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NICE CONNECTING PATHS IN CONNECTED COMPONENTS OF SETS OF ALGEBRAIC ELEMENTS IN A BANACH ALGEBRA

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Dedicated to the memory of Professor Miroslav Fiedler, from two grateful participants in mathematical olympiads

Abstract. Generalizing earlier results about the set of idempotents in a Banach algebra, or of self-adjoint idempotents in a C^*-algebra, we announce constructions of nice connecting paths in the connected components of the set of elements in a Banach algebra, or of self-adjoint elements in a C^*-algebra, that satisfy a given polynomial equation, without multiple roots. In particular, we prove that in the Banach algebra case every such non-central element lies on a complex line, all of whose points satisfy the given equation. We also formulate open questions.

Keywords: Banach algebra; C^*-algebra; (self-adjoint) idempotent; connected component of (self-adjoint) algebraic elements; (local) pathwise connectedness; similarity; analytic path; polynomial path; polygonal path; centre of a Banach algebra; distance of connected components

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1. Introduction

Let A be a unital complex Banach algebra. Sometimes we will assume that, moreover, A is a C^*-algebra.

We let

$$E(A) := \{a \in A\colon\ a^2 = a\}$$

be the set of *idempotents* of A, and

$$S(A) := \{a \in A\colon\ a^2 = a = a^*\}$$

the set of *self-adjoint idempotents* for the C^*-algebra case.

The *connected components of* $E(A)$ *and of* $S(A)$ have been investigated by many authors. To some of them we will refer later at the respective theorems. An ample literature is given in [1].

Let

$$p(\lambda) := \prod_{i=1}^{n} (\lambda - \lambda_i)$$

be a polynomial over $\mathbb{C}$*, with all* λ_i*'s mutually distinct. In the* C^**-algebra case, when considering self-adjoint elements, we will assume that all* λ_i*'s are real.* (In fact, if $q(\lambda) := \prod\{(\lambda - \lambda_i)\colon\ 1 \leqslant i \leqslant n,\ \lambda_i \in \mathbb{R}\}$, then $p(a) = 0$ and $a = a^*$ imply $q(a) = 0$. Thus below we could use $q(\lambda)$ rather than $p(\lambda)$.) The λ_i's are fixed throughout this paper.

We write

$$E_p(A) := \{a \in A\colon\ p(a) = 0\},$$

and

$$S_p(A) := \{a \in A\colon\ p(a) = 0,\ a = a^*\}$$

for the C^*-algebra case. Then $E(A)$ and $S(A)$ are special cases of $E_p(A)$ and $S_p(A)$: namely, for $p(\lambda) := \lambda(\lambda - 1)$.

We say that $\{e_1, \ldots, e_n\} \subset A$ is a *partition of unity*, or in the C^*-algebra case that $\{e_1, \ldots, e_n\} \subset A$ is a *self-adjoint partition of unity*, if

$$\begin{cases} \{e_1, \ldots, e_n\} \subset E(A), \text{ or } \{e_1, \ldots, e_n\} \subset S(A), \\ \text{and } e_i e_j = 0 \text{ for } 1 \leqslant i, j \leqslant n \text{ and } i \neq j, \\ \text{and } \sum\limits_{i=1}^{n} e_i = 1. \end{cases}$$

The detailed proofs of the statements announced in Section 2 will be published in [7]. The idea of this development originates from personal conversations of the authors at the conference Operator theory and applications: Perspectives and challenges, held in Jurata, Poland, March 18–28, 2010, and from the 2011 lecture by the first named author [6].

2. Theorems

The "only if" part of the following Proposition 1 comes from the Riesz decomposition theorem.

Proposition 1. *Let A be a unital complex Banach algebra (C^*-algebra). Let $a \in A$. Then $a \in E_p(A)$ ($a \in S_p(A)$) if and only if there exists a (self-adjoint) partition of unity $\{e_1, \ldots, e_n\}$ such that*

$$a = \sum_{i=1}^{n} \lambda_i e_i.$$

In the "only if" part, for $a \in E_p(A)$ (for $a \in S_p(A)$) one can choose the e_i's as polynomials of a, with complex (real) coefficients, which depend only on the λ_i's.

This representation provides the tool for reducing questions about $E_p(A)$ (about $S_p(A)$) to those about $E(A)$ (about $S(A)$). Of course, for the respective proofs for $E_p(A)$ (for $S_p(A)$) one still has substantial work to do. As an illustration, we include a sketch of proof of Theorem 7 in Section 3.

The distinctness of the λ_i's is essential in order that a should have such a simple form. For $T \in A := B(l^2 \oplus l^2)$, having a block matrix form $(T_{ij})_{i,j=1}^2$, which is subdiagonal (i.e., strictly lower triangular), we have $T^2 = 0$, but $T_{21} \in B(l^2)$ can be as complicated as an element of $B(l^2)$ can be.

A *path in a topological space* X is a continuous map $f\colon [0,1] \to X$. We will say that $f(0), f(1) \in X$ *are connected by this path* f. By a small abuse of language we will also say that $f([0,1]) \subset X$ *is a path in* X (e.g., for polygonal paths). A topological space X is *pathwise connected* if any two of its points are connected by a path in X. A topological space X is *locally pathwise connected* if each point $x \in X$ has a base of (not necessarily open) neighbourhoods consisting of pathwise connected sets.

