\mathcal{FM}_m$ the category of fibred manifolds with m-dimensional bases and fibred maps with embeddings as base maps, $\mathcal{G}r$ the category of all Lie groups and their homomorphisms, $\mathcal{PB}_m$ the category of all principal fiber bundles with m-dimensional bases and their principal bundle homomorphisms covering embeddings, $\mathcal{VB}$ the category of vector bundles and their vector bundle maps and $\mathcal{VB}_m$ the category of vector bundles with m-dimensional bases and their vector bundle maps covering embeddings.

By Definition 2.1 in [4], a gauge bundle functor on $\mathcal{PB}_m$ is a covariant functor $E\colon \mathcal{PB}_m \to \mathcal{FM}$ satisfying the following conditions:

(i) *Base preservation.* For any $\mathcal{PB}_m$-object $P = (p\colon P \to M)$ with the base M the induced $\mathcal{FM}$-object $EP = (\pi_P\colon EP \to M)$ is a fibred manifold over the same base M. For any $\mathcal{PB}_m$-morphism $f\colon P_1 \to P_2$ covering $\underline{f}\colon M_1 \to M_2$ the induced $\mathcal{FM}_m$-map $Ef\colon EP_1 \to EP_2$ is also over $\underline{f}$.

(ii) *Locality property.* For any $\mathcal{PB}_m$-object $p\colon P \to M$ and any open subset $U \subset M$ the $\mathcal{FM}$-map $Ei_U\colon E(P|_U) \to EP$ (induced by the inclusion $i_U\colon P|_U \to P$) is a diffeomorphism onto $\pi_P^{-1}(U)$.

(iii) *Regularity property.* E transforms smoothly parametrized families of $\mathcal{PB}_m$-morphisms into smoothly parametrized families of $\mathcal{FM}$-morphisms.

By Definition 2.2 and Lemma 2.3 in [4], a natural transformation $\eta\colon E \to E^1$ of gauge bundle functors on $\mathcal{PB}_m$ is a family of fibred maps $\eta_P\colon EP \to E^1P$ covering id_M for any $\mathcal{PB}_m$-object $P \to M$ such that $E^1 f \circ \eta_P = \eta_Q \circ Ef$ for any $\mathcal{PB}_m$-morphism $f\colon P \to Q$.

By Definition 2.5 in [4], a gauge bundle functor $E\colon \mathcal{PB}_m \to \mathcal{FM}$ is of order r if the following condition is satisfied:

For any $\mathcal{PB}_m$-morphisms $f_1, f_2\colon P \to Q$ between $\mathcal{PB}_m$-objects $P \to M$ and Q and any $x \in M$, from $j^r_x(f_1) = j^r_x(f_2)$ it follows that $(Ef_1)|_{E_xP} = (Ef_2)|_{E_xP}$.

A gauge bundle functor $E\colon \mathcal{PB}_m \to \mathcal{FM}$ is fiber product preserving if $(EP_1) \times_M (EP_2) = E(P_1 \times_M P_2)$ for any $\mathcal{PB}_m$-objects with the same base M (the identification is induced by the E-prolongation of the fiber product projections).

Given a $\mathcal{PB}_m$-object $P \to M$ with the structure Lie group G we have a principal connection bundle $QP := J^1P/G$ of P (sections of QP are in bijection with principal (right invariant) connections on P). Given a $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering the embedding $\underline{f}\colon M \to M_1$ with the Lie group homomorphism $\nu_f\colon G \to G_1$, the map $J^1f\colon J^1P \to J^1P_1$ factorizes into the $\mathcal{FM}$-map $Qf\colon QP \to QP_1$. In this way we obtain a gauge bundle functor $Q\colon \mathcal{PB}_m \to \mathcal{FM}$ of order 1. It is fiber product preserving.

Given a $\mathcal{PB}_m$-object $P \to M$ we have the r-th order principal prolongation $W^rP := P^rM \times_M J^rP$ (see Section 15 in [2]). Any $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ induces a fibred map $W^rf := P^r\underline{f} \times_{\underline{f}} J^rf\colon P^rM \times_M J^rP \to P^rM_1 \times_{M_1} J^rP_1$. In this way we obtain a gauge bundle functor $W^r\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r. The functor W^r is not fiber product preserving.

In the present paper, we describe all gauge bundle functors $\mathcal{PB}_m \to \mathcal{FM}$ of order r by means of the so called (r,m)-systems. Next, we describe fiber product preserving gauge bundle functors $E\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r by means of the product preserving (r,m)-systems.

All manifolds considered in the paper are assumed to be Hausdorff, finite dimensional, second countable, without boundary and smooth, i.e., of class C^∞. Maps between manifolds are assumed to be of class C^∞.

