

SOME RESULTS ON THE ANNIHILATOR GRAPH
OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring. The annihilator graph of R , denoted by $\text{AG}(R)$, is the undirected graph with all nonzero zero-divisors of R as vertex set, and two distinct vertices x and y are adjacent if and only if $\text{ann}_R(xy) \neq \text{ann}_R(x) \cup \text{ann}_R(y)$, where for $z \in R$, $\text{ann}_R(z) = \{r \in R : rz = 0\}$. In this paper, we characterize all finite commutative rings R with planar or outerplanar or ring-graph annihilator graphs. We characterize all finite commutative rings R whose annihilator graphs have clique number 1, 2 or 3. Also, we investigate some properties of the annihilator graph under the extension of R to polynomial rings and rings of fractions. For instance, we show that the graphs $\text{AG}(R)$ and $\text{AG}(T(R))$ are isomorphic, where $T(R)$ is the total quotient ring of R . Moreover, we investigate some properties of the annihilator graph of the ring of integers modulo n , where $n \geq 1$.

Keywords: annihilator graph; zero-divisor graph; outerplanar; ring-graph; cut-vertex; clique number; weakly perfect; chromatic number; polynomial ring; ring of fractions

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1. INTRODUCTION

Let R be a commutative ring with nonzero identity. We denote the sets of all zero-divisors and nilpotent elements of R by $Z(R)$ and $\text{Nil}(R)$, respectively. In 1999, Anderson and Livingston introduced the zero-divisor graph of R , denoted by $\Gamma(R)$, that is the graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$ and distinct vertices x and y being adjacent in $\Gamma(R)$ if and only if $xy = 0$. Beck introduced this concept in 1988 but he allowed all the elements of R as vertices and was mainly interested in colorings. Several other classes of graphs associated with algebraic structures have been defined and studied (cf. [2], [6], [10], [14], [13], [22]). One of the most important class of graphs associated with the algebraic structures is that of Cayley graphs (cf. [19],

[20], [21]). Recently, in [12], the concept of the annihilator graph has been defined and studied. The annihilator graph of R , denoted by $\text{AG}(R)$, is an undirected graph with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\text{ann}_R(xy) \neq \text{ann}_R(x) \cup \text{ann}_R(y)$, where for $z \in R$, $\text{ann}_R(z) = \{r \in R: rz = 0\}$. Let G be an additive abelian group and let S be a symmetric subset of G . The Cayley graph $\text{Cay}(G, S)$ is the graph with vertex set G and two vertices x and y are adjacent if and only if $x - y \in S$. It is easy to see that the induced subgraph of the Cayley graph $\text{Cay}(R, S)$, where $S = R \setminus Z(R)$, with vertex set $Z(R)^*$ is a subgraph of the annihilator graph $\text{AG}(R)$. Indeed, assume that $x - y \in S$, and suppose on the contrary that x and y are not adjacent in $\text{AG}(R)$. Then, by [12], Lemma 2.1, we have $\text{ann}_R(x) \subseteq \text{ann}_R(y)$ or $\text{ann}_R(y) \subseteq \text{ann}_R(x)$. Without loss of generality, we may assume that $\text{ann}_R(x) \subseteq \text{ann}_R(y)$. So $x - y \in Z(R)$, which is a contradiction.

By [12], Lemma 2.1, the zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\text{AG}(R)$. Many results on zero-divisor graphs of commutative rings have been obtained (cf. [3], [4], [5], [8], [9], [15], [17]). Let $\text{AG}_N(R)$ be the (induced) subgraph of $\text{AG}(R)$ with vertices $\text{Nil}(R)^* = \text{Nil}(R) \setminus \{0\}$. Recall that R is reduced if $\text{Nil}(R) = 0$. Also, $\text{Min}(R)$ is the set of all minimal prime ideals of R . In [2], the authors studied the situations that the unit, unitary and total graphs are ring-graph or outerplanar. Also, in [1], they studied the ring-graph and outerplanarity for co-maximal and zero-divisor graphs. In the second section of this paper, we completely characterize all finite commutative rings with planar or outerplanar or ring-graph annihilator graphs. In the third section we characterize all finite commutative rings R , whose annihilator graphs have clique number 1, 2 or 3. In the fourth section, we investigate the annihilator graph of the extension of R to polynomial rings and rings of fractions. Also, we show that the graphs $\text{AG}(R)$ and $\text{AG}(T(R))$ are isomorphic, where $T(R)$ is the total quotient ring of R . Finally, in the fifth section, we investigate some properties of the annihilator graph of the ring of integers modulo n , where $n \geq 1$. For instance, we study cut-vertices and cut-sets in $\text{AG}(\mathbb{Z}_n)$.

Now, we recall some definitions and notation on graphs. Let G be a simple graph with vertex set $V(G)$ and let C be a cycle of G . A chord in G is any edge joining two nonadjacent vertices in C . A primitive cycle is a cycle without chord. Moreover, if any two primitive cycles intersect in at most one edge, then we say G has the primitive cycle property (PCP). The number of primitive cycles of G is the free rank of G and is denoted by $\text{frank}(G)$. We have $\text{rank}(G) := q - n + r$, where q , n and r are the number of edges of G , the number of vertices of G and the number of connected components of G , respectively.

A graph G is called planar if it can be drawn in the plane without crossing edges. A graph G is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing.

The precise definition of a ring-graph can be found in section 2 of [18]. Also, in [18], the authors showed that the following conditions are equivalent:

- (i) G is a ring-graph,
- (ii) $\text{rank}(G) = \text{frank}(G)$,
- (iii) G satisfies PCP and G does not contain a subdivision of K^4 as a subgraph.

So every ring-graph is planar. Moreover, in [18], authors showed that every outer-planar graph is a ring-graph. A set $A \subset V(G)$ is said to be a cut-set if its removal increases the number of connected components of G and no proper subset of A satisfies the same condition. A cut-set consisting of only one element is called a cut-vertex of G . Suppose that $x, y \in V(G)$. If x is adjacent to y , then y is a neighbour of x . We use the notation $x - y$ to say that x and y are adjacent in a graph G . The girth of G , denoted by $\text{gr}(G)$, is the length of a shortest cycle in G ($\text{gr}(G) = \infty$ if G contains no cycles). Also we denote the complete graph with n vertices by K^n and we denote the complete bipartite graph by $K^{m,n}$. We denote the star graph by $K^{1,n}$. Let k be a positive integer. For a graph G , a k -coloring of the vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The chromatic number of G , denoted by $\chi(G)$, is the smallest number k such that G admits a k -coloring. Any subgraph of G is called a clique if it is complete and the size of a largest clique in a graph G is denoted by $\text{cl}(G)$. A graph G is called weakly perfect provided $\chi(G) = \text{cl}(G)$ (cf. [23]).

2. RING-GRAPHS AND OUTERPLANAR ANNIHILATOR GRAPHS

In this section, we investigate all finite commutative rings R such that their annihilator graphs are planar or outerplanar or ring-graph. Throughout this section, R is a finite commutative ring with nonzero identity and \mathbb{F} is a finite field. Specially, \mathbb{F}_4 is a field with four elements.

Theorem 2.1. *The annihilator graph $\text{AG}(R)$ is planar if and only if R is isomorphic to one of the following rings:*

- (i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
- (ii) $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$,
- (iii) \mathbb{Z}_4 , $\mathbb{Z}_2[x]/(x^2)$, \mathbb{Z}_8 , $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2[x, y]/(x, y)^2$, \mathbb{Z}_9 ,
 $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$, \mathbb{Z}_{25} , $\mathbb{Z}_5[x]/(x^2)$.

