

PSEUDO ALMOST PERIODICITY OF FRACTIONAL
INTEGRO-DIFFERENTIAL EQUATIONS WITH
IMPULSIVE EFFECTS IN BANACH SPACES

ZHINAN XIA, Hangzhou

Received July 22, 2015. First published February 24, 2017.

Abstract. In this paper, for the impulsive fractional integro-differential equations involving Caputo fractional derivative in Banach space, we investigate the existence and uniqueness of a pseudo almost periodic *PC*-mild solution. The working tools are based on the fixed point theorems, the fractional powers of operators and fractional calculus. Some known results are improved and generalized. Finally, existence and uniqueness of a pseudo almost periodic *PC*-mild solution of a two-dimensional impulsive fractional predator-prey system with diffusion are investigated.

Keywords: impulsive fractional integro-differential equation; pseudo almost periodicity; probability density; fractional powers of operator

MSC 2010: 34A37, 26A33, 34C27

1. INTRODUCTION

The concept of a pseudo almost periodic function was introduced by Zhang [26], [27] in the early nineties. It is an important generalization of an almost periodic function. Since then, this pioneer work has attracted more and more attention and many authors have made important contributions to this theory. For more details on pseudo almost periodic functions and related topics, one can see [5], [8], [9], [14], [12], [17] and the references therein.

This research is supported by the National Natural Science Foundation of China (Grant Nos. 11501507, 11426201).

In this paper, we investigate the existence and uniqueness of pseudo almost periodic mild solutions of impulsive fractional integro-differential equations

$$(1.1) \quad {}^c D^\alpha u(t) + Au(t) = f(t, u(t)) + (Ku)(t) + \sum_{k=-\infty}^{\infty} G_k(u(t))\delta(t - \tau_k),$$

where

$$(Ku)(t) = \int_{-\infty}^t k(t-s)g(s, u(s)) \, ds,$$

$0 < \alpha \leq 1$, $-A: \mathcal{D}(A) \subset X \rightarrow X$ is a linear infinitesimal operator of an analytic semigroup $S(t)$, f, g are pseudo almost periodic in $t \in \mathbb{R}$ uniformly in the second variable, $G_k: \mathcal{D}(G_k) \subset X \rightarrow X$ are continuous impulsive operators, $\delta(\cdot)$ is Dirac's delta-function, $\{\tau_k\} \in T$, where T will be defined later. Here the fractional derivative is understood in Caputo's sense. We notice that fractional order models have received much attention in recent years due to their extensive and efficient applications to nonlinear dynamics concerning fluid flows, electrical networks, viscoelasticity, biology and many other branches of science [1], [10], [20].

If (1.1) is without impulsive effects, then (1.1) becomes a fractional integro-differential equation. Existence of almost periodic mild solutions is studied in [7] by semigroup theory. By the Banach contraction mapping principle, pseudo almost periodic solutions are studied in [4].

If $(Ku)(t) = 0$ and $\alpha = 1$, then (1.1) becomes the impulsive differential equations

$$(1.2) \quad u'(t) + Au(t) = f(t, u(t)) + \sum_{k=-\infty}^{\infty} G_k(u(t))\delta(t - \tau_k).$$

For (1.2), the existence and uniqueness of almost periodic solution is investigated under the condition that A is the infinitesimal generator of an analytic semigroup by Stamov and Alzabout in [23]. Later, the results of [23] are generalized by Chérif in [6], where pseudo almost periodic solutions are studied. If A is the infinitesimal generator of a C_0 -semigroup, Liu and Zhang investigate the existence and uniqueness of almost periodic and pseudo almost periodic solutions in Banach space, see [15], [16], [18].

Notice that if $(Ku)(t) = 0$, then (1.1) becomes an impulsive fractional differential equations and existence and uniqueness of almost periodic solutions are investigated in [24]. However, for fractional integro-differential equations with impulsive effects, i.e., (1.1), the study of asymptotic behavior of solutions is rare; particularly for the pseudo almost periodicity of (1.1), it is an untreated topic and this is the main motivation of this paper. We will make use of the fixed point theorems and the

fractional powers of operators to derive some sufficient conditions guaranteeing the existence and uniqueness of pseudo almost periodic solution to (1.1).

The paper is organized as follows. In Section 2, we recall some fundamental results about the notion of piecewise pseudo almost periodic functions including the composition theorem. Sections 3 is devoted to the existence and uniqueness of pseudo almost periodic mild solution of (1.1) by fractional powers of operators and fixed point theorems. In Section 4, some interesting examples are presented to illustrate the main results.

2. PRELIMINARIES AND BASIC RESULTS

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces, Ω a subset of X and let \mathbb{N} , \mathbb{Z} , \mathbb{R} , and \mathbb{C} stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. For A being a linear operator on X , $\mathcal{D}(A)$, $\varrho(A)$, $R(\lambda, A)$ and $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of A . Let T be the set consisting of all real sequences $\{\tau_k\}_{k \in \mathbb{Z}}$ such that $\kappa = \inf_{k \in \mathbb{Z}} (\tau_{k+1} - \tau_k) > 0$. It is immediate that this condition implies that $\lim_{k \rightarrow \infty} \tau_k = \infty$ and $\lim_{k \rightarrow -\infty} \tau_k = -\infty$.

In order to facilitate the discussion below, we further introduce the following notations

- ▷ $C(\mathbb{R}, X)$ (or $C(\mathbb{R} \times \Omega, X)$): the set of continuous functions from \mathbb{R} to X (from $\mathbb{R} \times \Omega$ to X , respectively).
- ▷ $BC(\mathbb{R}, X)$ (or $BC(\mathbb{R} \times \Omega, X)$): the Banach space of bounded continuous functions from \mathbb{R} to X (from $\mathbb{R} \times \Omega$ to X , respectively) with the supremum norm.
- ▷ $PC(\mathbb{R}, X)$: the space formed by all piecewise continuous functions $f: \mathbb{R} \rightarrow X$ such that $f(\cdot)$ is continuous at t for any $t \notin \{\tau_k\}_{k \in \mathbb{Z}}$, $f(\tau_k^+)$, $f(\tau_k^-)$ exist, and $f(\tau_k^-) = f(\tau_k)$ for all $k \in \mathbb{Z}$.
- ▷ $PC(\mathbb{R} \times \Omega, X)$: the space formed by all piecewise continuous functions $f: \mathbb{R} \times \Omega \rightarrow X$ such that for any $x \in \Omega$, $f(\cdot, x) \in PC(\mathbb{R}, X)$ and for any $t \in \mathbb{R}$, $f(t, \cdot)$ is continuous at $x \in \Omega$.

Following [20], we recall the fractional integral of order $\alpha > 0$ as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

and the fractional Caputo's derivative of the function f of order $0 < \alpha < 1$ as

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

where $\Gamma(\alpha)$ is the classical Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Definition 2.1 ([11]). A function $f: \mathbb{R} \rightarrow X$ is said to be almost periodic if for each $\varepsilon > 0$ there exists an $l(\varepsilon) > 0$ such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t + \tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$. Denote by $AP(\mathbb{R}, X)$ the set of such functions.

