Czechoslovak Mathematical Journal, 68 (143) (2018), 491–496

EVERY 2-GROUP WITH ALL SUBGROUPS NORMAL-BY-FINITE IS LOCALLY FINITE

Enrico Jabara, Venezia

Received September 30, 2016. First published February 14, 2018.

Abstract. A group G has all of its subgroups normal-by-finite if H/H_G is finite for all subgroups H of G. The Tarski-groups provide examples of p-groups (p a "large" prime) of nonlocally finite groups in which every subgroup is normal-by-finite. The aim of this paper is to prove that a 2-group with every subgroup normal-by-finite is locally finite. We also prove that if $|H/H_G| \leqslant 2$ for every subgroup H of G, then G contains an Abelian subgroup of index at most 8.

Keywords: 2-group; locally finite group; normal-by-finite subgroup; core-finite group

MSC 2010: 20F50, 20F14, 20D15

1. Introduction

A group G is a **CF**-group (core-finite) if each of its subgroups is normal-by-finite, that is, H/H_G is finite for all $H \leqslant G$. The subgroup H_G, the core of H in G, defined by

$$H_G = \bigcap_{g \in G} H^g,$$

is the biggest normal subgroup of G contained in H.

If G is a **CF**-group, we denote by $\sigma(G)$ the $\sup\limits_{H \leqslant G} |H : H_G|$ and we say that G is **BCF** (boundedly core-finite) if $\sigma(G) < \infty$.

The main Theorem in [1] states that every locally finite **CF**-group is Abelian-by-finite and **BCF**. In order to prove the previous result, the hypothesis that G is locally finite is essential, as indicated by the existence of so-called Tarski-groups, for instance the examples due to Rips and Ol'shanskii (see [5]) of infinite groups all of whose nontrivial subgroups have prime order.

The aim of this paper is to provide an elementary proof of the following results.

Theorem A. *Let G be a 2-group. If G is a* **CF**-*group, then G is locally finite.*

As a consequence of Theorem A and the main result of [1] we have that every **CF** 2-group is Abelian-by-finite and **BCF**.

In the particular case of 2-groups G with $\sigma(G) \leqslant 2$ we can provide a proof that G is locally finite and hypercentral in a direct way (see Lemma 8). We also extend to all groups a result proved in [7] for finite groups.

Theorem B. *Let G be a group. If $\sigma(G) \leqslant 2$, then G contains an Abelian subgroup of index at most 8.*

2. Proofs

A quasicyclic 2-group is a group of type C_{2^∞}, that is, an Abelian group with generators $x_1, x_2, \ldots, x_n, \ldots$ and defining relations $x_1^2 = 1$ and $x_{i+1}^2 = x_i$ for all $i > 1$. A group of type C_{2^∞} is not finitely generated and has every proper subgroup cyclic of infinite index; in particular, if a **CF**-group G contains a quasicyclic 2-group C, then $C \trianglelefteq G$.

Let G be a group and let Λ be a well-ordered set; an indexed set $(G_\lambda)_{\lambda\in\Lambda}$ of subgroups of G is an ascending series if whenever $\lambda, \mu \in \Lambda$ and $\lambda \leqslant \mu$, then $G_\lambda \leqslant G_\mu$.

Lemma 1. *If a group G is the union of an ascending series $(G_\lambda)_{\lambda\in\Lambda}$ of locally finite groups, then G is locally finite.*

P r o o f. Trivial (see the proof of Lemma 1.A.2 in [3]). □

Lemma 2. *Every periodic hyperabelian group is locally finite.*

P r o o f. It is clear that a periodic Abelian group is locally finite. Since extensions of locally finite groups by locally finite groups are locally finite (Lemma 1.A.2 of [3] or 14.3.1 of [6]), the conclusion follows by Lemma 1. □

The following result is the key to prove our Theorem A.

Lemma 3. *Let G be an infinite 2-group containing no subgroups of type C_{2^∞}. Then the centralizer of every finite subgroup of G is infinite.*

P r o o f. This is Theorem C of [4]. □

P r o o f of Theorem A. Let G be an infinite 2-group which is **CF**. Since every quotient of a **CF**-group is trivially a **CF**-group, in order to prove that G is locally finite, by Lemma 2 it suffices to prove that G contains a nontrivial normal Abelian subgroup. We must distinguish between two cases.

