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APPROXIMATION BY SUBLINEAR OPERATORS

G. A. ANASTASSIOU

Abstract. In this paper, we study the approximation of functions by positive sublinear operators under differentiability. We produce general Jackson type inequalities under initial conditions. We apply them to a series of well-known Max-product operators. So our approach is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of a high order derivative of the function under approximation.

1. Introduction

The main motivation here is the monograph by B. Bede, L. Coroianu, and S. Gal [**4**], 2016.

Let $N \in \mathbb{N}$, the well-known Bernstein polynomials ([**6**]) be positive linear operators, defined by the formula

$$B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f\Big(\frac{k}{N}\Big), \quad x \in [0,1]\,,\ f \in C([0,1]). \tag{1}$$

T. Popoviciu in [**6**] 1935 proved for $f \in C([0,1])$ that

$$|B_N(f)(x) - f(x)| \le \frac{5}{4}\omega_1\Big(f, \frac{1}{\sqrt{N}}\Big) \qquad \text{for all } x \in [0,1], \tag{2}$$

where

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in[0,1]:\\ |x-y|\le\delta}} |f(x)-f(y)|, \qquad \delta > 0, \tag{3}$$

is the first modulus of continuity.

G. G. Lorentz in [**5**, p. 21, 1986] proved for $f \in C^1([0,1])$ that

$$|B_N(f)(x) - f(x)| \le \frac{3}{4\sqrt{N}}\omega_1\Big(f', \frac{1}{\sqrt{N}}\Big) \qquad \text{for all } x \in [0,1]. \tag{4}$$

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In [**4**, p. 10], the authors introduced the basic Max-product Bernstein operators

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \qquad N \in \mathbb{N}, \tag{5}$$

where $\bigvee$ stands for maximum, $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$ and $f\colon [0,1] \to \mathbb{R}_+ = [0,\infty)$.

These operators are nonlinear and piecewise rational.

The authors [**4**] studied similar nonlinear operators such as: the Max-product Favard-Szász-Mirakjan operators and their truncated version, the Max-product Baskakov operators and their truncated version, and also many other similar specific operators. The study [**4**] is based on presenting the general theory of sublinear operators. These Max-product operators tend to converge faster to the on hand function.

So as mentioned in [**4**, p. 30] for $f\colon [0,1] \to \mathbb{R}_+$ continuous, we have the estimate

$$|B_N^{(M)}(f)(x) - f(x)| \le 12\omega_1\Big(f, \frac{1}{\sqrt{N+1}}\Big) \qquad \text{for all } N \in \mathbb{N},\ x \in [0,1]. \tag{6}$$

Also from [**4**, p. 36], we mention that for $f\colon [0,1] \to \mathbb{R}_+$ being concave function, we get

$$|B_N^{(M)}(f)(x) - f(x)| \le 2\omega_1\Big(f, \frac{1}{N}\Big) \qquad \text{for all } x \in [0,1], \tag{7}$$

a much faster convergence.

In this article, we expand the study [**4**] by considering smoothness of functions, which is not done in [**4**]. So our inequalities are with respect to $\omega_1\left(f^{(n)}, \delta\right)$, $\delta > 0$, $n \in \mathbb{N}$.

At first,we present general related theory of sublinear operators, and then we apply it to specific as above Max-product operators.

2. Main Results

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, $n \in \mathbb{N}$, and

$$\begin{aligned} CB_+^n(I) = \big\{ f\colon I \to \mathbb{R}_+ : f^{(i)} &\text{ is continuous and bounded on } I, \\ &\text{for all } i = 0,1,\ldots,n \big\}. \end{aligned} \tag{8}$$

Let $f \in CB_+^n(I)$ and any $x, y \in I$. By Taylor's formula, we have

$$f(y) = \sum_{i=0}^{n} f^{(i)}(x) \frac{(y-x)^i}{i!} + \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \big(f^{(n)}(t) - f^{(n)}(x)\big) \mathrm{d}t. \tag{9}$$

We define for

$$f \in CB_+(I) = \{ f\colon I \to \mathbb{R}_+ : f \text{ is continuous and bounded on } I \},$$

the first modulus of continuity

$$\omega_1(f,\delta) = \sup_{\substack{x,y \in I: \\ |x-y| \le \delta}} |f(x) - f(y)|,$$

where $0 < \delta \leq \text{diameter}(I)$.