Proof. Clearly, by [12], Lemma 2.1, the zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\text{AG}(R)$. Hence if $\Gamma(R)$ is not planar, then $\text{AG}(R)$ is not planar either. So in order to investigate the planarity of $\text{AG}(R)$, we need only

to study the rings R whose zero-divisor graphs are planar. In [3] and [16] it was shown that $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:

- (i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$,
- (ii) $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$,
- (iii) \mathbb{Z}_4 , $\mathbb{Z}_2[x]/(x^2)$, \mathbb{Z}_8 , $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, \mathbb{Z}_9 , $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 - 2)$, $\mathbb{Z}_4[x]/(x^2 + 2x + 2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$, \mathbb{Z}_{25} , $\mathbb{Z}_5[x]/(x^2)$, \mathbb{Z}_{16} , $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x, y]/(x^2 - y^2, xy)$, $\mathbb{Z}_2[x, y]/(x^2, y^2)$, $\mathbb{Z}_4[x]/(2x, x^3 - 2)$, $\mathbb{Z}_4[x, y]/(x^2 - 2, xy, y^2 - 2, 2x)$, $\mathbb{Z}_4[x, y]/(x^2, xy - 2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 - 2x)$, $\mathbb{Z}_8[x]/(2x, x^2 - 4)$, \mathbb{Z}_{27} , $\mathbb{Z}_9[x]/(x^2 - 3, 3x)$, $\mathbb{Z}_9[x]/(x^2 - 6, 3x)$, $\mathbb{Z}_3[x]/(x^3)$.

Now we study the planarity of $\text{AG}(R)$, when R is one of the above rings. By Figure 1, it is easy to see that $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is planar.

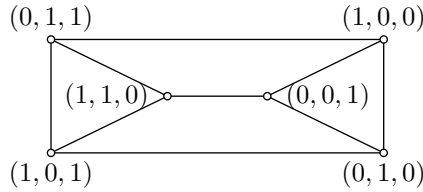


Figure 1. $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

In Figure 2, the graph $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ has a copy of $K^{3,3}$, and so it is not planar.

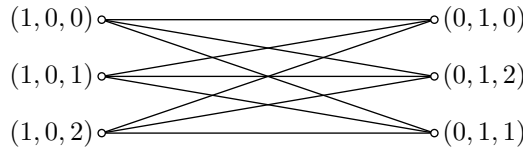


Figure 2.

If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}$ with $|\mathbb{F}| = m$, then $\text{AG}(R) \cong K^{1, m-1}$. Hence $\text{AG}(\mathbb{Z}_2 \times \mathbb{F})$ is planar. Also if R is isomorphic to $\mathbb{Z}_3 \times \mathbb{F}$ with $|\mathbb{F}| = m$, then one can easily check that $\text{AG}(R) \cong K^{2, m-1}$. Thus $\text{AG}(\mathbb{Z}_3 \times \mathbb{F})$ is planar.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, then we have $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Then, by Figure 3, it is obvious that $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is planar.

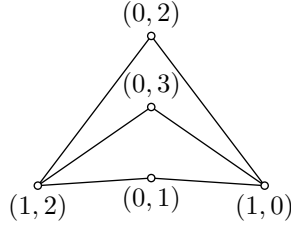


Figure 3. $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, then we have $\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_4) \cong \text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))$. Now, one can easily find a copy of $K^{3,3}$ with vertex set $\{(1,0), (2,0), (1,2), (0,1), (0,2), (0,3)\}$ in $\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_4)$ (see Figure 4), and so it is not planar.

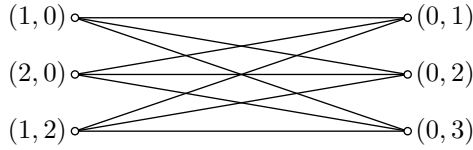


Figure 4.

If R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2 - 2)$, then we have $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_8) \cong \text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)) \cong \text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2 - 2))$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_8$. By Figure 5, the graph $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_8)$ has a subdivision of K^5 . So $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_8)$ is not planar.

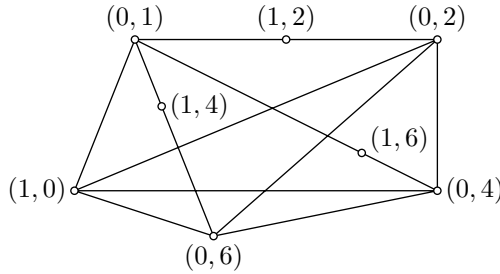


Figure 5.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_9$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, then we have $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_9) \cong \text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2))$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_9$. Then, by Figure 6, one can find a copy of $K^{3,3}$. Hence $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_9)$ is not planar.

Also, if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$, then $\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_9) \cong \text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2))$. Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_9$. Then one can find a copy of K^5 with vertex set $\{(0,3), (0,6), (1,3), (1,6), (2,3)\}$ in $\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_9)$, so $\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_9)$ is not planar.

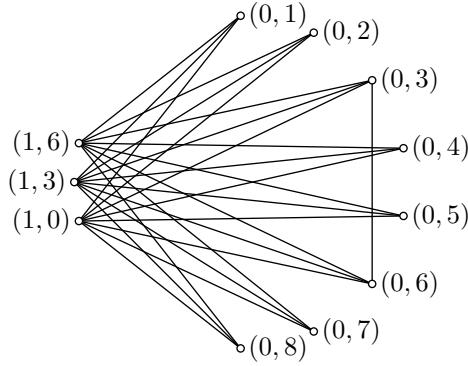


Figure 6. $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_9)$.

Now, we study the situation when R is a local ring. Badawi in [12] proved that $\text{AG}_N(R)$ is a complete graph. Since for finite local rings we have $Z(R) = \text{Nil}(R)$ if $|Z(R)| \geq 6$, hence $\text{AG}(R)$ contains a copy of K^5 , and so it is not planar. It is easy to see that the following rings have $|Z(R)| = 8$, and hence their annihilator graphs are not planar:

$$\begin{aligned} &\mathbb{Z}_{16}, \mathbb{Z}_2[x]/(x^4), \mathbb{Z}_2[x, y]/(x^2 - y^2, xy), \mathbb{Z}_2[x, y]/(x^2, y^2), \mathbb{Z}_4[x]/(2x, x^3 - 2), \\ &\mathbb{Z}_8[x]/(2x, x^2 - 4), \mathbb{Z}_4[x, y]/(x^2 - 2, xy, y^2 - 2, 2x), \mathbb{Z}_4[x, y]/(x^2, xy - 2, y^2), \\ &\mathbb{Z}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 - 2x), \mathbb{Z}_4[x]/(x^2 - 2), \mathbb{Z}_4[x]/(x^2 + 2x + 2). \end{aligned}$$

Also in the following rings we have $|Z(R)| = 9$, and hence their annihilator graphs are not planar:

$$\mathbb{Z}_{27}, \mathbb{Z}_9[x]/(x^2 - 3, 3x), \mathbb{Z}_9[x]/(x^2 - 6, 3x), \mathbb{Z}_3[x]/(x^3).$$

Now, one can easily check that the following isomorphisms hold:

$$\begin{aligned} \text{AG}(\mathbb{Z}_4) &\cong \text{AG}(\mathbb{Z}_2[x]/(x^2)) \cong K^1, \\ \text{AG}(\mathbb{Z}_8) &\cong \text{AG}(\mathbb{Z}_2[x]/(x^3)) \cong \text{AG}(\mathbb{Z}_4[x]/(2x, x^2 - 2)) \\ &\cong \text{AG}(\mathbb{Z}_2[x, y]/(x, y)^2) \cong \text{AG}(\mathbb{Z}_4[x]/(2x, x^2)) \\ &\cong \text{AG}(\mathbb{F}_4[x]/(x^2)) \cong \text{AG}(\mathbb{Z}_4[x]/(x^2 + x + 1)) \cong K^3, \\ \text{AG}(\mathbb{Z}_9) &\cong \text{AG}(\mathbb{Z}_3[x]/(x^2)) \cong K^2, \\ \text{AG}(\mathbb{Z}_{25}) &\cong \text{AG}(\mathbb{Z}_5[x]/(x^2)) \cong K^4. \end{aligned}$$

By the above discussion the result holds. □

In the next theorem, we characterize all rings with ring-graph annihilator graphs.