(I) G contains a subgroup A of type C_{2^∞}.

Since $|A : A_G|$ is finite and A does not contain proper subgroups of finite index, we have $A_G = A$ and hence A is a nontrivial normal Abelian subgroup of G.

(II) G contains no subgroups of type C_{2^∞}.

By the Zorn's Lemma in G there is a maximal Abelian subgroup A. If A is finite, then by Lemma 3, $C_G(A)$ is infinite. Let $g \in C_G(A) \setminus A$, then clearly $\langle A, g\rangle$ is Abelian, against the hypothesis that A is maximal. Hence A is infinite and since $|A : A_G| < \infty$, we have that A_G is infinite. So A_G is a nontrivial normal Abelian subgroup of G.

This proves Theorem A. □

Remark 4. Let G be a finite 2-group.

(a) If $\sigma(G) = 1$, then G is Abelian or Hamiltonian (for the structure of Hamiltonian groups see 5.3.7 of [6]), in particular, G contains an Abelian normal subgroup of index 2.

(b) In [7] it is proved that if $\sigma(G) \leqslant 2$, then G contains an Abelian normal subgroup of index at most 4 and this bound is sharp.

(c) In [2] it is proved that if $\sigma(G) \leqslant 2^s$, then G contains an Abelian subgroup of index at most 2^t with

$$t \leqslant \tfrac{1}{2}(s+1)(2s^3+7s^2+9s+2)(s^3+2s^2+3s+3).$$

Using an inverse limit argument (see Proposition 1.K.2 of [3] and the Remark that follows) it is not difficult to check that all previous results are valid if G is locally finite, and hence, by our Theorem A, for all 2-groups.

By proving Theorem B we extend the Corollary of [7] to infinite groups. In the proof we freely make use of the fact that if G is a **BCF**-group and $H \leqslant G$, then

$$\sigma(H) \leqslant \sigma(G),$$

moreover if $H \trianglelefteq G$, then

$$\sigma(G/H) \leqslant \sigma(G).$$

Lemma 5. *Let G be a group with $\sigma(G) \leqslant 2$ and let A be the subgroup of G generated by all infinite cyclic normal subgroups and by all cyclic subgroups of odd order. Then A is Abelian.*

P r o o f. Finite odd-order subgroups of G are normal and also Dedekind groups, which implies that the elements of G of odd order generate an abelian normal subgroup of G.

Let $x \in G$ with $o(x) = \infty$ and $\langle x \rangle \trianglelefteq G$. The only nonidentity element α of $\mathrm{Aut}(\langle x \rangle)$ inverts x; if $a \in G$ and $x^a = x^{-1}$, then $\langle x \rangle \cap \langle a \rangle \leqslant C_{\langle x \rangle}(\alpha) = 1$. Thus, x is centralized by every odd order element and by every infinite cyclic normal subgroup of G. This shows that A is abelian. $\square$

Lemma 6. *Let G be a nonperiodic group with $\sigma(G) \leqslant 2$ and let A be the subgroup generated by all normal infinite cyclic subgroups of G and by all cyclic subgroups of odd order of G. Then for each $g \in G$ either $a^g = a$ for all $a \in A$ or $a^g = a^{-1}$ for all $a \in A$. Therefore $|G : C_G(A)| \leqslant 2$ and moreover $g \in C_G(A)$ for every $g \in G$ of infinite order.*

P r o o f. By Lemma 5, A is Abelian and hence if $g \in G$ induces by conjugation a nontrivial automorphism on A, then $g^2 \in A$.