Definition 2.2 ([21]). A sequence $\{x_n\}$ is called almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its ε -periods, i.e., there exists a natural number $l = l(\varepsilon)$ such that for $k \in \mathbb{Z}$ there is at least one number p in $[k, k + l]$ for which inequality $\|x_{n+p} - x_n\| < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, X)$ the set of such sequences.

For $\{\tau_k\}_{k \in \mathbb{Z}} \in T$, $\{\tau_k^j\}$ is defined by

$$\{\tau_k^j = \tau_{k+j} - \tau_k\}, \quad k \in \mathbb{Z}, \quad j \in \mathbb{Z}.$$

It is easy to verify that the numbers τ_k^j satisfy

$$\tau_{k+i}^j - \tau_k^j = \tau_{k+j}^i - \tau_k^i, \quad \tau_k^j - \tau_k^i = \tau_{k+i}^{j-i} \quad \text{for } i, j, k \in \mathbb{Z}.$$

Definition 2.3 ([21]). A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

- (1) $\{\tau_k^j = \tau_{k+j} - \tau_k\}$, $k, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$ there exists a relatively dense set in \mathbb{R} of ε -almost periods common for all of the sequences $\{\tau_k^j\}$.
- (2) For any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $\|f(t') - f(t'')\| < \varepsilon$.
- (3) For any $\varepsilon > 0$ there exists a relatively dense set Ω_ε in \mathbb{R} such that if $\tau \in \Omega_\varepsilon$, then

$$\|f(t + \tau) - f(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$ which satisfy the condition $|t - \tau_k| > \varepsilon$, $k \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, X)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, X)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{\tau_k^j\}$ are equipotentially almost periodic. Let $\mathcal{UPC}(\mathbb{R}, X)$ be the space of all functions $f \in PC(\mathbb{R}, X)$ such that f satisfies the condition (2) in Definition 2.3.

Definition 2.4. A function $f \in PC(\mathbb{R} \times \Omega, X)$ is said to be piecewise almost periodic in t uniformly in $x \in \Omega$ if for each compact set $K \subseteq \Omega$, $\{f(\cdot, x): x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that $\|f(t + \tau, x) - f(t, x)\| \leq \varepsilon$ for all $x \in K$, $\tau \in \Omega_\varepsilon$ and $t \in \mathbb{R}$, $|t - \tau_k| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times \Omega, X)$ the set of all such functions.

Lemma 2.1 ([21]). *If the sequences $\{\tau_k^j\}$ are equipotentially almost periodic, then for each $j > 0$ there exists a positive integer N such that on each interval of length j there are no more than N elements of the sequence $\{\tau_k\}$, i.e.,*

$$i(t, s) \leq N(t - s) + N,$$

where $i(t, s)$ is the number of the points $\{\tau_k\}$ in the interval $[s, t]$.

Lemma 2.2 ([21]). *Assume that $f \in AP_T(\mathbb{R}, X)$, $\{x_k\}_{k \in \mathbb{Z}} \in AP(\mathbb{Z}, X)$, and $\{\tau_k^j\}$, $j \in \mathbb{Z}$ are equipotentially almost periodic. Then for each $\varepsilon > 0$ there exist relatively dense sets Ω_ε of \mathbb{R} and Q_ε of \mathbb{Z} such that*

- (i) $\|f(t + \tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$, $|t - \tau_k| > \varepsilon$, $\tau \in \Omega_\varepsilon$ and $k \in \mathbb{Z}$;
- (ii) $\|x_{k+q} - x_k\| < \varepsilon$ for all $q \in Q_\varepsilon$ and $k \in \mathbb{Z}$;
- (iii) $|\tau_k^q - \tau| < \varepsilon$ for all $q \in Q_\varepsilon$, $\tau \in \Omega_\varepsilon$ and $k \in \mathbb{Z}$.

Define

$$\begin{aligned} PAP_T^0(\mathbb{R}, X) &= \left\{ f \in PC(\mathbb{R}, X): \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|f(t)\| dt = 0 \right\}, \\ PAP_T^0(\mathbb{R} \times \Omega, X) &= \left\{ f \in PC(\mathbb{R} \times \Omega, X): \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|f(t, x)\| dt = 0 \right. \\ &\quad \text{uniformly with respect to } x \in K, \\ &\quad \left. \text{where } K \text{ is an arbitrary compact subset of } \Omega \right\}. \end{aligned}$$

Definition 2.5 ([16]). A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_T(\mathbb{R}, X)$ and $\varphi \in PAP_T^0(\mathbb{R}, X)$. Denote by $PAP_T(\mathbb{R}, X)$ the set of all such functions. $PAP_T(\mathbb{R}, X)$ is a Banach space when endowed with the supremum norm.

Definition 2.6 ([16]). Let $PAP_T(\mathbb{R} \times \Omega, X)$ consist of all functions $f \in PC(\mathbb{R} \times \Omega, X)$ such that $f = g + \varphi$, where $g \in AP_T(\mathbb{R} \times \Omega, X)$ and $\varphi \in PAP_T^0(\mathbb{R} \times \Omega, X)$.

Remark 2.1. The set $PAP_T^0(\mathbb{R}, X)$ is a translation invariant subset of $PC(\mathbb{R}, X)$.

The following composition theorem holds for piecewise pseudo almost periodic functions.

Theorem 2.3 ([16]). *Let $f \in PAP_T(\mathbb{R} \times \Omega, X)$, $\varphi \in PAP_T(\mathbb{R}, X)$ and $\mathcal{R}(\varphi) \subset \Omega$. Assume that there exists a constant $L_f > 0$ such that*

$$\|f(t, u) - f(t, v)\| \leq L_f \|u - v\|, \quad t \in \mathbb{R}, \quad u, v \in \Omega.$$

Then $f(\cdot, \varphi) \in PAP_T(\mathbb{R}, X)$.

Next, we introduce the concept of a generalized pseudo almost periodic function (sequence) which is more general than a pseudo almost periodic function (sequence), see [2], [13].

Define

$$\begin{aligned} \tilde{P}AP_0(\mathbb{Z}, X) &= \left\{ x: \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{k=-n}^n \|x_k\| = 0 \right\}. \\ \tilde{P}AP_0(\mathbb{R}, X) &= \left\{ f: \mathbb{R} \rightarrow X \text{ is measure and } \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|f(t)\| dt = 0 \right\}. \end{aligned}$$

Definition 2.7 ([2]). A measurable function $f: \mathbb{R} \rightarrow X$ is called generalized pseudo-almost periodic if $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$, $\varphi \in \tilde{P}AP_0(\mathbb{R}, X)$. Denote by $\tilde{P}AP(\mathbb{R}, X)$ the set of all such functions.

Definition 2.8 ([13]). A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is called generalized pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, X)$, $x_n^2 \in \tilde{P}AP_0(\mathbb{Z}, X)$. Denote by $\tilde{P}AP(\mathbb{Z}, X)$ the set of such sequences.