Theorem 2.2. *The annihilator graph $\text{AG}(R)$ is a ring-graph if and only if R is isomorphic to one of the following rings:*

- (i) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$,
- (ii) \mathbb{Z}_4 , $\mathbb{Z}_2[x]/(x^2)$, \mathbb{Z}_8 , $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2[x, y]/(x, y)^2$,
 $\mathbb{Z}_4[x]/(2x, x^2)$, \mathbb{Z}_9 , $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$.

Proof. Since every ring-graph is planar, it is enough to study the rings with planar annihilator graphs. Since

$$\text{frank}(\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 7 \quad \text{and} \quad \text{rank}(\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 4,$$

by Figure 1, $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not a ring-graph. If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, by Figure 3, we have

$$\text{rank}(\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 6 - 5 + 1 = 2 \quad \text{and} \quad \text{frank}(\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 3.$$

Thus $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is not a ring-graph. Also $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$, and so $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$ is not a ring-graph. If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}$, then it is easy to see that $\text{AG}(\mathbb{Z}_2 \times \mathbb{F})$ is a star graph. Hence $\text{AG}(\mathbb{Z}_2 \times \mathbb{F})$ is a ring-graph. If R is isomorphic to $\mathbb{Z}_3 \times \mathbb{F}$, then $\text{AG}(R)$ is isomorphic to $K^{2, m-1}$, where $|\mathbb{F}| = m$. Thus $\text{rank}(\text{AG}(\mathbb{Z}_3 \times \mathbb{F})) = m - 2$ and $\text{frank}(\text{AG}(\mathbb{Z}_3 \times \mathbb{F})) = (m - 1)(m - 2)/2$. Therefore $\text{AG}((\mathbb{Z}_3 \times \mathbb{F}))$ is a ring-graph if and only if $(m - 1)(m - 2)/2 = m - 2$, which implies that $m = 2$ or $m = 3$. So $\text{AG}(\mathbb{Z}_3 \times \mathbb{F})$ is a ring-graph if and only if $\mathbb{F} \cong \mathbb{Z}_2$ or $\mathbb{F} \cong \mathbb{Z}_3$. Also, in view of the proof of Theorem 2.1, the annihilator graphs of all rings

$$\mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2 - 2), \mathbb{Z}_2[x, y]/(x, y)^2, \\ \mathbb{Z}_4[x]/(2x, x^2), \mathbb{Z}_9, \mathbb{Z}_3[x]/(x^2), \mathbb{F}_4[x]/(x^2) \text{ and } \mathbb{Z}_4[x]/(x^2 + x + 1)$$

are ring-graphs. The graphs $\text{AG}(\mathbb{Z}_{25})$ and $\text{AG}(\mathbb{Z}_5[x]/(x^2))$ are isomorphic to K^4 , and so they are not ring-graphs. \square

In the next theorem, by using the fact that every outerplanar graph is a ring-graph in conjunction with Theorem 2.2, we determine all rings R with outerplanar annihilator graphs.

Theorem 2.3. *The annihilator graph $\text{AG}(R)$ is outerplanar if and only if R is isomorphic to one of the following rings:*

- (i) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$,
- (ii) \mathbb{Z}_4 , $\mathbb{Z}_2[x]/(x^2)$, \mathbb{Z}_8 , $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_2[x, y]/(x, y)^2$,
 $\mathbb{Z}_4[x]/(2x, x^2)$, \mathbb{Z}_9 , $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$.

P r o o f. Since a graph G is outerplanar if and only if it does not contain a subdivision of a complete graph K^4 or a complete bipartite graph $K^{2,3}$, one can check that no ring-graph annihilator graphs contain a subdivision of a complete graph K^4 or a complete bipartite graph $K^{2,3}$. Now the result follows immediately from Theorem 2.2. \square

3. CLIQUE NUMBERS OF THE ANNIHILATOR GRAPHS

We begin this section with a lemma which shows that the annihilator graph of the product of three fields is weakly perfect.

Lemma 3.1. *Let K_1, K_2 and K_3 be fields. Then $\text{cl}(\text{AG}(K_1 \times K_2 \times K_3)) = \chi(\text{AG}(K_1 \times K_2 \times K_3)) = 3$.*

P r o o f. Suppose that (a, b, c) is in $Z(K_1 \times K_2 \times K_3)$. Then at least one of a, b or c is zero. So, if

$$\begin{aligned} A_1 &= \{(a, b, c) \in K_1 \times K_2 \times K_3 : a = 0 \text{ and } b \neq 0 \neq c\}, \\ A_2 &= \{(a, b, c) \in K_1 \times K_2 \times K_3 : b = 0 \text{ and } a \neq 0 \neq c\}, \\ A_3 &= \{(a, b, c) \in K_1 \times K_2 \times K_3 : c = 0 \text{ and } a \neq 0 \neq b\}, \\ A_4 &= \{(a, b, c) \in K_1 \times K_2 \times K_3 : a = b = 0 \text{ and } c \neq 0\}, \\ A_5 &= \{(a, b, c) \in K_1 \times K_2 \times K_3 : a = c = 0 \text{ and } b \neq 0\}, \\ A_6 &= \{(a, b, c) \in K_1 \times K_2 \times K_3 : b = c = 0 \text{ and } a \neq 0\}, \end{aligned}$$

then $Z(K_1 \times K_2 \times K_3)^* = \bigcup_{i=1}^6 A_i$. Now, by the definition of the annihilator graph $\text{AG}(R)$, every vertex in A_1 is adjacent to every vertex in A_2, A_3 and A_6 , every vertex in A_2 is adjacent to every vertex in A_1, A_3 and A_5 , every vertex in A_3 is adjacent to every vertex in A_1, A_2 and A_4 , every vertex in A_4 is adjacent to every vertex in A_3, A_5 and A_6 , every vertex in A_5 is adjacent to every vertex in A_2, A_4 and A_6 and every vertex in A_6 is adjacent to every vertex in A_1, A_4 and A_5 . Also, each A_i for $i = 1, \dots, 6$ is an independent set. Hence $\text{cl}(\text{AG}(K_1 \times K_2 \times K_3)) = 3$, and so $\chi(\text{AG}(K_1 \times K_2 \times K_3)) \geq 3$. Since each A_i for $i = 1, \dots, 6$ is an independent set, we can color every vertex in A_1 by λ_1 . Now, since every vertex in A_2 is adjacent to every vertex in A_1 , we color every vertex in A_2 by λ_2 . Also, every vertex in A_3 is adjacent to every vertex in A_1 and A_2 . Therefore we need another color λ_3 for every vertex in A_3 . Every vertex in A_4 is adjacent to every vertex in A_3, A_5 and A_6 , and so we can color the vertices in A_4 by λ_1 or λ_2 . Without loss of generality, we color

every vertex in A_4 by λ_1 . Since every vertex in A_5 is adjacent to A_2 and A_4 , we cannot color the vertices in A_5 by λ_1 and λ_2 . So we color every vertex in A_5 by λ_3 . Finally, since every vertex in A_6 is adjacent to A_1, A_4 and A_5 , we color the vertices in A_6 by λ_2 . Hence $\chi(\text{AG}(K_1 \times K_2 \times K_3)) = 3$. \square

In the next theorem we characterize all finite rings R whose annihilator graphs have clique number 1, 2 or 3.