Let $x \in G$ with $o(x) = \infty$ and let $u \in A$ of odd order. We have $\langle x^2 \rangle \trianglelefteq G$, hence $[u, x^2] = 1$ and we can write $(x^2u)^2 = x^4u^2$, so $o(x^2u) = \infty$ and $\langle (x^2u)^2 \rangle \trianglelefteq G$. Now

$$\langle u \rangle = \langle u^2 \rangle \leqslant \langle x^2, (x^2u)^2 \rangle.$$

Accordingly, if G is a nonperiodic group, then the subgroup A is the one generated by the normal infinite cyclic subgroups of G. Let $\langle x \rangle \trianglelefteq G$ and $\langle y \rangle \trianglelefteq G$ with $o(x) = \infty = o(y)$. If there is $g \in G$ such that $x^g = x^{-1}$ and $y^g = y$, then $\langle x, y \rangle = \langle x \rangle \times \langle y \rangle$, so $o(xy) = \infty$, but $\langle (xy)^2 \rangle$ is not normal in G. This shows that $a^g = a^{-1}$ for every $a \in A$ and hence $|G : C_G(A)| \leqslant 2$. $\square$

Remark 7. From Lemmas 5 and 6 it follows that every group G with $\sigma(G) \leqslant 2$ and without elements of order 2 is Abelian.

Lemma 8. *Let G be a 2-group. If $\sigma(G) \leqslant 2$, then G is hypercentral.*

P r o o f. Suppose $G \neq 1$. If G has no elements of order 4, then G is elementary Abelian and the claim is proved. If there is an element $g \in G$ with $o(g) = 4$, then $\langle g^2 \rangle$ is a normal subgroup of order 2 of G and hence $g^2 \in Z(G)$, so $Z(G) \neq 1$. $\square$

Lemma 9. *Let G be a hypercentral periodic group. If $\sigma(G) \leqslant 2$, then G contains an Abelian normal subgroup of index at most 4.*

P r o o f. Since G is hypercentral, it is locally finite and, by Lemma 5, we can write $G = R_1 \times R_2$ with R_1 an Abelian $2'$-group and R_2 a locally finite (hypercentral) group. By Remark 4(b), R_2 contains a normal Abelian subgroup S of index at most 4. We can conclude that $R_1 \times S$ is a normal Abelian subgroup of index at most 4 in G.

$\square$

P r o o f of Theorem B. Suppose that G is a periodic group and let A be the subgroup of G generated by all elements of odd order. By Lemma 5, A is Abelian, and hence locally finite. Since G/A is a 2-group with $\sigma(G/A) \leqslant 2$, it is locally finite and hence we can deduce that G is locally finite. By Corollary of [7], every finite group X with $\sigma(X) \leqslant 2$ contains an Abelian subgroup of index at most 8, applying Proposition 1.K.2 of [3] we obtain that also G has an Abelian subgroup of index at most 8. So in this case the claim is proved.

From now we suppose that G contains elements of infinite order.

Let A be the subgroup of G generated by all normal infinite cyclic subgroups of G and by all cyclic subgroups of odd order of G and let $C = C_G(A)$ (by the proof of Lemma 6, A is actually generated by the normal infinite cyclic subgroups of G). By construction we have that A is normal and G/A is a 2-group, by Lemma 5 we have that A is Abelian and by Lemma 6 that $|G : C| \leqslant 2$.

First of all we prove that if $G = C$, then G contains a normal Abelian subgroup of index at most 4. In this case G is hypercentral because G/A is a 2-group, which is hypercentral by Lemma 8, and $A \leqslant Z(G)$ by hypothesis.

Let $a \in A$ be such that $\langle a \rangle \trianglelefteq G$ and $o(a) = \infty$.

Let $h \in G$, then $[h, a] = 1$, moreover if $o(h) < \infty$, then $o(ah) = \infty$ and $a^2h^2 = (ah)^2 \in A$, so $h^2 \in A$. Hence for every $g \in G$ we have $g^2 \in A$ and G/A is an elementary Abelian 2-group. Since $A \leqslant Z(G)$ for $g, h \in G$, we can write

$$[g, h]^2 = [g^2, h] = 1$$

and this shows G' to be an elementary Abelian 2-subgroup of A. By the Zorn's Lemma in A there is a free Abelian subgroup of maximal rank Q, moreover we can suppose $Q \trianglelefteq G$ because since $|Q : Q_G| \leqslant 2$, we can consider Q_G instead of Q. Then A/Q is an Abelian periodic group. Let T be the full preimage in G of the $2'$-Hall subgroup of A/Q, then T does not have elements of finite even order and G/T is a 2-group. Moreover, $G' \cap T = 1$ and hence the full preimage of any Abelian subgroup of G/T is an Abelian subgroup of G. Since G/T is periodic and hypercentral, then, by Lemma 9, it contains a normal Abelian subgroup S/T of index at most 4. The full preimage S of S/T in G is a normal Abelian subgroup of index at most 4 of G.