Lemma 2.4 ([13]). *If $\{x_n\}_{n \in \mathbb{Z}}$ is a $\tilde{P}AP_0(\mathbb{Z}, X)$ sequence, then there exists a function $g \in \tilde{P}AP_0(\mathbb{R}, X)$ such that $g(n) = x_n$, $n \in \mathbb{Z}$.*

Similarly to the proof in [16], one has

Theorem 2.5. *Assume that a sequence of vector-valued functions $\{G_k\}_{k \in \mathbb{Z}}$ is generalized pseudo almost periodic, and there exists a constant $L_1 > 0$ such that*

$$\|G_k(u) - G_k(v)\| \leq L_1 \|u - v\|, \quad u, v \in \Omega, \quad k \in \mathbb{Z}.$$

If $\varphi \in \tilde{P}AP(\mathbb{R}, X)$, then $G_k(\varphi(\tau_k))$ is generalized pseudo almost periodic.

3. IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

In this section, we investigate the existence and uniqueness of piecewise pseudo almost periodic mild solutions of (1.1).

Let $t_0 \in \mathbb{R}$, denote by $u(t) = u(t, t_0, u_0)$, $u_0 \in X$, the solution of (1.1) with an initial condition

$$(3.1) \quad u(t_0) = u_0.$$

The solution $u(t) = u(t, t_0, u_0)$ of problem (1.1) and (3.1) is a piecewise continuous function with points of discontinuity at the moments τ_k , $k \in \mathbb{Z}$, at which it is continuous from the left, i.e. the following relations hold:

$$u(\tau_k^-) = u(\tau_k), \quad u(\tau_k^+) = u(\tau_k) + G_k(u(\tau_k)), \quad k \in \mathbb{Z},$$

that is $u \in PC(\mathbb{R}, X)$. With respect to the norm $\|u\| = \sup_{t \in \mathbb{R}} \|u(t)\|$, one can easily see that $PC(\mathbb{R}, X)$ is a Banach space.

First, we make the following assumptions:

(H₁) $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ such that

$$\|S(t)\| \leq M e^{-\omega t} \quad \text{for } t \geq 0,$$

where $\omega > 0$.

(H₂) $k \in C(\mathbb{R}^+, \mathbb{R})$ and $|k(t)| \leq C_k e^{-\eta t}$ for some positive constants C_k, η .

(H₃) $f \in PAP_T(\mathbb{R} \times X_\beta, X)$ and there exist constants $L_f > 0$, $0 < \beta < 1$ such that

$$\|f(t, u) - f(t, v)\| \leq L_f \|u - v\|_\beta, \quad t \in \mathbb{R}, \quad u, v \in X_\beta,$$

where X_β , $\|\cdot\|_\beta$ are defined later.

(H₄) $g \in PAP_T(\mathbb{R} \times X_\beta, X)$ and there exists a constant $L_g > 0$ such that

$$\|g(t, u) - g(t, v)\| \leq L_g \|u - v\|_\beta, \quad t \in \mathbb{R}, \quad u, v \in X_\beta.$$

(H₅) $G_k \in \tilde{PAP}(\mathbb{Z}, X)$ and there exists a constant $L_1 > 0$ such that

$$\|G_k(u) - G_k(v)\| \leq L_1 \|u - v\|_\beta, \quad t \in \mathbb{R}, \quad u, v \in X_\beta, \quad k \in \mathbb{Z}.$$

Definition 3.1 ([25]). By a *PC*-mild solution of (1.1) and (3.1) we mean a function $u \in PC(\mathbb{R}, X)$ which satisfies the following integral equation:

$$(3.2) \quad u(t) = \begin{cases} \mathcal{T}(t-t_0)u_0 + \int_{t_0}^t (t-s)^{\alpha-1} \mathcal{S}(t-s)(f(s, u(s)) + (Ku)(s)) \, ds & \text{for } t \in [t_0, \tau_1], \\ \mathcal{T}(t-t_0)u_0 + \int_{t_0}^t (t-s)^{\alpha-1} \mathcal{S}(t-s)(f(s, u(s)) + (Ku)(s)) \, ds \\ \quad + (Ku)(s) \, ds + \mathcal{T}(t-\tau_1)y_1 & \text{for } t \in (\tau_1, \tau_2], \\ \vdots \\ \mathcal{T}(t-t_0)u_0 + \int_{t_0}^t (t-s)^{\alpha-1} \mathcal{S}(t-s)(f(s, u(s)) + (Ku)(s)) \, ds \\ \quad + \sum_{t_0 < \tau_k < t} \mathcal{T}(t-\tau_k)y_k & \text{for } t \in (\tau_k, \tau_{k+1}], \end{cases}$$

where

$$\begin{aligned} y_k &= G_k(u(\tau_k)), \quad (Ku)(t) = \int_{-\infty}^t k(t-s)g(s, u(s)) \, ds, \\ \mathcal{T}(t) &= \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) \, d\theta, \quad \mathcal{S}(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) S(t^\alpha \theta) \, d\theta, \\ \xi_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-1/\alpha} \varpi_\alpha(\theta^{-1/\alpha}) \geq 0, \\ \varpi_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty), \end{aligned}$$

ξ_α is a probability density function defined on $(0, \infty)$, that is

$$\xi_\alpha \geq 0, \quad \theta \in (0, \infty) \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta) \, d\theta = 1.$$

Note that when (H_1) holds, we deduce that if $u(t)$ is a bounded *PC*-mild solution of (1.1) on \mathbb{R} , then we take the limit as $t_0 \rightarrow -\infty$ and using (3.2), we obtain

$$(3.3) \quad u(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s)(f(s, u(s)) + (Ku)(s)) \, ds + \sum_{\tau_k < t} \mathcal{T}(t-\tau_k)y_k.$$

Let the operator $-A$ in (1.1) and (3.1) be an infinitesimal operator of an analytic semigroup $S(t)$ in the Banach space X and $0 \in \varrho(A)$. For any $\beta > 0$, we define the fractional power $A^{-\beta}$ of the operator A by

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} S(t) \, dt.$$

$A^{-\beta}$ is bounded, bijective and $A^\beta = (A^{-\beta})^{-1}$, $\beta > 0$ is a closed linear operator such that $\mathcal{D}(A^\beta) = \mathcal{R}(A^{-\beta})$. The operator A^0 is the identity operator in X and for $0 \leq \beta \leq 1$, the space $X_\beta = \mathcal{D}(A^\beta)$ with the norm $\|x\|_\beta = \|A^\beta x\|$ is a Banach space.