Theorem 3.2. *Let R be a finite commutative ring and let K_1, K_2 and K_3 be finite fields. Also, let \mathbb{F}_4 be a field with four elements. Then the following statements hold:*

- (a) $\text{cl}(\text{AG}(R)) = 1$ if and only if R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[x]/(x^2)$.
- (b) $\text{cl}(\text{AG}(R)) = 2$ if and only if R is isomorphic to one of the following rings:

$$K_1 \times K_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_9, \mathbb{Z}_3[x]/(x^2).$$

- (c) $\text{cl}(\text{AG}(R)) = 3$ if and only if R is isomorphic to one of the following rings:

$$K_1 \times K_2 \times K_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2, x)^2, \\ \mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 + x + 1), \mathbb{Z}_2[x, y]/(x, y)^2, \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

Proof. (a) Clearly, by [12], Lemma 2.1, the zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\text{AG}(R)$. Hence if $\text{cl}(\text{AG}(R)) = n$, then $\text{cl}(\Gamma(R)) \leq n$. So $\text{cl}(\text{AG}(R)) = 1$ if and only if $\text{cl}(\Gamma(R)) = 1$. Also by [15], Proposition 2.2, $\text{cl}(\Gamma(R)) = 1$ if and only if R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[x]/(x^2)$.

(b) In order to characterize all rings R with $\text{cl}(\text{AG}(R)) = 2$, we need only to study the rings R with $\text{cl}(\Gamma(R)) = 1$ or 2. It is easy to see that if $\text{cl}(\Gamma(R)) = 1$, then $\text{cl}(\text{AG}(R)) = 1$. Now, by [15], page 226, $\text{cl}(\Gamma(R)) = 2$ if and only if R is isomorphic to one of the following rings:

$$K_1 \times K_2, K_1 \times \mathbb{Z}_4, K_1 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_9, \\ \mathbb{Z}_3[x]/(x^2), \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

If $R \cong K_1 \times K_2$ with $|K_1| = n$ and $|K_2| = m$, then one can easily check that $\text{AG}(R) \cong K^{n-1, m-1}$. Thus $\text{cl}(\text{AG}(K_1 \times K_2)) = 2$.

If $R \cong K_1 \times \mathbb{Z}_4$ or $K_1 \times \mathbb{Z}_2[x]/(x^2)$ with $|K_1| = n$, then $\text{AG}(K_1 \times \mathbb{Z}_4) \cong \text{AG}(K_1 \times \mathbb{Z}_2[x]/(x^2))$. Let $R \cong K_1 \times \mathbb{Z}_4$. Then $\text{AG}(K_1 \times \mathbb{Z}_4)$ contains a complete graph K^n with vertex set $\{(0, 2), (r_1, 2), (r_2, 2), \dots, (r_{n-1}, 2)\}$, where $r_i \neq 0$ for $i = 1, \dots, n-1$, and so $\text{cl}(\text{AG}(R)) \geq n$. Thus, for $n \geq 3$ we have that $\text{cl}(\text{AG}(K_1 \times \mathbb{Z}_4)) \geq 3$. Now,

if $n = 2$, then $K_1 \cong \mathbb{Z}_2$, and so $\text{cl}(\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 2$. If $R \cong \mathbb{Z}_8$, then we have $\text{AG}(\mathbb{Z}_8) \cong K^3$. Thus $\text{cl}(\text{AG}(\mathbb{Z}_8)) = 3$.

If $R \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$, then we have $\text{AG}(\mathbb{Z}_9) \cong \text{AG}(\mathbb{Z}_3[x]/(x^2)) \cong K^2$. Thus $\text{cl}(\text{AG}(\mathbb{Z}_9)) = \text{cl}(\text{AG}(\mathbb{Z}_3[x]/(x^2))) = 2$. If $R \cong \mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, then we have $\text{AG}(\mathbb{Z}_2[x]/(x^3)) \cong \text{AG}(\mathbb{Z}_4[x]/(2x, x^2 - 2)) \cong K^3$. Therefore, for clique subgraph is $\text{cl}(\text{AG}(\mathbb{Z}_2[x]/(x^3))) = \text{cl}(\text{AG}(\mathbb{Z}_4[x]/(2x, x^2 - 2))) = 3$.

(c) In order to characterize all rings R with $\text{cl}(\text{AG}(R)) = 3$, we need only to study the rings R with $\text{cl}(\Gamma(R)) = 1, 2$ or 3 . In view of the proof of part (b), we have $\text{cl}(\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_4)) = \text{cl}(\text{AG}(\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))) = \text{cl}(\text{AG}(\mathbb{Z}_8)) = \text{cl}(\text{AG}(\mathbb{Z}_2[x]/(x^3))) = \text{cl}(\text{AG}(\mathbb{Z}_4[x]/(2x, x^2 - 2))) = 3$. Now we study the rings R with $\text{cl}(\Gamma(R)) = 3$. By [9], Theorem 4.4, $\text{cl}(\Gamma(R)) = 3$ if and only if R is isomorphic to one of the following rings:

$$\begin{aligned} & \mathbb{Z}_4 \times \mathbb{Z}_4, \quad \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2), \quad \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2), \\ & K_1 \times K_2 \times K_3, \quad K_1 \times K_2 \times \mathbb{Z}_4, \quad K_1 \times K_2 \times \mathbb{Z}_2[x]/(x^2), \\ & K_1 \times \mathbb{Z}_8, \quad K_1 \times \mathbb{Z}_9, \quad K_1 \times \mathbb{Z}_3[x]/(x^2), \quad K_1 \times \mathbb{Z}_2[x]/(x^3), \\ & K_1 \times \mathbb{Z}_4[x]/(2x, x^2 - 2), \quad \mathbb{Z}_{16}, \quad \mathbb{Z}_2[x]/(x^4), \quad \mathbb{Z}_4[x]/(2x, x^3 - 2), \\ & \mathbb{Z}_4[x]/(x^2 - 2), \quad \mathbb{Z}_4[x]/(x^2 + 2x + 2), \quad \mathbb{F}_4[x]/(x^2), \quad \mathbb{Z}_4[x]/(x^2 + x + 1), \\ & \mathbb{Z}_2[x, y]/(x, y)^2, \quad \mathbb{Z}_4[x]/(2, x)^2, \quad \mathbb{Z}_{27}, \quad \mathbb{Z}_3[x]/(x^3), \quad \mathbb{Z}_9[x]/(3x, x^2 - 3), \\ & \mathbb{Z}_9[x]/(3x, x^2 - 6), \quad \mathbb{Z}_2[x, y]/(x^2, y^2 - xy), \quad \mathbb{Z}_2[x, y]/(x^2, y^2), \\ & \mathbb{Z}_8[x]/(2x - 4, x^2), \quad \mathbb{Z}_4[x]/(x^2), \quad \mathbb{Z}_4[x]/(x^2 - 2x), \\ & \mathbb{Z}_4[x, y]/(x^2, xy - 2, y^2, 2x, 2y) \quad \text{or} \quad \mathbb{Z}_4[x, y]/(x^2, xy - 2, x^2 - xy, 2x, 2y). \end{aligned}$$

If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, then $\text{AG}(\mathbb{Z}_4 \times \mathbb{Z}_4)$ contains a complete graph K^5 with vertex set $\{(2, 1), (2, 2), (2, 3), (1, 2), (3, 2)\}$. Thus $\text{cl}(\text{AG}(\mathbb{Z}_4 \times \mathbb{Z}_4)) \geq 5$. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$, then $\text{AG}(R)$ contains a complete graph K^5 with vertex set $\{(2, 1), (2, x), (2, 1 + x), (1, x), (3, x)\}$. Therefore $\text{cl}(\text{AG}(\mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2))) \geq 5$. If $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$, then $\text{AG}(R)$ contains a complete graph K^5 with vertex set $\{(x, 1), (x, x), (x, 1 + x), (1 + x, x), (1, x)\}$. Thus $\text{cl}(\text{AG}(R)) \geq 5$. If $R \cong K_1 \times K_2 \times K_3$, then by Lemma 3.1, $\text{cl}(\text{AG}(R)) = 3$. If $R \cong K_1 \times K_2 \times \mathbb{Z}_4$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(0, 1, 2), (1, 1, 0), (1, 0, 2), (0, 0, 2)\}$. Therefore $\text{cl}(\text{AG}(R)) \geq 4$. If $R \cong K_1 \times K_2 \times \mathbb{Z}_2[x]/(x^2)$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(0, 1, x), (1, 1, 0), (1, 0, x), (0, 0, x)\}$. Hence $\text{cl}(\text{AG}(R)) \geq 4$. If $R \cong K_1 \times \mathbb{Z}_8$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(0, 1), (1, 2), (1, 4), (1, 6)\}$. Thus $\text{cl}(\text{AG}(R)) \geq 4$. If $R \cong K_1 \times \mathbb{Z}_9$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1, 3), (1, 6), (0, 3), (0, 6)\}$. So $\text{cl}(\text{AG}(R)) \geq 4$. If $R \cong K_1 \times \mathbb{Z}_3[x]/(x^2)$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1, x), (1, 2x), (0, x), (0, 2x)\}$. Thus $\text{cl}(\text{AG}(R)) \geq 4$.