Theorem 2. *Let A be a unital complex Banach algebra and C a connected component of $E_p(A)$. Then C is a relatively open subset of $E_p(A)$. Further, C is locally pathwise connected via each of the following types of paths:*

1) *similarity via an exponential function, i.e., $t \mapsto \mathrm{e}^{-ct} a \mathrm{e}^{ct}$;*
2) *a polynomial path of degree at most three;*
3) *a polygonal path of n segments.*

For $E(A)$, relative openness of C was proved by Zemánek [11], 1) was proved by Zemánek [11], 2) was proved by Esterle [3] and Tremon [10], 3) was proved by Kovarik [4] (cf. also [11]).

Theorem 3. *Under the hypotheses of Theorem* 2, C *is pathwise connected via each of the following types of paths:*

1) *similarity via a finite product of exponential functions, i.e.,* $t \mapsto \mathrm{e}^{-c_m t} \ldots \mathrm{e}^{-c_1 t} \times a\mathrm{e}^{c_1 t} \ldots \mathrm{e}^{c_m t}$*;*
2) *a polynomial path;*
3) *a polygonal path.*

In fact, there is a path satisfying 1) *and* 2) *simultaneously.*

For $E(A)$, 1) was proved by Zemánek [11], 2) was proved by Esterle [3] and Tremon [10], 3) was proved by Kovarik [4] (cf. also [11]), and the last sentence was proved by [3] and [10].

Problem. Does there exist a uniform bound on the "minimum degree" of these polynomial connections, possibly depending on n, valid for all Banach algebras? Does such a bound exist, depending on n and on A (or even on C)? Even the case of a uniform bound for polynomial connections of idempotents is open, even if we allow dependence of the bound on A (or even on C). For some particular cases, see [10] and [8]. ([9] announced a further partial result, but his proof seems to be incorrect.)

Even the "simplest" case $A := B(l^2)$ is open. (The case $A =: B(\mathbb{C}^n)$ is solved affirmatively by [10], the uniform bound being 3, which is sharp. Here the connected components of $E(A)$ consist of the projections of the same rank.) For $A = B(l^2)$, the connected components of $E(A)$ are $\{e \in A\colon \dim N(e) = \alpha,\ \dim R(e) = \beta\}$, where $0 \leqslant \alpha,\ \beta \leqslant \aleph_0$ are cardinalities with $\alpha + \beta = \aleph_0$, cf. [1] ($N(\cdot)$ is the null-space and $R(\cdot)$ is the range). By [8], for $\min\{\alpha, \beta\} < \aleph_0$, in the respective connected component there exists an at most third degree polynomial path between any two elements of that component. But even the case $\alpha = \beta = \aleph_0$ here is open.

Theorem 4. *Let* A *be a unital complex* C^**-algebra, and* C *a connected component of* $S_p(A)$*. Then* C *is a relatively open subset of* $S_p(A)$*. Further,* C *is locally pathwise connected by similarities via exponential functions, i.e.,* $t \mapsto \mathrm{e}^{-\mathrm{i}ct} a \mathrm{e}^{\mathrm{i}ct}$*, where additionally* $c = c^*$*.*

For $S(A)$, Theorem 4 was proved by Maeda [5] (cf. also [11]).

Theorem 5. *Under the hypotheses of Theorem* 4, C *is pathwise connected by similarities via finite products of exponential functions, i.e.,* $t \mapsto \mathrm{e}^{-\mathrm{i}c_m t} \ldots \mathrm{e}^{-\mathrm{i}c_1 t} a \times \mathrm{e}^{\mathrm{i}c_1 t} \ldots \mathrm{e}^{\mathrm{i}c_m t}$*, where additionally* $c_1 = c_1^*, \ldots, c_m = c_m^*$*.*

For $S(A)$, Theorem 5 was proved by Maeda [5] (cf. also [11]).

For the C^*-algebra case, the analogues of 2) and 3) from Theorems 2 and 3 are false for $S_p(A)$. In fact, already the connected component of $S(B(\mathbb{C}^2))$ consisting of all rank-one orthogonal projections does not contain any non-constant polynomial path. (The connected components of $S(B(\mathbb{C}^n))$ consist of orthogonal projections of the same rank.)

Theorem 6. *Let A be a unital complex Banach algebra (C^*-algebra). Let $a \in E_p(A)$ (let $a \in S_p(A)$). Then a belongs to the centre of A if and only if its connected component in $E_p(A)$ (in $S_p(A)$) is $\{a\}$.*

Theorem 6 for $E(A)$ was proved by Zemánek [11], for $S(A)$ by Maeda [5]. In Theorem 6, of course, the "only if" part for $S_p(A)$ follows from the "only if" part for $E_p(A)$.

Theorem 7. *Let A be a unital complex Banach algebra, and C a connected component of $E_p(A)$. If C is disjoint from the centre of A, then any element of C belongs to a complex line entirely contained in C. In particular, C is unbounded.*

For $E(A)$, Theorem 7 was proved by Zemánek [11].