Lemma 3.1 ([19]). *Let $-A$ be an infinitesimal operator of an analytic semi-group $S(t)$. Then*

- (i) $S(t): X \rightarrow \mathcal{D}(A^\beta)$ for every $t > 0$ and $\beta \geq 0$;
- (ii) for every $x \in \mathcal{D}(A^\beta)$, it follows that $S(t)A^\beta x = A^\beta S(t)x$;
- (iii) for every $t > 0$, the operator $A^\beta S(t)$ is bounded and

$$(3.4) \quad \|A^\beta S(t)\| \leq M_\beta t^{-\beta} e^{-\lambda t}, \quad M_\beta > 0, \lambda > 0;$$

- (iv) for $0 < \beta \leq 1$ and $x \in \mathcal{D}(A^\beta)$, we have

$$\|S(t)x - x\| \leq C_\beta t^\beta \|A^\beta x\|, \quad C_\beta > 0.$$

Lemma 3.2. *Assume that (H_1) , (H_2) , (H_4) hold. If $u \in PAP_T(\mathbb{R}, X_\beta)$, then*

$$(Ku)(t) = \int_{-\infty}^t k(t-s)g(s, u(s)) ds \in PAP_T(\mathbb{R}, X).$$

Proof. For $u \in PAP_T(\mathbb{R}, X_\beta)$, it is not difficult to see that $\varphi(\cdot) = g(\cdot, u(\cdot)) \in PAP_T(\mathbb{R}, X)$ by Theorem 2.3. Let $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP_T(\mathbb{R}, X)$, $\varphi_2 \in PAP_T^0(\mathbb{R}, X)$, then

$$\int_{-\infty}^t k(t-s)g(s, u(s)) ds = \int_{-\infty}^t k(t-s)\varphi_1(s) ds + \int_{-\infty}^t k(t-s)\varphi_2(s) ds := \Psi_1(t) + \Psi_2(t),$$

where

$$\Psi_1(t) = \int_{-\infty}^t k(t-s)\varphi_1(s) ds, \quad \Psi_2(t) = \int_{-\infty}^t k(t-s)\varphi_2(s) ds.$$

(i) $\Psi_1 \in AP_T(\mathbb{R}, X)$. It is not difficult to see that $\Psi_1 \in UPC(\mathbb{R}, X)$. Since $\varphi_1 \in AP_T(\mathbb{R}, X)$, for $\varepsilon > 0$, let Ω_ε be a relatively dense subset of \mathbb{R} formed by the ε -periods of φ_1 . If $\tau \in \Omega_\varepsilon$, $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$, then

$$\|\varphi_1(t + \tau) - \varphi_1(t)\| < \varepsilon.$$

Hence, by (H₂), for $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$, one has

$$\begin{aligned} \|\Psi_1(t + \tau) - \Psi_1(t)\| &= \left\| \int_{-\infty}^{t+\tau} k(t + \tau - s)\varphi_1(s) \, ds - \int_{-\infty}^t k(t - s)\varphi_1(s) \, ds \right\| \\ &= \left\| \int_{-\infty}^t k(t - s)(\varphi_1(s + \tau) - \varphi_1(s)) \, ds \right\| \\ &\leq \int_{-\infty}^t C_k e^{-\eta(t-s)} \|\varphi_1(s + \tau) - \varphi_1(s)\| \, ds < \frac{C_k}{\eta} \varepsilon, \end{aligned}$$

which implies that $\Psi_1 \in AP_T(\mathbb{R}, X)$.

(ii) $\Psi_2 \in PAP_T^0(\mathbb{R}, X)$. In fact, for $r > 0$, one has

$$\begin{aligned} \frac{1}{2r} \int_{-r}^r \|\Psi_2(t)\| \, dt &= \frac{1}{2r} \int_{-r}^r \left\| \int_{-\infty}^t k(t - s)\varphi_2(s) \, ds \right\| \, dt \\ &= \frac{1}{2r} \int_{-r}^r \left\| \int_0^\infty k(s)\varphi_2(t - s) \, ds \right\| \, dt \\ &\leq \frac{1}{2r} \int_{-r}^r \int_0^\infty C_k e^{-\eta s} \|\varphi_2(t - s)\| \, ds \, dt \\ &\leq \int_0^\infty C_k e^{-\eta s} \Phi_r(s) \, ds, \end{aligned}$$

where

$$\Phi_r(s) = \frac{1}{2r} \int_{-r}^r \|\varphi_2(t - s)\| \, dt.$$

Since $\varphi_2 \in PAP_T^0(\mathbb{R}, X)$, it follows that $\varphi_2(\cdot - s) \in PAP_T^0(\mathbb{R}, X)$ for each $s \in \mathbb{R}$ by Remark 2.1, hence $\lim_{r \rightarrow \infty} \Phi_r(s) = 0$ for all $s \in \mathbb{R}$. By using the Lebesgue dominated convergence theorem, we have $\Psi_2 \in PAP_T^0(\mathbb{R}, X)$. This completes the proof. \square

Lemma 3.3. Assume that (H₁)–(H₄) hold. If $u \in PAP_T(\mathbb{R}, X_\beta)$, then

$$(\Lambda u)(t) := \int_{-\infty}^t (t - s)^{\alpha-1} \mathcal{J}(t - s)(f(s, u(s)) + (Ku)(s)) \, ds$$

lies in $PAP_T(\mathbb{R}, X_\beta)$.

Proof. If $u \in PAP_T(\mathbb{R}, X_\beta)$, $Ku \in PAP_T(\mathbb{R}, X)$ by Lemma 3.2, and $f(\cdot, u(\cdot)) \in PAP_T(\mathbb{R}, X)$ by Theorem 2.3. Hence $h(\cdot) = (Ku)(\cdot) + f(\cdot, u(\cdot)) \in PAP_T(\mathbb{R}, X)$. Let $h = h_1 + h_2$, where $h_1 \in AP_T(\mathbb{R}, X)$, $h_2 \in PAP_T^0(\mathbb{R}, X)$, then

$$(\Lambda u)(t) = \int_{-\infty}^t (t - s)^{\alpha-1} \mathcal{J}(t - s)h(s) \, ds := \Lambda_1(t) + \Lambda_2(t),$$

where

$$\begin{aligned}\Lambda_1(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s) h_1(s) \, ds, \\ \Lambda_2(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s) h_2(s) \, ds.\end{aligned}$$

(i) $\Lambda_1 \in AP_T(\mathbb{R}, X_\beta)$. It is not difficult to see that $\Lambda_1 \in \mathcal{UPC}(\mathbb{R}, X)$. Since $h_1 \in AP_T(\mathbb{R}, X)$, for $\varepsilon > 0$ there exists a relatively dense set Ω_ε such that for $\tau \in \Omega_\varepsilon$, $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$,

$$\|h_1(t + \tau) - h_1(t)\| < \varepsilon.$$

Hence, by Lemma 3.1, for $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$, one has

$$\begin{aligned}\|\Lambda_1(t + \tau) - \Lambda_1(t)\|_\beta &= \|A^\beta(\Lambda_1(t + \tau) - \Lambda_1(t))\| \\ &\leq \int_{-\infty}^t (t-s)^{\alpha-1} \|A^\beta \mathcal{S}(t-s)\| \|h_1(s + \tau) - h_1(s)\| \, ds \\ &\leq \alpha \varepsilon M_\beta \int_{-\infty}^t \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) (t-s)^{-\alpha\beta+\alpha-1} e^{-\lambda\theta(t-s)^\alpha} \, d\theta \, ds \\ &= \alpha \varepsilon M_\beta \int_0^\infty \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} \, d\theta \, d\sigma,\end{aligned}$$

where $\sigma = t - s$. Note that

$$\begin{aligned}&\alpha \int_0^\infty \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} \, d\theta \, d\sigma \\ &= \alpha \int_0^\infty \xi_\alpha(\theta) \int_0^\infty \theta^{1-\beta} \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} \, d\sigma \, d\theta \\ &= \frac{1}{\lambda^{1-\beta}} \int_0^\infty \xi_\alpha(\theta) \int_0^\infty (\lambda\theta\sigma^\alpha)^{-\beta} e^{-\lambda\theta\sigma^\alpha} \, d(\lambda\theta\sigma^\alpha) \, d\theta = \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}.\end{aligned}$$