Now, if $R \cong K_1 \times \mathbb{Z}_2[x]/(x^3)$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1, x), (1, x^2 + x), (0, x), (0, x^2 + x)\}$, and so $\text{cl}(\text{AG}(R)) \geq 4$. If $R \cong K_1 \times \mathbb{Z}_4[x]/(2x, x^2 - 2)$, then $\text{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1, 2), (1, 2 + x), (0, 2), (0, 2 + x)\}$. So $\text{cl}(\text{AG}(R)) \geq 4$. If R is isomorphic to one of the following rings:

$$\begin{aligned} & \mathbb{Z}_{16}, \mathbb{Z}_2[x]/(x^4), \mathbb{Z}_4[x]/(2x, x^3 - 2), \mathbb{Z}_4[x]/(x^2 - 2), \\ & \mathbb{Z}_4[x]/(x^2 + 2x + 2), \mathbb{Z}_2[x, y]/(x^2, y^2 - xy), \mathbb{Z}_2[x, y]/(x^2, y^2), \\ & \mathbb{Z}_8[x]/(2x - 4, x^2), \mathbb{Z}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 - 2x), \\ & \mathbb{Z}_4[x, y]/(x^2, xy - 2, y^2, 2x, 2y) \quad \text{or} \quad \mathbb{Z}_4[x, y]/(x^2, xy - 2, x^2 - xy, 2x, 2y), \end{aligned}$$

then its annihilator graph is isomorphic to K^7 . Thus, for clique subgraph is $\text{cl}(\text{AG}(R)) = 7$. If $R \cong \mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$, $\mathbb{Z}_2[x, y]/(x, y)^2$ or $\mathbb{Z}_4[x]/(2, x)^2$, then its annihilator graph is isomorphic to K^3 . Hence $\text{cl}(\text{AG}(R)) = 3$. If $R \cong \mathbb{Z}_{27}$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 3)$ or $\mathbb{Z}_9[x]/(3x, x^2 - 6)$, then its annihilator graph is isomorphic to K^8 . Therefore $\text{cl}(\text{AG}(R)) = 8$. Now by the above discussion the result holds. \square

4. EXTENSION RINGS

In this section, we compare some properties of the annihilator graph $\text{AG}(R)$ with the graphs $\text{AG}(R[x])$ and $\text{AG}(S^{-1}R)$. Note that McCoy's theorem states that $f(x) \in R[x]$ is a zero-divisor if and only if there is a nonzero element $r \in R$ such that $rf(x) = 0$. Also it is proved that a polynomial $f(x)$ over a commutative ring R is nilpotent if and only if each coefficient of $f(x)$ is nilpotent (cf. [11]).

Proposition 4.1. *Let R be a finite commutative ring with $|Z(R)^*| > 1$ and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the annihilator graph $\text{AG}(R)$ is a complete graph if and only if R is a local ring.*

Proof. First assume that the annihilator graph $\text{AG}(R)$ is a complete graph. We shall show that R is a local ring. If R is a finite commutative ring and $Z(R)$ is an ideal of R , then R is a local ring with $Z(R) = \text{Nil}(R)$ its unique maximal ideal. So it is enough to show that $Z(R)$ is an ideal of R . Let $|Z(R)^*| = 2$. So $Z(R) = \{0, x, y\}$ where $x \neq y$. If $xy \neq 0$, then $x^2 = y^2 = 0$. Hence $Z(R) = \text{Nil}(R)$. Therefore $Z(R)$ is an ideal of R . Now, suppose that $xy = 0$. Then the zero-divisor graph $\Gamma(R)$ is a complete graph. Moreover, in [8], Theorem 2.10, it was shown that for any finite commutative ring R , if $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is a local

ring with characteristic p or p^2 and $|\Gamma(R)| = p^s - 1$, where p is a prime and $s \geq 1$. Thus the result follows. Now assume that $|Z(R)^*| \geq 3$. Let x, y be distinct elements in $Z(R)^*$. It is enough to show that $x + y \in Z(R)$. Since $\Gamma(R)$ is connected, there is a nonzero element $r \in R$ such that $rx = 0$ (or $ry = 0$). Now, because $\text{AG}(R)$ is a complete graph, $r - y$ is an edge of $\text{AG}(R)$. So $\text{ann}_R(ry) \neq \text{ann}_R(r) \cup \text{ann}_R(y)$. Thus there exists $r' \in R$ such that $r'ry = 0$ and $r'r \neq 0$ and $r'y \neq 0$. Hence $r'r(x + y) = 0$. Therefore $x + y \in Z(R)$.

Conversely, since for a finite local ring we have $Z(R) = \text{Nil}(R)$, the result follows from [12], Theorem 3.10. \square

Theorem 4.2. *Let R be a finite commutative ring with $|Z(R)^*| > 1$ and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\text{AG}(R)$ is complete, then $\text{AG}(R[x])$ is also complete.*

Proof. It is enough to show that every zero-divisor element in $R[x]$ is nilpotent. Then, by [12], Theorem 3.10, $\text{AG}(R[x])$ is complete. Since $\text{AG}(R)$ is complete, by Proposition 4.1, R is a local ring. So $Z(R) = \text{Nil}(R)$. Now, let $f(x) \in Z(R[x])$, where $f(x) = a_0 + a_1x + \dots + a_nx^n$. Thus $a_0, \dots, a_n \in \text{Nil}(R)$, which implies that $f(x)$ is nilpotent. \square

Recall that the diameter of a graph G , denoted by $\text{diam}(G)$, is equal to $\sup\{d(a, b) : a, b \in V(G)\}$, where $d(a, b)$ is the length of the shortest path connecting a and b .

Corollary 4.3. *Let R be a finite commutative ring with $|Z(R)^*| > 1$ and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\text{diam}(\text{AG}(R)) = \text{diam}(\text{AG}(R[x]))$.*

Proof. By [12], Theorem 2.2, we have $\text{diam}(\text{AG}(R)) = 1$ or 2 . Assume that $\text{diam}(\text{AG}(R)) = 1$. So $\text{AG}(R)$ is a complete graph. Therefore, by Theorem 4.2, the annihilator graph $\text{AG}(R[x])$ is also a complete graph. Hence $\text{diam}(\text{AG}(R[x])) = 1$. Now if $\text{diam}(\text{AG}(R)) = 2$, then there are distinct elements $a, b \in Z(R)^*$, such that $a - b$ is not an edge of $\text{AG}(R)$. We show that $a - b$ is not an edge of $\text{AG}(R[x])$ either. Since $a - b$ is not an edge of $\text{AG}(R)$, $\text{ann}_R(ab) = \text{ann}_R(a) \cup \text{ann}_R(b)$. So, $\text{ann}_R(a) \subseteq \text{ann}_R(b)$ or $\text{ann}_R(b) \subseteq \text{ann}_R(a)$. Now (without loss of generality), we set $\text{ann}_R(a) \subseteq \text{ann}_R(b)$. So $\text{ann}_R(ab) = \text{ann}_R(b)$. Suppose that $f(x) \in \text{ann}_{R[x]}(ab)$ where $f(x) = a_0 + a_1x + \dots + a_nx^n$. Hence $aba_0 = aba_1 = \dots = aba_n = 0$. Since $\text{ann}_R(ab) = \text{ann}_R(b)$, $ba_0 = ba_1 = \dots = ba_n = 0$. Thus $f(x) \in \text{ann}_{R[x]}(b)$. Hence $\text{ann}_{R[x]}(ab) \subseteq \text{ann}_{R[x]}(b)$. Therefore $\text{ann}_{R[x]}(ab) = \text{ann}_{R[x]}(a) \cup \text{ann}_{R[x]}(b)$, so $a - b$ is not an edge of $\text{AG}(R[x])$, and since $\text{diam}(\text{AG}(R[x])) \leq 2$, we conclude that $\text{diam}(\text{AG}(R[x])) = 2$. \square