In the C^*-algebra case even the entire $S_p(A)$ has a distance $\max\{|\lambda_i|\colon 1 \leqslant i \leqslant n\}$ from 0, so the analogue of Theorem 7 for $S_p(A)$ is false for *each* C^*-algebra A.

Theorem 6 and Theorem 7 yield the next Corollary 8.

Corollary 8. *Let A be a unital complex Banach algebra. Then $E_p(A)$ is a union of its isolated points and of complex lines.*

Theorem 9. *There exists an explicit constant $c(\lambda_1, \ldots, \lambda_n) > 0$ (depending on $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and invariant under any map $(\lambda_1, \ldots, \lambda_n) \mapsto (a + b\lambda_1, \ldots, a + b\lambda_n)$ with $a, b \in \mathbb{R}$ and $b \neq 0$) such that the following holds. If A is a unital complex C^*-algebra, and C_1, C_2 are distinct connected components of $S_p(A)$, then the distance of C_1 and C_2 is at least $c(\lambda_1, \ldots, \lambda_n) \min\{|\lambda_i - \lambda_j|\colon 1 \leqslant i, j \leqslant n,\ i \neq j\}$.*

Conjecture. Let A be a unital complex Banach algebra (C^*-algebra) and C_1, C_2 distinct connected components of $E_p(A)$ (of $S_p(A)$). Then the distance of C_1 and C_2 is at least $\min\{|\lambda_i - \lambda_j|\colon 1 \leqslant i, j \leqslant n,\ i \neq j\}$.

For $n = 2$ this conjecture is equivalent to the statement that this distance for $E_p(A) := E(A)$ (for $S_p(A) := S(A)$) is at least 1, which is due to Zemánek [11] (due to Maeda [5]). For $n \geqslant 3$ we do not even know whether this distance for the Banach algebra case is positive.

If true, this conjecture would be sharp, for any Banach algebra: consider $\lambda_i \cdot 1$ and $\lambda_j \cdot 1$.

The Conjecture for the case of $S_p(A)$ would follow from the Conjecture in the case of $E_p(A)$. In fact, different connected components of $S_p(A)$ are subsets of different connected components of $E_p(A)$, by [2], Section 1, Applications, 2), also taking into consideration our Proposition 1 and Theorem 3.

3. A proof

P r o o f of Theorem 7. The proof of Theorem 7 follows from Theorem 3 and Theorem 6. If C is disjoint from the centre, then by Theorem 6 it has more than one elements. Let $a_0 \in C$ be an arbitrary element of C, and let $a_1 \in C$, with $a_1 \neq a_0$. Then, by Theorem 3, 3), there exists a non-constant polygonal path connecting a_0 to a_1 in C. Its first non-constant segment (counted from a_0) is the graph of a non-constant polynomial of degree 1, say of

$$\lambda \mapsto a_0 + b\lambda, \quad \text{from} \quad [0,1] \quad \text{to} \quad C \ (\subset E_p(A) \subset A).$$

Hence

$$b \neq 0 \quad \text{and we have for all } \lambda \in [0,1] \text{ identically } p(a_0 + b\lambda) = 0. \tag{1}$$

Then the equation in (1) is a polynomial equation, with coefficients from A and of degree at most n, for $\lambda \in \mathbb{C}$. (Attention: here the coefficient of λ^n is b^n, which may be 0 even for $b \neq 0$.)

We make an indirect assumption. If the polynomial

$$\mathbb{C} \ni \lambda \mapsto p(a_0 + b\lambda) \in A \tag{2}$$

were not identically 0 *for all* $\lambda \in \mathbb{C}$*,* then for some $\lambda_0 \in \mathbb{C}$ we would have

$$p(a_0 + b\lambda_0) \neq 0.$$

Then for some continuous linear functional a' on A we would have

$$\langle p(a_0 + b\lambda_0), a' \rangle \neq 0.$$

The polynomial

$$\mathbb{C} \ni \lambda \mapsto \langle p(a_0 + b\lambda), a' \rangle \in \mathbb{C} \tag{3}$$

is a $\mathbb{C}$-valued polynomial on $\mathbb{C}$ of degree at most n, which would not vanish at $\lambda_0 \in \mathbb{C}$. Hence *the polynomial in* (3) *would have at most* n *distinct roots.*

However, by (1) we have that *the polynomial in* (3) *vanishes for all* $\lambda \in [0,1]$ *identically.* This is a contradiction, showing that our indirect assumption is false.

That is, the polynomial in (2) is identically 0 for all $\lambda \in \mathbb{C}$. In other words, for all $\lambda \in \mathbb{C}$ we have

$$p(a_0 + b\lambda) = 0, \quad \text{i.e.,} \quad a_0 + b\lambda \in E_p(A),$$

which implies by connectedness of $\mathbb{C}$ that for all $\lambda \in \mathbb{C}$ we have even

$$a_0 + b\lambda \in C.$$

Since by (1) $b \neq 0$, we see that C *contains a complex line passing through its arbitrary point* a_0. $\square$

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