1. A characterization of gauge bundle functors on $\mathcal{PB}_m$ of order r by means of (r,m)-systems

Using the results of Section 15 in [2] we see that in fact we have $W^r\colon \mathcal{PB}_m \to \mathcal{PB}_m$. Indeed, we have a functor $W^r_m\colon \mathcal{G}r \to \mathcal{G}r$ sending any Lie group G into its r-th order prolongation group $W^r_m G = G^r_m \rtimes T^r_m G$ in dimension m (see Section 15 in [2]) and any Lie group homomorphism $\nu\colon G \to G_1$ into a Lie group homomorphisms $W^r_m\nu := \mathrm{id}_{G^r_m} \times T^r_m\nu\colon W^r_m G \to W^r_m G_1$ (that $W^r_m\nu$ is a Lie group homomorphism follows from the formula on prolongation group multiplication from Section 15 in [2]). Now, given a $\mathcal{PB}_m$-object $P \to M$ with the structure Lie group G, $W^r P$ is again a $\mathcal{PB}_m$-object with the structure Lie group $W^r_m G$ (see Section 15 in [2]). Moreover, given a $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ and with the Lie group homomorphism $\nu_f\colon G \to G_1$, $W^r f\colon W^r P \to W^r P_1$ is a principal bundle homomorphism with the Lie group homomorphism $W^r_m\nu_f\colon W^r_m G \to W^r_m G_1$ (which follows from the formula on the principal prolongation bundle right actions from Section 15 in [2]). The above fact is a particular case of a more general result of [1], too.

Suppose we have a system (F,α) consisting of a regular functor $F\colon \mathcal{G}r \to \mathcal{M}f$ sending a Lie group G into a manifold FG and a Lie group homomorphism ν: $G \to G_1$ into an induced map $F\nu\colon FG \to FG_1$, and of a family α of smooth left actions $\alpha_G\colon W^r_m G \times FG \to FG$ for any Lie group G. The regularity means that F transforms smoothly parametrized families of Lie group homomorphisms into smoothly parametrized families of maps.

Definition 1. A system (F,α) as above is called an (r,m)-system if for any Lie group homomorphism $\nu\colon G \to G_1$ the map $F\nu\colon FG \to FG_1$ is $(W^r_m G, W^r_m G_1)$-invariant over $W^r_m\nu\colon W^r_m G \to W^r_m G_1$, i.e., $F\nu(g\cdot v) = W^r_m\nu(g)\cdot F\nu(v)$ for any $v \in FG$ and any $g \in W^r_m G$.

The system (W^r_m,β) consisting of the functor $W^r_m\colon \mathcal{G}r \to \mathcal{G}r$ (mentioned above) treated as the functor $W^r_m\colon \mathcal{G}r \to \mathcal{M}f$ and the collection β of actions $\beta_G\colon W^r_m G \times W^r_m G \to W^r_m G$ (defined by the prolongation group multiplication) for any Lie group G is an example of an (r,m)-system.

Given an (r,m)-system (F,α) we can construct a gauge bundle functor $E^{(F,\alpha)}\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r as follows.

Example 1. For any $\mathcal{PB}_m$-object P with the structure Lie group G we put

$$E^{(F,\alpha)}P = W^r P[FG,\alpha_G].$$

For any $\mathcal{PB}_m$-map $f\colon P \to P_1$ with the homomorphism $\nu_f\colon G \to G_1$ we put

$$E^{(F,\alpha)}f = W^r f[F\nu_f]\colon W^r P[FG,\alpha_G] \to W^r P_1[FG_1,\alpha_{G_1}].$$

If $\mu\colon (F,\alpha) \to (F^1,\alpha^1)$ is a homomorphism of (r,m)-systems (i.e., $\mu\colon F \to F^1$ is a functor transformation such that $\mu_G\colon FG \to F^1G$ is a smooth W^r_mG-invariant map for any Lie group G) we have a natural transformation $\eta^{(\mu)}\colon E^{(F,\alpha)} \to E^{(F^1,\alpha^1)}$ given by

$$\eta^{(\mu)}_P := W^r(\mathrm{id}_P)[\mu_G]\colon E^{(F,\alpha)}P \to E^{(F^1,\alpha^1)}P$$

for any $\mathcal{PB}_m$-object $P \to M$ with the structure Lie group G.

Conversely, suppose we have a gauge bundle functor $E\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r. We construct an (r,m)-system $(F^{(E)},\alpha^{(E)})$ as follows.