Theorem 4.4. *Let R be a commutative ring. If R is not an integral domain, then the annihilator graph $\text{AG}(R[x])$ is not planar.*

Proof. First suppose that R is not a reduced ring. So there exists a nonzero nilpotent element $a \in R$. Let n be the least positive integer such that $a^n = 0$. Then one can find a copy of K^5 with vertex set $\{a, ax, ax^2, ax^3, ax^4\}$ in $\text{AG}(R[x])$, and so $\text{AG}(R[x])$ is not planar. Now assume that R is a reduced ring. Hence there exist $a, b \in R$ such that $a \neq b$ and $ab = 0$. Then, by Figure 7, the graph $\text{AG}(R[x])$ has a copy of $K^{3,3}$, and so $\text{AG}(R[x])$ is not planar. \square

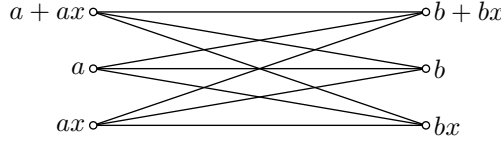


Figure 7.

The following corollary immediately follows from Theorem 4.4.

Corollary 4.5. *Let R be a commutative ring. If R is not an integral domain, then $\text{gr}(\text{AG}(R[x])) \in \{3, 4\}$.*

In the rest of the section, we study the annihilator graph of the ring of fractions $S^{-1}R$, where S is a multiplicatively closed subset of R . It is obvious that if $r \in Z(R)$, then $r/s \in Z(S^{-1}R)$ for every $s \in S$. Now, let $r/s \in Z(S^{-1}R)$. Thus there is a nonzero element $r'/s' \in S^{-1}R$ such that $(r/s) \cdot (r'/s') = 0/1$. So there exists $u \in S$ such that $urr' = 0$. Clearly $ur' \neq 0$, because otherwise $r'/s' = 0/1$. Thus $r \in Z(R)$.

Proposition 4.6. *Let R be a commutative ring. If r and r' are arbitrary elements of R such that $\text{ann}_R(r) \subseteq \text{ann}_R(r')$, then $\text{ann}_{S^{-1}R}(r/s) \subseteq \text{ann}_{S^{-1}R}(r'/s')$ for every $s, s' \in S$.*

Proof. Assume that $\text{ann}_R(r) \subseteq \text{ann}_R(r')$, and suppose on the contrary that $\text{ann}_{S^{-1}R}(r/s) \not\subseteq \text{ann}_{S^{-1}R}(r'/s')$. Then there is $r''/s'' \in S^{-1}R$ such that $(r/s) \times (r''/s'') = 0/1$ and $(r'/s')(r''/s'') \neq 0/1$. So there exists $u \in S$ such that $urr'' = 0$, and, for every $v \in S$, $vr'r'' \neq 0$. So $ur'' \in \text{ann}_R(r)$. Thus $ur'' \in \text{ann}_R(r')$. So we have $ur'r'' = 0$, which is the required contradiction. \square

Lemma 4.7. *Let R be a commutative ring. If a_1/s_1 is adjacent to a_2/s_2 in $\text{AG}(S^{-1}R)$, then either a_1 is adjacent to a_2 or a_1s_2 is adjacent to a_2s_1 in $\text{AG}(R)$ for every $s_1, s_2 \in S$.*

Proof. First assume that $a_1 \neq a_2$. Since $a_1/s_1 - a_2/s_2$ is an edge in $\text{AG}(S^{-1}R)$, there is b/s in $\text{AG}(S^{-1}R)$ such that $(b/s)(a_1a_2/s_1s_2) = 0$, $(b/s)(a_1/s_1) \neq 0$ and $(b/s)(a_2/s_2) \neq 0$. Hence there exists $v \in S$ such that $vba_1a_2 = 0$, $vba_1 \neq 0$ and

$vba_2 \neq 0$, and so a_1 is adjacent to a_2 in $\text{AG}(R)$. Now assume that $a_1 = a_2$. Since $a_1/s_1 \neq a_2/s_2$, we have $a_1s_2 \neq a_2s_1$. Also a_1s_2/s_1s_2 is adjacent to a_2s_1/s_1s_2 in $\text{AG}(S^{-1}R)$, and so a_1s_2 is adjacent to a_2s_1 in $\text{AG}(R)$. \square

By Lemma 4.7, one can see that if $\text{AG}(S^{-1}R)$ is a complete graph, then $\text{AG}(R)$ is complete.

Lemma 4.8. *Let R be a commutative ring. If $\text{ann}_R(x_1) = \text{ann}_R(x_2)$, then x_1 and x_2 have the same neighbours in $\text{AG}(R)$.*

Proof. Suppose that x is adjacent to x_1 in $\text{AG}(R)$. So we have $\text{ann}_R(xx_1) \neq \text{ann}_R(x) \cup \text{ann}_R(x_1)$. Hence there is x' such that $x'xx_1 = 0$, $x'x_1 \neq 0$ and $x'x \neq 0$. Now, since $\text{ann}_R(x_1) = \text{ann}_R(x_2)$, we have $x'xx_2 = 0$ and $x'x_2 \neq 0$. Therefore $\text{ann}_R(xx_2) \neq \text{ann}_R(x) \cup \text{ann}_R(x_2)$, and so x is adjacent to x_2 in $\text{AG}(R)$. Also, if x is adjacent to x_2 in $\text{AG}(R)$, then similarly x is adjacent to x_1 in $\text{AG}(R)$. So x_1 and x_2 have the same neighbours in $\text{AG}(R)$. \square

Lemma 4.9. *Let R be a commutative ring. Suppose that s is an arbitrary element in S . If $r \in Z(R)$, then r/s and $r/1$ have the same neighbours in $\text{AG}(S^{-1}R)$.*

Proof. By Lemma 4.8, it is enough to show that $\text{ann}_{S^{-1}R}(r/s) = \text{ann}_{S^{-1}R}(r/1)$. So if $a/t \in \text{ann}_{S^{-1}R}(r/s)$, then we have $(a/t)(r/s) = 0/1$. Hence there exists $u \in S$ such that $uar = 0$. Also $(a/t)(r/1) = ar/t = aru/(tu) = 0/1$, and so $a/t \in \text{ann}_{S^{-1}R}(r/1)$. Now, if $a/t \in \text{ann}_{S^{-1}R}(r/1)$, then there exists $u \in S$ such that $uar = 0$. Also $(a/t)(r/s) = ar/(ts) = aru/(tsu) = 0/1$. Therefore $a/t \in \text{ann}_{S^{-1}R}(r/s)$. \square

Let $T(R) = S^{-1}R$ be the total quotient ring of R , where $S = R - Z(R)$. In [7], Theorem 2.2, Anderson and Shapiro showed that the graphs $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic. For $x, y \in R$, they defined a relation \sim as follows: $x \sim y$ if $\text{ann}_R(x) = \text{ann}_R(y)$. Clearly \sim is an equivalence relation on R . Let $T = T(R)$. Denote the equivalence relations on $Z(R)^*$ and $Z(T)^*$ by \sim_R and \sim_T , respectively, and denote their equivalence classes by $[a]_R$ and $[a]_T$, respectively. They proved that there is a bijection between equivalence classes of $\Gamma(T(R))$ and $\Gamma(R)$, and they defined a bijection $\varphi: Z(R)^* \rightarrow Z(T)^*$ by $\varphi(x) = \varphi_\alpha(x)$, where $\varphi_\alpha: [a_\alpha] \rightarrow [a_\alpha/1]$ is a bijection and $x \in [a_\alpha]$. In the next theorem, using the above notation, we show that $\text{AG}(R)$ is isomorphic to $\text{AG}(T(R))$.