Suppose now $|G : C| = 2$. For the previous argument we know that in C there is an Abelian (normal) subgroup S of index at most 4 and hence $|G : S| \leqslant 8$. $\square$

Remark 10. The bound found in Theorem B is best possible, as has been shown in the proof of Corollary of [7].

Example (i). Let $G = \langle x_i,\ y;\ x_i^y = x_i^{-1}\rangle$ be the semidirect product of the quasicyclic 2-group $\langle x_1, x_2, \ldots, x_n, \ldots\rangle$ by the infinite cyclic group $\langle y\rangle$, then G is a nonperiodic **CF**-group. The element $x_n y^2 \in G$ has infinite order and it is easy to prove that $|\langle x_n y^2\rangle : \langle x_n y^2\rangle_G| = 2^{n-1}$, so G is not a **BCF**-group. This shows that the main theorem of [1] cannot be extended to nonperiodic groups.

Remark 11. Let G be a group and suppose that for every $g \in G$ we have $|\langle g\rangle : \langle g\rangle_G| \leqslant 2$. Then the subgroup $N = \langle g^2;\ g \in G\rangle$ is Abelian or Hamiltonian and G/N has exponent 2, so it is elementary Abelian. Hence $G' \leqslant N$ and $|G''| \leqslant 2$ (in particular if G is periodic, then it is locally finite), but it can happen that $\sigma(G) = \infty$, as shown in Example (ii).

Example (ii). Let

$$G = \bigoplus_{i\in\mathbb{N}} D_i$$

with $D_i = \langle x_i, y_i; x_i^4, y_i^2, x_i^{y_i} x_i\rangle$ isomorphic to the dihedral group of order 8. It is clear that if $g \in G$, then $|\langle g\rangle : \langle g\rangle_G| \leqslant 2$, but G does not possess Abelian subgroups of finite index. Moreover, if $n \in \mathbb{N}$, the elementary Abelian subgroup $H = \langle y_1, y_2, \ldots, y_n\rangle$ of G is such that $|H| = 2^n$ and $H_G = 1$, in particular $\sigma(G) = \infty$.

Acknowledgments. The author would like to thank the referee for many helpful suggestions.

References

[1] *J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith, J. Wiegold*: Groups with all subgroups normal-by-finite. J. Aust. Math. Soc., Ser. A *59* (1995), 384–398. zbl MR
[2] *G. Cutolo, E. I. Khukhro, J. C. Lennox, S. Rinauro, H. Smith, J. Wiegold*: Locally finite groups all of whose subgroups are boundedly finite over their cores. Bull. Lond. Math. Soc. *29* (1997), 563–570. zbl MR doi
[3] *O. H. Kegel, B. A. F. Wehrfritz*: Locally Finite Groups. North-Holland Mathematical Library 3, North-Holland Publishing, Amsterdam, 1973. zbl MR
[4] *J. C. Lennox, A. Mohammadi Hassanabadi, A. G. R. Stewart, J. Wiegold*: Nilpotent extensibility and centralizers in infinite 2-groups. Proceedings of the Second International Group Theory Conference (Bressanone, 1989). Rend. Circ. Mat. Palermo (2) Suppl. No. 23, 1990, pp. 209–219. zbl MR
[5] *A. Yu. Ol'shanskiĭ*: Geometry of Defining Relations in Groups. Mathematics and Its Applications. Soviet Series 70, Kluwer Academic Publishers, Dordrecht, 1991. zbl MR doi
[6] *D. J. S. Robinson*: A Course in the Theory of Groups. Graduate Texts in Mathematics 80, Springer, New York, 1996. zbl MR doi
[7] *B. Wilkens*: More on core-2 2-groups. J. Group Theory *20* (2017), 193–225. zbl MR doi

Author's address: Enrico Jabara, DFBC—Università di Venezia, Dorsoduro 3484/D—30123 Venezia, Italy, e-mail: `jabara@unive.it`.