Hence, one has

$$\|\Lambda_1(t + \tau) - \Lambda_1(t)\|_\beta \leq \frac{\Gamma(1-\beta) M_\beta \varepsilon}{\lambda^{1-\beta}},$$

which implies that $\Lambda_1 \in AP_T(\mathbb{R}, X_\beta)$.

(ii) $\Lambda_2 \in PAP_T^0(\mathbb{R}, X_\beta)$. In fact, for $r > 0$ one has

$$\begin{aligned}
\frac{1}{2r} \int_{-r}^r \|\Lambda_2(t)\|_\beta dt &= \frac{1}{2r} \int_{-r}^r \|A^\beta \Lambda_2(t)\| dt \\
&\leq \frac{1}{2r} \int_{-r}^r \int_{-\infty}^t (t-s)^{\alpha-1} \|A^\beta \mathcal{S}(t-s)\| \|h_2(s)\| ds dt \\
&\leq \frac{\alpha M_\beta}{2r} \int_{-r}^r \int_{-\infty}^t \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) (t-s)^{-\alpha\beta+\alpha-1} e^{-\lambda\theta(t-s)^\alpha} \|h_2(s)\| d\theta ds dt \\
&= \frac{\alpha M_\beta}{2r} \int_{-r}^r \int_0^\infty \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} \|h_2(t-\sigma)\| d\theta d\sigma dt \\
&= \alpha M_\beta \int_0^\infty \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} H_r(\sigma) d\theta d\sigma,
\end{aligned}$$

where

$$H_r(\sigma) = \frac{1}{2r} \int_{-r}^r \|h_2(t-\sigma)\| dt.$$

Since $h_2 \in PAP_T^0(\mathbb{R}, X)$, it follows that $h_2(\cdot - \sigma) \in PAP_T^0(\mathbb{R}, X)$ for each $\sigma \in \mathbb{R}$ by Remark 2.1, then $\lim_{r \rightarrow \infty} H_r(\sigma) = 0$ for all $\sigma \in \mathbb{R}$. Hence $\Lambda_2 \in PAP_T^0(\mathbb{R}, X_\beta)$. \square

Theorem 3.4. Assume that (H₁)–(H₅) hold. If $\Theta < 1$, where

$$\Theta = M_\beta(L_g C_k \eta^{-1} + L_f) \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}} + 2L_1 N M_\beta \left(\frac{1}{m^\beta} + \frac{1}{e^\lambda - 1} \right),$$

then (1.1) has a unique PC-mild solution $u \in PAP_T(\mathbb{R}, X_\beta)$.

Proof. Let $\mathcal{F}: PAP_T(\mathbb{R}, X_\beta) \rightarrow PC(\mathbb{R}, X_\beta)$ be the operator defined by

$$\begin{aligned}
(3.5) \quad (\mathcal{F}u)(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s) (f(s, u(s)) + (Ku)(s)) ds \\
&\quad + \sum_{\tau_k < t} \mathcal{T}(t-\tau_k) G_k(u(\tau_k)).
\end{aligned}$$

We will show that \mathcal{F} has a fixed point in $PAP_T(\mathbb{R}, X_\beta)$ and divide the proof into several steps.

(i) $\mathcal{F}u \in PAP_T(\mathbb{R}, X_\beta)$. For $u \in PAP_T(\mathbb{R}, X_\beta)$, by Lemma 3.3, one has

$$(\Lambda u)(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s) (f(s, u(s)) + (Ku)(s)) ds \in PAP_T(\mathbb{R}, X_\beta).$$

It remains to show that

$$(3.6) \quad \sum_{\tau_k < t} \mathcal{T}(t-\tau_k) G_k(u(\tau_k)) \in PAP_T(\mathbb{R}, X_\beta).$$

By Theorem 2.5, $G_k(u(\tau_k)) \in \tilde{P}AP(\mathbb{Z}, X)$. Let $G_k(u(\tau_k)) = \beta_k + \gamma_k$, where $\beta_k \in AP(\mathbb{Z}, X)$ and $\gamma_k \in \tilde{P}AP_0(\mathbb{Z}, X)$, then

$$\sum_{\tau_k < t} \mathcal{T}(t - \tau_k) G_k(u(\tau_k)) = \sum_{\tau_k < t} \mathcal{T}(t - \tau_k) \beta_k + \sum_{\tau_k < t} \mathcal{T}(t - \tau_k) \gamma_k := \Phi_1(t) + \Phi_2(t).$$

Since $\{\tau_k^j\}$, $k, j \in \mathbb{Z}$ are equipotentially almost periodic, hence by Lemma 2.2, for any $\varepsilon > 0$ there exist relative dense sets of real numbers Ω_ε and integers Q_ε such that for $\tau_k < t \leq \tau_{k+1}$, $\tau \in \Omega_\varepsilon$, $q \in Q_\varepsilon$, $|t - \tau_k| > \varepsilon$, $|t - \tau_{k+1}| > \varepsilon$, $j \in \mathbb{Z}$, one has