Theorem 4.10. *Let R be a commutative ring. Then the graphs $\text{AG}(R)$ and $\text{AG}(T(R))$ are isomorphic.*

Proof. By the proof of [7], Theorem 2.2, we have the bijection $\varphi: Z(R)^* \rightarrow Z(T)^*$ defined by $\varphi(x) = \varphi_\alpha(x)$, where $\varphi_\alpha: [a_\alpha] \rightarrow [a_\alpha/1]$ is a bijection and $x \in [a_\alpha]$. Thus we only need to show that x and y are adjacent in $\text{AG}(R)$ if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in $\text{AG}(T(R))$; i.e., $\text{ann}_R(xy) \neq \text{ann}_R(x) \cup \text{ann}_R(y)$ if and only if $\text{ann}_T(\varphi(x)\varphi(y)) \neq \text{ann}_T(\varphi(x)) \cup \text{ann}_T(\varphi(y))$. Let $x \in [a]_R, y \in [b]_R, l \in [c]_R, w \in [a/1]_T, z \in [b/1]_T$ and $t \in [c/1]_T$. We need only to show that $xy l = 0, x l \neq 0$ and $y l \neq 0$ if and only if $wzt = 0, wt \neq 0$ and $zt \neq 0$. Note that $\text{ann}_T(x) = \text{ann}_T(a) = \text{ann}_T(w), \text{ann}_T(y) = \text{ann}_T(b) = \text{ann}_T(z)$ and $\text{ann}_T(l) = \text{ann}_T(c) = \text{ann}_T(t)$. Hence

$$\begin{aligned} xy l = 0 &\Leftrightarrow xy \in \text{ann}_T(l) = \text{ann}_T(t) \Leftrightarrow xyt = 0 \Leftrightarrow xt \in \text{ann}_T(y) = \text{ann}_T(z) \\ &\Leftrightarrow xtz = 0 \Leftrightarrow tz \in \text{ann}_T(x) = \text{ann}_T(w) \Leftrightarrow wzt = 0. \end{aligned}$$

Since φ is an isomorphism between the graphs $\Gamma(R)$ and $\Gamma(T(R))$, we have $xl \neq 0$ and $yl \neq 0$ if and only if $wt \neq 0$ and $zt \neq 0$. \square

Theorem 4.11. *Let R be a finite commutative ring. If \mathfrak{p} is a prime ideal of R , then the annihilator graph $\text{AG}(S^{-1}R)$ is complete, where $S = R - \mathfrak{p}$.*

Proof. Since $R_{\mathfrak{p}}$ is a finite local ring, $Z(R_{\mathfrak{p}}) = \text{Nil}(R_{\mathfrak{p}})$. So, by [12], Theorem 3.10, $\text{AG}(R_{\mathfrak{p}})$ is complete. \square

5. ANNIHILATOR GRAPHS OF \mathbb{Z}_n

In this section, we examine the existence of cut-vertices and cut-sets in $\text{AG}(\mathbb{Z}_n)$. We determine the reduced rings whose annihilator graphs are bipartite graphs. Also we show that $\text{AG}(\mathbb{Z}_n)$ is bipartite for a certain n .

Theorem 5.1. *Let $n \geq 6$. Then $r \in \mathbb{Z}_n$ is a cut-vertex of $\text{AG}(\mathbb{Z}_n)$ if and only if $2r = n$ and r is a prime integer.*

Proof. First assume that r is a cut-vertex of $\text{AG}(\mathbb{Z}_n)$. Since the zero-divisor graph is the (spanning) subgraph of the annihilator graph $\text{AG}(R)$, every cut-vertex in $\text{AG}(R)$ is a cut-vertex in $\Gamma(R)$. Therefore, by [17], Lemma 2.2, $n = 2r$. Now, we prove that r is a prime integer. Since r is a cut-vertex of the annihilator graph $\text{AG}(\mathbb{Z}_n)$, there exist vertices α and β such that $\alpha - r - \beta$ is the only path that connects α to β . Now, since α is not adjacent to β in $\text{AG}(\mathbb{Z}_n)$, hence by [12], Lemma 2.1, $\text{ann}_R(\alpha) \subseteq \text{ann}_R(\beta)$ or $\text{ann}_R(\beta) \subseteq \text{ann}_R(\alpha)$. Without loss of generality, we may assume $\text{ann}_R(\alpha) \subseteq \text{ann}_R(\beta)$. Let t be a nonzero element in $\mathbb{Z}_n \setminus \{r\}$. If $t\alpha = 0$, then $t\beta = 0$. So we have the path $\alpha - t - \beta$, which is impossible. Now, since $\alpha \in Z(\mathbb{Z}_n)$, we have $\alpha r = 0$. Therefore $\text{ann}_R(\alpha) = \{0, r\}$. If r is not prime, then there are positive

integers $q, q' \neq 1$ such that $qq' = r$. Hence we have $\text{ann}_R(\alpha q) \neq \text{ann}_R(\alpha) \cup \text{ann}_R(q)$ and $\text{ann}_R(\alpha q') \neq \text{ann}_R(\alpha) \cup \text{ann}_R(q')$, which implies that α is adjacent to q and q' in $\text{AG}(\mathbb{Z}_n)$. If $q\beta = 0$ or $q'\beta = 0$, then we have the path $\alpha - q - \beta$ or $\alpha - q' - \beta$, which is impossible. So $q\beta \neq 0$ and $q'\beta \neq 0$. In this situation one can easily check that $\text{ann}_R(\beta q) \neq \text{ann}_R(\beta) \cup \text{ann}_R(q)$ and $\text{ann}_R(\beta q') \neq \text{ann}_R(\beta) \cup \text{ann}_R(q')$, and therefore we have the paths $\alpha - q - \beta$ and $\alpha - q' - \beta$, which is again impossible. Thus r is prime.

Conversely, by [12], Theorem 3.8, $\text{AG}(\mathbb{Z}_n) \cong K^{1,m}$ for some $m \geq 1$. Thus it contains a cut-vertex. \square

Theorem 5.2. *If $|\text{Min}(\mathbb{Z}_n)| = 2$ and every minimal prime ideal has at least 3 elements, then the nonzero elements in a minimal prime ideal of \mathbb{Z}_n with minimal cardinality form a cut-set of $\text{AG}(\mathbb{Z}_n)$.*

Proof. Since $|\text{Min}(\mathbb{Z}_n)| = 2$ and every minimal prime ideal has at least 3 elements, we have $n = pq$, where p and q are distinct prime integers with $p, q \neq 2$. So let $A = \{x \in \mathbb{Z}_n : p \mid x, q \nmid x\}$ and $B = \{x \in \mathbb{Z}_n : q \mid x, p \nmid x\}$. One can easily see that $A \cup \{0\}$ and $B \cup \{0\}$ are minimal prime ideals of \mathbb{Z}_n and $\text{AG}(\mathbb{Z}_n) \cong K^{m,n}$, where $|A| = m$ and $|B| = n$ and A, B are two parts in $\text{AG}(\mathbb{Z}_n)$. \square

In the next theorem, we determine some conditions under which $\text{AG}(\mathbb{Z}_n)$ is weakly perfect.