Example 2. We define a functor $F^{(E)}\colon \mathcal{G}r \to \mathcal{M}f$ by

$$F^{(E)}G := E_0(\mathbb{R}^m \times G) \quad \text{and} \quad F^{(E)}\nu := E_0(\mathrm{id}_{\mathbb{R}^m} \times \nu)$$

for any Lie group G and any Lie group homomorphism $\nu\colon G \to G_1$. For any Lie group G we define an action $\alpha^{(E)}_G\colon W^r_mG \times F^{(E)}G \to F^{(E)}G$ by

$$\alpha^{(E)}_G(g,v) = E_0\varphi(v), \quad g = j^r_{(0,e)}\varphi \in W^r_mG, \quad v \in F^{(E)}G$$

(we identify elements of W^r_mG with r-jets at 0 of (local) principal bundle isomorphisms with id_G as Lie group homomorphisms and covering embeddings preserving 0 as in Section 15 in [2]).

If $\eta\colon E \to E^1$ is a natural transformation of gauge bundle functors $E, E^1\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r we have a homomorphism $\mu^{(\eta)}\colon (F^{(E)},\alpha^{(E)}) \to (F^{(E^1)},\alpha^{(E^1)})$ of (r,m)-systems given by

$$\mu^{(\eta)}_G := (\eta_{\mathbb{R}^m \times G})_0\colon F^{(E)}G \to F^{(E^1)}G.$$

Clearly, the above constructions from Examples 1 and 2 are mutually inverse. In particular, a $\mathcal{PB}_m$-natural isomorphism $\Theta\colon E^{(F^{(E)},\alpha^{(E)})} \to E$ can be given by

$$\Theta_P\colon E^{(F^{(E)},\alpha^{(E)})}P \to EP,\ \Theta_P([g,v]) := E\varphi(v), \quad g = j^r_0\varphi \in W^rP, \quad v \in F^{(E)}G$$

for any $\mathcal{PB}_m$-object $P \to M$ with the structure Lie group G (we identify elements of W^rP with r-jets at 0 of (local) principal bundle isomorphisms $\mathbb{R}^m \times G \to P$ with id_G as the Lie group homomorphisms as in Section 15 in [2]).

In general, categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are weak equivalent if there are functors $H_1\colon \mathcal{K}_1 \to \mathcal{K}_2$ and $H_2\colon \mathcal{K}_2 \to \mathcal{K}_1$ such that $H_2 \circ H_1 \cong \mathrm{id}_{\mathcal{K}_1}$ and $H_1 \circ H_2 \cong \mathrm{id}_{\mathcal{K}_2}$.

Thus we have proved the following theorem.

Theorem 1. *The category of gauge bundle functors $E\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r and their natural transformations is weak equivalent to the category of (r,m)-systems (F,α) and their homomorphisms.*

2. The case of fiber product preserving gauge bundle functors on $\mathcal{PB}_m$ of order r

Many important gauge bundle functors on $\mathcal{PB}_m$ are fiber product preserving. We present several examples of such functors.

(a) The functor $J^r\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ into its r-jet prolongation bundle $J^rP = \{j^r_x\sigma;\ \sigma\colon M \to P$ is a locally defined section of $P \to M\}$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ into $J^rf\colon J^rP \to J^rP_1$ (given by $J^rf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$) is a fiber product preserving gauge bundle functor of order r.

(b) The functor $J^r_v\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ into its vertical r-jet prolongation bundle $J^r_vP = \{j^r_x\sigma;\ \sigma\colon M \to P_x\}$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ into $J^r_vf\colon J^r_vP \to J^r_vP_1$, given by $J^r_vf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f_x \circ \sigma \circ \underline{f}^{-1})$, is a fiber product preserving gauge bundle functor of order r.

(c) Let A be a Weil algebra of order r. The functor $V^A\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ into its A-vertical bundle $V^AP = \bigcup\limits_{x\in M} T^AP_x$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ into $V^Af = \bigcup\limits_{x\in M} T^A(f_x)\colon V^AP \to V^AP_1$ is a fiber product preserving gauge bundle functor of order r. In particular, if $A = \mathbf{D}$ is the algebra of dual numbers, then $T^A = T$ is the tangent functor and $V^A = V\colon \mathcal{PB}_m \to \mathcal{FM}$ is the vertical functor.

(d) The above functors are particular cases of product preserving bundle functors $E\colon \mathcal{FM}_m \to \mathcal{FM}$ applied to $\mathcal{PB}_m$-objects and $\mathcal{PB}_m$-maps treated as $\mathcal{FM}_m$-objects and $\mathcal{FM}_m$-maps, respectively. The full description of fiber product preserving bundle functors $E\colon \mathcal{FM}_m \to \mathcal{FM}$ can be found in [3].