$$t + \tau > \tau_k + \varepsilon + \tau > \tau_{k+q},$$

and

$$\tau_{k+q+1} > \tau_{k+1} + \tau - \varepsilon > t + \tau,$$

that is $\tau_{k+q} < t + \tau < \tau_{k+q+1}$. Then

$$\begin{aligned} \|\Phi_1(t + \tau) - \Phi_1(t)\|_\beta &= \left\| \sum_{\tau_k < t + \tau} \mathcal{T}(t + \tau - \tau_k) \beta_k - \sum_{\tau_k < t} \mathcal{T}(t - \tau_k) \beta_k \right\|_\beta \\ &\leq \sum_{\tau_k < t} \|\mathcal{T}(t - \tau_k)(\beta_{k+q} - \beta_k)\|_\beta \\ &= \sum_{\tau_k < t} \|A^\beta \mathcal{T}(t - \tau_k)\| \|(\beta_{k+q} - \beta_k)\| \\ &\leq M_\beta \varepsilon \sum_{\tau_k < t} \int_0^\infty \theta^{-\beta} \xi_\alpha(\theta) (t - \tau_k)^{-\alpha\beta} e^{-\lambda\theta(t - \tau_k)^\alpha} d\theta \\ &\leq M_\beta \varepsilon \int_0^\infty \xi_\alpha(\theta) \left(\sum_{0 < \theta(t - \tau_k)^\alpha \leq 1} (\theta(t - \tau_k)^\alpha)^{-\beta} e^{-\lambda\theta(t - \tau_k)^\alpha} \right. \\ &\quad \left. + \sum_{j=1}^\infty \sum_{j < \theta(t - \tau_k)^\alpha \leq j+1} (\theta(t - \tau_k)^\alpha)^{-\beta} e^{-\lambda\theta(t - \tau_k)^\alpha} \right) d\theta \\ &\leq M_\beta \varepsilon \int_0^\infty \xi_\alpha(\theta) \left(\frac{2N}{m^\beta} + \frac{2N}{e^\lambda - 1} \right) d\theta \\ &= 2M_\beta N \varepsilon \left(\frac{1}{m^\beta} + \frac{1}{e^\lambda - 1} \right), \end{aligned}$$

where $m = \min\{\theta(t - \tau_k)^\alpha : 0 < \theta(t - \tau_k)^\alpha \leq 1\}$. Hence $\Phi_1 \in AP_T(\mathbb{R}, X_\beta)$.

Next, we show that $\Phi_2 \in PAP_T^0(\mathbb{R}, X_\beta)$. Since $\gamma_k \in \tilde{P}AP_0(\mathbb{Z}, X)$, by Lemma 2.4 and [13] there exists $g(t) = \gamma_k$, $t \in [k, k+1)$ such that $g \in \tilde{P}AP_0(\mathbb{R}, X)$ and $g(k) = \gamma_k$, $k \in \mathbb{Z}$.

By Lemma 3.1, one has

$$\begin{aligned}
\frac{1}{2r} \int_{-r}^r \|\Phi_2(t)\|_\beta dt &= \frac{1}{2r} \int_{-r}^r \left\| \sum_{\tau_k < t} \mathcal{T}(t - \tau_k) \gamma_k \right\|_\beta dt \\
&\leq \frac{1}{2r} \int_{-r}^r \sum_{\tau_k < t} \|A^\beta \mathcal{T}(t - \tau_k)\| \|\gamma_k\| \\
&\leq \frac{M_\beta}{2r} \int_{-r}^r \sum_{\tau_k < t} \int_0^\infty \theta^{-\beta} \xi_\alpha(\theta) (t - \tau_k)^{-\alpha\beta} e^{-\lambda\theta(t-\tau_k)^\alpha} \|g(k)\| d\theta dt \\
&\leq \frac{M_\beta}{2r} \int_{-r}^r \sum_{\tau_k < t} \int_0^\infty \xi_\alpha(\theta) (\theta(t - \tau_k)^\alpha)^{-\beta} e^{-\lambda\theta(t-\tau_k)^\alpha} \|g(t)\| d\theta dt \\
&\leq \frac{M_\beta}{2r} \int_{-r}^r \int_0^\infty \xi_\alpha(\theta) \left(\sum_{0 < \theta(t-\tau_k)^\alpha \leq 1} (\theta(t - \tau_k)^\alpha)^{-\beta} e^{-\lambda\theta(t-\tau_k)^\alpha} \right. \\
&\quad \left. + \sum_{j=1}^\infty \sum_{j < \theta(t-\tau_k)^\alpha \leq j+1} (\theta(t - \tau_k)^\alpha)^{-\beta} e^{-\lambda\theta(t-\tau_k)^\alpha} \right) \|g(t)\| d\theta dt \\
&\leq \frac{M_\beta}{2r} \int_{-r}^r \int_0^\infty \xi_\alpha(\theta) \left(\frac{2N}{m^\beta} + \frac{2N}{e^\lambda - 1} \right) \|g(t)\| d\theta dt \\
&\leq 2NM_\beta \left(\frac{1}{m^\beta} + \frac{1}{e^\lambda - 1} \right) \frac{1}{2r} \int_{-r}^r \|g(t)\| dt,
\end{aligned}$$

where N is the constant in Lemma 2.1. Hence

$$\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|\Phi_2(t)\|_\beta dt = 0,$$

then $\Phi_2 \in PAP_T^0(\mathbb{R}, X_\beta)$. So $\mathcal{F}u \in PAP_T(\mathbb{R}, X_\beta)$.

(ii) \mathcal{F} is a contraction. For $u, v \in PAP_T(\mathbb{R}, X_\beta)$,

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_\beta &\leq \int_{-\infty}^t \|(t-s)^{\alpha-1} \mathcal{S}(t-s)[((Ku)(s) + f(s, u(s))) \\
&\quad - ((Kv)(s) + f(s, v(s)))]\|_\beta ds \\
&\quad + \sum_{\tau_k < t} \|\mathcal{T}(t - \tau_k)[G_k(u(\tau_k)) - G_k(v(\tau_k))]\|_\beta \\
&\leq \int_{-\infty}^t (t-s)^{\alpha-1} \|A^\beta \mathcal{S}(t-s)\| \|[(Ku)(s) + f(s, u(s))] \\
&\quad - ((Kv)(s) + f(s, v(s)))]\| ds \\
&\quad + \sum_{\tau_k < t} \|A^\beta \mathcal{T}(t - \tau_k)\| \|G_k(u(\tau_k)) - G_k(v(\tau_k))\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha M_\beta (L_g C_k \eta^{-1} + L_f) \|u - v\|_\beta \\
&\quad \times \int_{-\infty}^t \int_0^\infty \theta^{1-\beta} \xi_\alpha(\theta) (t-s)^{-\alpha\beta+\alpha-1} e^{-\lambda\theta(t-s)^\alpha} d\theta ds \\
&\quad + L_1 M_\beta \|u - v\|_\beta \sum_{\tau_k < t} \int_0^\infty \theta^{-\beta} \xi_\alpha(\theta) (t-\tau_k)^{-\alpha\beta} e^{-\lambda\theta(t-\tau_k)^\alpha} d\theta \\
&\leq M_\beta (L_g C_k \eta^{-1} + L_f) \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}} \|u - v\|_\beta \\
&\quad + 2L_1 N M_\beta \left(\frac{1}{m^\beta} + \frac{1}{e^\lambda - 1} \right) \|u - v\|_\beta \\
&= \Theta \|u - v\|_\beta.
\end{aligned}$$

Since $\Theta < 1$, \mathcal{F} is a contraction.