Theorem 5.3. *Suppose that \mathbb{Z}_n is a finite reduced ring. If $|\text{Min}(\mathbb{Z}_n)| \leq 3$, then $\text{AG}(\mathbb{Z}_n)$ is weakly perfect.*

Proof. Since $|\text{Min}(\mathbb{Z}_n)| \leq 3$ and \mathbb{Z}_n is a reduced ring, we have $n = p, pq$ or pqr where p, q and r are distinct prime integers. If $n = p$, then $Z(\mathbb{Z}_n) = \{0\}$. If $n = pq$, then $\mathbb{Z}_n = \mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong K_1 \times K_2$, where K_1 and K_2 are finite fields with p and q elements, respectively. So $\text{AG}(K_1 \times K_2) \cong K^{p-1, q-1}$. Hence $\text{cl}(\text{AG}(\mathbb{Z}_{pq})) = \chi(\text{AG}(\mathbb{Z}_{pq})) = 2$. If $n = pqr$, then $\mathbb{Z}_n = \mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \cong K_1 \times K_2 \times K_3$, where K_1, K_2 and K_3 are finite fields with p, q and r elements, respectively. Therefore, by Lemma 3.1, $\text{cl}(\text{AG}(\mathbb{Z}_{pqr})) = \chi(\text{AG}(\mathbb{Z}_{pqr})) = 3$. \square

Theorem 5.4. *Let R be a commutative reduced ring. Then $\text{AG}(R)$ is bipartite if and only if there exist two distinct prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 of R such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. In addition, if $\text{AG}(R)$ is bipartite, then it is a complete bipartite graph.*

Proof. Suppose that \mathfrak{p}_1 and \mathfrak{p}_2 are distinct prime ideals of R such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. Then, in view of the proof of [3], Theorem 2.4, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Set $V_1 = \mathfrak{p}_1 \setminus \{0\}$ and $V_2 = \mathfrak{p}_2 \setminus \{0\}$. Now, we show that $\text{AG}(R)$ is bipartite

with two parts V_1 and V_2 . If $a, b \in V_1$ and a is adjacent to b , then $\text{ann}_R(ab) \neq \text{ann}_R(a) \cup \text{ann}_R(b)$. Hence there exists a nonzero element $c \in R$ such that $abc = 0$, $ac \neq 0$ and $bc \neq 0$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$ and $a, b \in V_1$, we have $c \in \mathfrak{p}_2$. So $ac \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, which is a contradiction. Also, for every $a \in V_1$ and $b \in V_2$, we have $ab = 0$ since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. Therefore $\text{AG}(R)$ is a complete bipartite graph.

Conversely, suppose that $\text{AG}(R)$ is bipartite. Since, by [12], Lemma 2.1, $\Gamma(R)$ is a (spanning) subgraph of $\text{AG}(R)$, we have that $\Gamma(R)$ is bipartite. Hence, by [3], Theorem 2.4, there exist two distinct prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 of R such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. \square

Theorem 5.5. *The graph $\text{AG}(\mathbb{Z}_n)$ is bipartite if and only if one of the following holds.*

- (i) $n = 4$ or 9 .
- (ii) $n = p_1 p_2$, where p_1 and p_2 are distinct prime integers.
- (iii) $n = 4p$, where p is a prime integer and $p \neq 2$.

Proof. First assume that the graph $\text{AG}(\mathbb{Z}_n)$ is bipartite. If there exist at least three distinct prime integers in divisors of n , say p_1, p_2 and p_3 , then p_1 is adjacent to p_2 and p_3 and p_2 is adjacent to p_3 . So, we have the cycle $p_1 - p_2 - p_3 - p_1$ of length three. Therefore $\text{AG}(\mathbb{Z}_n)$ is not bipartite. Hence there exist at most two distinct prime integers in divisors of n . Now let $n = p_1^r p_2^s$ for some distinct prime integers p_1, p_2 and nonzero positive integers r and s . Set $A_{p_1 p_2} = \{r \in \mathbb{N} : r \mid n, p_1 \mid r, p_2 \mid r\}$. Since $A_{p_1 p_2}$ is a complete subgraph of $\text{AG}(\mathbb{Z}_n)$ and $|A_{p_1 p_2}| = p_1^{r-1} p_2^{s-1}$, we have $r, s \leq 2$. If $r = s = 2$, then $n = p_1^2 p_2^2$, and so p_1 is adjacent to p_2^2 and $p_1 p_2$, and p_2^2 is adjacent to $p_1 p_2$. Thus $\text{AG}(\mathbb{Z}_n)$ is not bipartite. If $r = 2$ and $s = 1$, then $n = p_1^2 p_2$. Since $A_{p_1 p_2}$ is a complete subgraph of $\text{AG}(\mathbb{Z}_n)$ and $|A_{p_1 p_2}| = p_1$, we have $p_1 = 2$. So $n = 4p_2$, where $p_2 \neq 2$. Also if $r = 1$ and $s = 2$, then similarly $n = 4p_1$, where $p_1 \neq 2$. If $r = 1$ and $s = 1$, then $n = p_1 p_2$. If $n = p_1^r$, then $|Z(\mathbb{Z}_n)| = p_1^{r-1}$. Also, $\text{AG}(\mathbb{Z}_n)$ is a complete graph with $p_1^{r-1} - 1$ vertices. So $\text{AG}(\mathbb{Z}_n)$ is bipartite if and only if $p_1^{r-1} - 1 \leq 2$. Hence $p_1^{r-1} = 3$ or $p_1^{r-1} = 2$. If $p_1^{r-1} = 3$, then $p_1 = 3$ and $r = 2$. Thus $n = 9$. If $p_1^{r-1} = 2$, then $p_1 = 2$ and $r = 2$. Thus $n = 4$.

Conversely, one can easily check that $\text{AG}(\mathbb{Z}_4) \cong K^1$, $\text{AG}(\mathbb{Z}_9) \cong K^2$, $\text{AG}(\mathbb{Z}_{p_1 p_2}) \cong K^{p_1-1, p_2-1}$, where p_1 and p_2 are distinct prime integers and $\text{AG}(\mathbb{Z}_{4p}) \cong K^{3, 2p-2}$, where p is prime a integer and $p \neq 2$. \square

Lemma 5.6. *The graph $\text{AG}(\mathbb{Z}_{p^n})$ is weakly perfect, where p is a prime integer. In addition, $\text{AG}(\mathbb{Z}_{p^n})$ has chromatic number $p^{n-1} - 1$.*

Proof. Clearly $Z(\mathbb{Z}_{p^n})$ is an ideal of \mathbb{Z}_{p^n} . Then $Z(\mathbb{Z}_{p^n}) = \text{Nil}(\mathbb{Z}_{p^n})$. So by [12], Theorem 3.10, $\text{AG}(\mathbb{Z}_{p^n})$ is a complete graph with $p^{n-1} - 1$ vertices. Hence $\text{AG}(\mathbb{Z}_{p^n})$ is weakly perfect and has chromatic number $p^{n-1} - 1$. \square

Lemma 5.7. *The graphs $\text{AG}(\mathbb{Z}_{p_1 p_2})$ and $\text{AG}(\mathbb{Z}_{p_1 p_2 p_3})$ for some distinct prime integers p_1, p_2, p_3 are weakly perfect. In addition,*

$$\chi(\text{AG}(\mathbb{Z}_{p_1 p_2})) = 2 \quad \text{and} \quad \chi(\text{AG}(\mathbb{Z}_{p_1 p_2 p_3})) = 3.$$

Proof. Let $R \cong \mathbb{Z}_{p_1 p_2}$. Then one can easily check that $\text{AG}(R)$ is bipartite. So $\text{AG}(R)$ is weakly perfect and $\chi(\text{AG}(R)) = 2$. Let $R \cong \mathbb{Z}_{p_1 p_2 p_3}$. Then $R \cong K_1 \times K_2 \times K_3$, where K_1, K_2 and K_3 are fields with $|K_1| = p_1$, $|K_2| = p_2$ and $|K_3| = p_3$. Hence, by Lemma 3.1, $\text{AG}(R)$ is weakly perfect and $\chi(\text{AG}(R)) = 3$. \square

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