(e) Let $E\colon \mathcal{FM}_m \to \mathcal{FM}$ be a fiber product preserving bundle functor. The right action of the structure Lie group G on an $\mathcal{PB}_m$-object P (treated as an $\mathcal{FM}_m$-object) induces (in an obvious way) a right action of G on EP. Thus we have the functor $Q^E\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ into $Q^EP := EP/G$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ into the quotient $Q^Ef\colon Q^EP \to Q^EP_1$ of $Ef\colon EP \to EP_1$. The functor $Q^E\colon \mathcal{PB}_m \to \mathcal{FM}$ is again a fiber product preserving gauge bundle functor. In particular, if $E = J^1$, then $Q^E = Q\colon \mathcal{PB}_m \to \mathcal{FM}$ is the principal connection bundle functor mentioned in the introduction. Below, we consider the right invariant vertical vector field functor $Q^V\colon \mathcal{PB}_m \to \mathcal{FM}$ (sections of $Q^V P$ are in bijection with right invariant vertical vector fields on P). In fact, $Q^V\colon \mathcal{PB}_m \to \mathcal{VB}$.

(f) Let $G\colon \mathcal{M}f_m \to \mathcal{VB}$ be a vector bundle functor (for example the p-form bundle functor $\wedge^p T^*\colon \mathcal{M}f_m \to \mathcal{VB}$). Thus we have the functor $\mathcal{K}^G\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ into (vector bundle) $\mathcal{K}^G P := GM \otimes Q^V P$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ into (vector bundle) map $\mathcal{K}^G f := G\underline{f} \otimes Q^V f\colon \mathcal{K}^G P \to \mathcal{K}^G P_1$. The functor $\mathcal{K}^G\colon \mathcal{PB}_m \to \mathcal{FM}$ is a fiber product preserving gauge bundle functor. In fact, $\mathcal{K}^G\colon \mathcal{PB}_m \to \mathcal{VB}$. In particular, if $G = \wedge^2 T^*\colon \mathcal{M}f_m \to \mathcal{VB}$ we obtain the principal connection curvature functor $\mathcal{K}^G = \mathcal{K}\colon \mathcal{PB}_m \to \mathcal{FM}$ (the curvature tensor of a principal connection on P can be treated as a section of $\mathcal{K}P$).

(g) Let $E\colon \mathcal{FM}_m \to \mathcal{FM}$ be a fiber product preserving bundle functor. Thus we have the functor $E'\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ with the structure Lie group G into $E'P := E(M \times G)$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ and with the Lie group homomorphism $\nu_f\colon G \to G_1$ into $E'f := E(\underline{f} \times \nu_f)\colon E'P \to E'P_1$. The functor $E'\colon \mathcal{PB}_m \to \mathcal{FM}$ is a fiber product preserving gauge bundle functor.

(h) Let $E\colon \mathcal{VB}_m \to \mathcal{FM}$ be a fiber product preserving gauge bundle functor (a full description of such functors can be found in [5]). Thus we have the functor $E^o\colon \mathcal{PB}_m \to \mathcal{FM}$ sending any $\mathcal{PB}_m$-object $P \to M$ with the structure Lie group G with the Lie algebra $\mathcal{L}(G)$ into $E^o P := E(M \times \mathcal{L}(G))$ and any $\mathcal{PB}_m$-map $f\colon P \to P_1$ covering $\underline{f}\colon M \to M_1$ with the Lie group homomorphism $\nu_f\colon G \to G_1$ into $E^o f := (\underline{f} \times \mathcal{L}(\nu_f))\colon E^o P \to E^o P_1$. The functor $E^o\colon \mathcal{PB}_m \to \mathcal{FM}$ is a fiber product preserving gauge bundle functor.

Definition 2. An (r, m)-system (F, α) is product preserving if $F\colon \mathcal{G}r \to \mathcal{M}f$ is product preserving.

It is easily seen that if (F, α) is a product preserving (r, m)-system, then the gauge bundle functor $E^{(F,\alpha)}\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r (from Example 1) is fiber product preserving. Conversely, if $E\colon \mathcal{PB}_m \to \mathcal{FM}$ is a fiber product preserving gauge bundle functor of order r, then the (r, m)-system $(F^{(E)}, \alpha^{(E)})$ (from Example 2) is product preserving. Thus we have proved the following fact.

Theorem 2. *The category of fiber product preserving gauge bundle functors $E\colon \mathcal{PB}_m \to \mathcal{FM}$ of order r and their natural transformations is weak equivalent to the category of product preserving (r, m)-systems (F, α) and their homomorphisms.*

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