By (i), $\mathcal{F}(PAP_T(\mathbb{R}, X_\beta)) \subset PAP_T(\mathbb{R}, X_\beta)$. Since (ii) holds, by the Banach contraction mapping principle, \mathcal{F} has a unique fixed point in $PAP_T(\mathbb{R}, X_\beta)$, which is the unique piecewise pseudo almost periodic *PC*-mild solution of (1.1). \square

If $(Ku)(t) = 0$, then (1.1) is an impulsive fractional differential equation

$$(3.7) \quad {}^c D^\alpha u(t) + Au(t) = f(t, u(t)) + \sum_{k=-\infty}^{\infty} G_k(u(t)) \delta(t - \tau_k).$$

By Theorem 3.4, one has the following result:

Corollary 3.5. *Assume that (H_1) , (H_3) , (H_5) hold. If $M_\beta L_f \Gamma(1-\beta)/\lambda^{1-\beta} + 2L_1 N M_\beta (1/m^\beta + 1/(e^\lambda - 1)) < 1$, then (3.7) has a unique *PC*-mild solution $u \in PAP_T(\mathbb{R}, X_\beta)$.*

4. EXAMPLES

In this section, we provide some examples to illustrate our main results.

Example 4.1. Consider the fractional partial differential equation with impulsive effects

$$(4.1) \quad \left\{ \begin{aligned} & {}^c D^\alpha w(t, x(t)) - \frac{\partial^2 w(t, x)}{\partial x^2} \\ &= \int_{-\infty}^t k(t-s) g(s, x, w(s, x)) ds + \gamma F(t) \cos(w(t, x)), \\ &\quad t \in \mathbb{R}, t \neq \tau_k, k \in \mathbb{Z}, x \in (0, 1), \\ & w(\tau_k^+, x) = (\beta_k + 1)w(\tau_k, x), \quad k \in \mathbb{Z}, x \in [0, 1], \\ & w(t, 0) = w(t, 1) = 0, \end{aligned} \right.$$

where $1 < \alpha \leq 1$, $\tau_k = k + |\sin k + \sin \sqrt{2}k|/4$, $F \in PAP_T(\mathbb{R}, X)$, $\beta_k \in PAP(\mathbb{Z}, \mathbb{R})$. Note that integers $\{\tau_k^j\}$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$ are equipotentially almost periodic and $\kappa = \inf_{i \in \mathbb{Z}} (\tau_{k+1} - \tau_k) > 0$; one can see [16], [21] for more details.

Let $X = (L^2[0, 1], \|\cdot\|_{L^2})$, define the linear operator $-A$ by

$$D(-A) = \{u \in X: u'' \in X, u(0) = u(1) = 0\} \quad \text{and} \quad -Au = \Delta u = u'', u \in D(-A).$$

It is well known, see [19] that $-A$ is the infinitesimal generator of a semigroup $S(t)$ on X with $\|S(t)\|_{L^2} \leq e^{-t}$ for $t \geq 0$, hence (H_1) holds. Let $u(t)x = w(t, x)$, $t \in \mathbb{R}$, $x \in [0, 1]$, then (4.8) can be rewritten in the abstract form (1.1). Since $G_k(u) = \beta_k u$ and $\beta_k \in PAP(\mathbb{Z}, \mathbb{R})$, (H_5) holds with $L_1 = \sup_{k \in \mathbb{Z}} \|\beta_k\|$. By Theorem 3.4, one has

Theorem 4.1. *Under assumptions (H_2) , (H_4) , if $L = \max\{\gamma, L_g, L_1\}$ is sufficiently small, then (4.8) has a unique PC-mild PAP_T solution.*

Example 4.2. Consider a two-dimensional impulsive fractional predator-prey system with diffusion

$$(4.2) \quad \begin{cases} {}^c D^\alpha u(t, x(t)) = \mu_1 \Delta u + u \left[a_1(t, x) - b(t, x)u - \frac{c_1(t, x)v}{r(t, x)v + u} \right], \\ \hspace{15em} t \in \mathbb{R}, t \neq \tau_k, k \in \mathbb{Z}, \\ {}^c D^\alpha v(t, x(t)) = \mu_2 \Delta v + v \left[-a_2(t, x) + \frac{c_2(t, x)u}{r(t, x)u + v} \right], \\ \hspace{15em} t \in \mathbb{R}, t \neq \tau_k, k \in \mathbb{Z}, \\ u(\tau_k^+, x) = u(\tau_k, x)I_k(x, u(\tau_k, x), v(\tau_k, x)), \quad k \in \mathbb{Z}, \\ v(\tau_k^+, x) = v(\tau_k, x)J_k(x, u(\tau_k, x), v(\tau_k, x)), \quad k \in \mathbb{Z}, \\ \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0, \quad \left. \frac{\partial v}{\partial n} \right|_{\partial \Omega} = 0, \end{cases}$$

where $0 < \alpha \leq 1$, in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$, nonuniformly distributed in the domain $\overline{\Omega} = \Omega \times \partial \Omega$; $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \dots + \partial^2/\partial x_n^2$ is the Laplace operator and $\partial/\partial n$ is the outward normal derivative. $\mu_1 > 0$, $\mu_2 > 0$ are diffusion coefficients, the positive functions a_1, a_2, c_1 and c_2 stand for prey intrinsic growth rate, capturing rate of the predator, death rate of the predator and conversion rate, respectively; one can see [3] for more details.

Let

$$\tau_k = k + \alpha_k, \quad k \in \mathbb{Z},$$

where $\{\alpha_k\}$, $\alpha_k \in \mathbb{R}$, $k \in \mathbb{Z}$ is an almost periodic sequence such that

$$\sup_{k \in \mathbb{Z}} |\alpha_k| = \alpha < \frac{1}{2},$$

then $\{\tau_k^j\}$, $k, j \in \mathbb{Z}$ are equipotentially almost periodic and $\kappa = \inf_{k \in \mathbb{Z}} (\tau_{k+1} - \tau_k) > 0$; one can see [23], [24] for more details.

Let $w = (u, v)$ and

$$\begin{aligned} A &= \begin{bmatrix} \lambda - \mu_1 \Delta & 0 \\ 0 & \lambda - \mu_2 \Delta \end{bmatrix}, \\ f(t, w) &= \begin{bmatrix} u \left[a_1(t, x) - b(t, x)u - \frac{c_1(t, x)v}{r(t, x)v + u} \right] + \lambda u \\ v \left[-a_2(t, x) + \frac{c_2(t, x)u}{r(t, x)u + v} \right] + \lambda v \end{bmatrix}, \\ G_k(w(\tau_k)) &= \begin{bmatrix} u(\tau_k, x)I_k(x, u(\tau_k, x), v(\tau_k, x)) - u(\tau_k, x) \\ v(\tau_k, x)J_k(x, u(\tau_k, x), v(\tau_k, x)) - v(\tau_k, x) \end{bmatrix}, \end{aligned}$$

where $\lambda > 0$, then (4.9) can be rewritten in the form (3.7):

$${}^c D^\alpha w(t) + Aw(t) = f(t, w(t)) + \sum_{k=-\infty}^{\infty} G_k(w(t))\delta(t - \tau_k).$$

It is well known [22] that the operator A is sectorial and $\operatorname{Re} \sigma(A) \leq -\lambda$, the analytic semigroup of the operator A is e^{-At} and

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-At} dt.$$

Assume that

- (A₁) $a_i(t, x)$, $c_i(t, x)$, $i = 1, 2$, $b(t, x)$ and $r(t, x)$ are piecewise pseudo almost periodic functions with respect to t , uniformly for $x \in \overline{\Omega}$, and positive-valued on $\mathbb{R} \times \overline{\Omega}$.
- (A₂) The sequences of functions $\{I_k(x, u, v)\}$, $\{J_k(x, u, v)\}$, $k \in \mathbb{Z}$ are generalized pseudo almost periodic with respect to k , uniformly for $x, u, v \in \overline{\Omega}$.

By Corollary 3.5, one has

Theorem 4.2. *Under assumptions (A₁), (A₂), (H₃), (H₅), if $L = \max\{L_f, L_1\}$ is sufficiently small, then (4.9) has a unique PC-mild PAP_T solution.*

References

- [1] *K. Adolfsson, J. Enelund, P. Olsson*: On the fractional order model of viscoelasticity. *Mech. Time-Depend. Mat.* **9** (2005), 15–34. [doi](#)
- [2] *E. Ait Dads, O. Arino*: Exponential dichotomy and existence of pseudo almost-periodic solutions of some differential equations. *Nonlinear Anal., Theory Methods Appl.* **27** (1996), 369–386. [zbl](#) [MR](#) [doi](#)
- [3] *M. U. Akhmet, M. Beklioglu, T. Ergenc, V. I. Tkachenko*: An impulsive ratio-dependent predator-prey system with diffusion. *Nonlinear Anal., Real World Appl.* **7** (2006), 1255–1267. [zbl](#) [MR](#) [doi](#)
- [4] *J. Cao, Q. Yang, Z. Huang*: Optimal mild solutions and weighted pseudo-almost periodic classical solutions of fractional integro-differential equations. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* **74** (2011), 224–234. [zbl](#) [MR](#) [doi](#)
- [5] *Y. K. Chang, R. Zhang, G. M. N'Guérékata*: Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. *Comput. Math. Appl.* **64** (2012), 3160–3170. [zbl](#) [MR](#) [doi](#)
- [6] *F. Chérif*: Pseudo almost periodic solutions of impulsive differential equations with delay. *Differ. Equ. Dyn. Syst.* **22** (2014), 73–91. [zbl](#) [MR](#) [doi](#)
- [7] *A. Debbouche, M. M. El-borai*: Weak almost periodic and optimal mild solutions of fractional evolution equations. *Electron. J. Differ. Equ. (electronic only)* **2009** (2009), No. 46, 8 pages. [zbl](#) [MR](#)
- [8] *T. Diagana, G. M. N'Guérékata*: Pseudo almost periodic mild solutions to hyperbolic evolution equations in intermediate Banach spaces. *Appl. Anal.* **85** (2006), 769–780. [zbl](#) [MR](#) [doi](#)
- [9] *H.-S. Ding, J. Liang, G. M. N'Guérékata, T. J. Xiao*: Mild pseudo-almost periodic solutions of nonautonomous semilinear evolution equations. *Math. Comput. Modelling* **45** (2007), 579–584. [zbl](#) [MR](#) [doi](#)
- [10] *M. Enelund, P. Olsson*: Damping described by fading memory-analysis and application to fractional derivative models. *Int. J. Solids Struct.* **36** (1999), 939–970. [zbl](#) [MR](#) [doi](#)
- [11] *A. M. Fink*: Almost Periodic Differential Equations. *Lecture Notes in Mathematics* **377**, Springer, New York, 1974. [zbl](#) [MR](#) [doi](#)
- [12] *H. R. Henríquez, B. de Andrade, M. Rabelo*: Existence of almost periodic solutions for a class of abstract impulsive differential equations. *ISRN Math. Anal.* **2011** (2011), Article ID 632687, 21 pages. [zbl](#) [MR](#) [doi](#)
- [13] *J. Hong, R. Obaya, A. Sanz*: Almost-periodic-type solutions of some differential equations with piecewise constant argument. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* **45** (2001), 661–688. [zbl](#) [MR](#) [doi](#)
- [14] *Y. Li, C. Wang*: Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales. *Adv. Difference Equ.* **2012** (2012), Article ID 77, 24 pages. [zbl](#) [MR](#) [doi](#)
- [15] *J. Liu, C. Zhang*: Existence and stability of almost periodic solutions for impulsive differential equations. *Adv. Difference Equ.* **2012** (2012), Article ID 34, 14 pages. [zbl](#) [MR](#) [doi](#)
- [16] *J. Liu, C. Zhang*: Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations. *Adv. Difference Equ.* **2013** (2013), 2013:11, 21 pages. [MR](#) [doi](#)
- [17] *J. Liu, C. Zhang*: Existence and stability of almost periodic solutions to impulsive stochastic differential equations. *Cubo* **15** (2013), 77–96. [zbl](#) [MR](#) [doi](#)
- [18] *J. Liu, C. Zhang*: Existence of almost periodic solutions for impulsive neutral functional differential equations. *Abstr. Appl. Anal.* **2014** (2014), Article ID 782018, 11 pages. [MR](#) [doi](#)
- [19] *A. Pazy*: Semigroup of Linear Operators and Applications to Partial Differential Equations. *Applied Mathematical Sciences* **44**, Springer, New York, 1983. [zbl](#) [MR](#) [doi](#)

- [20] *I. Podlubny*: Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering 198, Academic Press, San Diego, 1999. [zbl](#) [MR](#)
- [21] *A. M. Samoilenko, N. A. Perestyuk*: Impulsive Differential Equations. World Scientific Series on Nonlinear Science. Series A. 14, World Scientific, Singapore, 1995. [zbl](#) [MR](#) [doi](#)
- [22] *G. T. Stamov*: Almost Periodic Solutions of Impulsive Differential Equations. Lecture Notes in Mathematics 2047, Springer, Berlin, 2012. [zbl](#) [MR](#) [doi](#)
- [23] *G. T. Stamov, J. O. Alzabut*: Almost periodic solutions for abstract impulsive differential equations. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 72 (2010), 2457–2464. [zbl](#) [MR](#) [doi](#)
- [24] *G. T. Stamov, I. M. Stamova*: Almost periodic solutions for impulsive fractional differential equations. Dyn. Syst. 29 (2014), 119–132. [zbl](#) [MR](#) [doi](#)
- [25] *J. R. Wang, M. Fečkan, Y. Zhou*: On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Partial Differ. Equ. 8 (2011), 345–361. [zbl](#) [MR](#) [doi](#)
- [26] *C. Zhang*: Pseudo almost periodic solutions of some differential equations. J. Math. Anal. Appl. 181 (1994), 62–76. [zbl](#) [MR](#) [doi](#)
- [27] *C. Zhang*: Pseudo almost periodic solutions of some differential equations. II. J. Math. Anal. Appl. 192 (1995), 543–561. [zbl](#) [MR](#) [doi](#)

Author's address: Z h i n a n X i a, Department of Applied Mathematics, Zhejiang University of Technology, Liuhe Road 288, Hangzhou 310023, Zhejiang, China, e-mail: xiazn299@zjut.edu.cn.