We call the remainder of (9)

$$R_n(x,y) = \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \big(f^{(n)}(t) - f^{(n)}(x)\big) \mathrm{d}t \qquad \text{for all } x, y \in I. \tag{10}$$

By [**1**, p. 217] and [**2**, Chapter 7, (7.27), p. 194], we get

$$|R_n(x,y)| \leq \frac{\omega_1(f^{(n)},\delta)}{n!} |x-y|^n \Big(1 + \frac{|x-y|}{(n+1)\delta}\Big) \qquad \text{for all } x, y \in I,\ \delta > 0. \tag{11}$$

We may rewrite (11) as

$$|R_n(x,y)| \leq \frac{\omega_1(f^{(n)},\delta)}{n!} \Big[|x-y|^n + \frac{|x-y|^{n+1}}{(n+1)\delta}\Big] \qquad \text{for all } x, y \in I,\ \delta > 0. \tag{12}$$

That is,

$$\Big| f(y) - \sum_{i=0}^{n} f^{(i)}(x) \frac{(y-x)^i}{i!} \Big| \leq \frac{\omega_1(f^{(n)},\delta)}{n!} \Big[|x-y|^n + \frac{|x-y|^{n+1}}{(n+1)\delta}\Big] \tag{13}$$

for all $x, y \in I$, $\delta > 0$.

Furthermore, it holds

$$|f(y) - f(x)| \leq \sum_{i=1}^{n} \Big| f^{(i)}(x) \Big| \frac{|y-x|^i}{i!} + \frac{\omega_1(f^{(n)},\delta)}{n!} \Big[|x-y|^n + \frac{|x-y|^{n+1}}{(n+1)\delta}\Big] \tag{14}$$

for all $x, y \in I$, $\delta > 0$.

In case of $f^{(i)}(x) = 0$ for $i = 1, \ldots, n$, for a specific $x \in I$, we get

$$|f(y) - f(x)| \leq \frac{\omega_1(f^{(n)},\delta)}{n!} \Big[|x-y|^n + \frac{|x-y|^{n+1}}{(n+1)\delta}\Big] \qquad \text{for all } y \in I\ \delta > 0. \tag{15}$$

In case of $n = 1$, we derive

$$|f(y) - f(x)| \leq |f'(x)||y-x| + \omega_1(f',\delta) \Big[|x-y| + \frac{(x-y)^2}{2\delta}\Big] \tag{16}$$

for all $x, y \in I$, $\delta > 0$.

In case of $n = 1$ and $f'(x) = 0$, for a specific $x \in I$, we get

$$|f(y) - f(x)| \leq \omega_1(f',\delta) \Big[|x-y| + \frac{(x-y)^2}{2\delta}\Big] \qquad \text{for all } y \in I,\ \delta > 0. \tag{17}$$

Call $C_+(I) = \{f \colon I \to \mathbb{R}_+ : f \text{ is continuous on } I\}$.

Let $L_N \colon C_+(I) \to CB_+(I)$, $n, N \in \mathbb{N}$, be a sequence of operators satisfying the following properties (see also [**4**, p. 17]):

(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f) \qquad \text{for all } \alpha \geq 0,\ f \in C_+(I), \tag{18}$$

(ii) (Monotonicity)

$$\text{if } f, g \in C_+(I) \text{ satisfy } f \leq g, \text{ then } L_N(f) \leq L_N(g) \text{ for all } N \in \mathbb{N}, \tag{19}$$

and

(iii) (Subadditivity)

$$L_N(f+g) \le L_N(f) + L_N(g) \text{ for all } f, g \in C_+(I). \tag{20}$$

We call L_N positive sublinear operators.

In particular, we study the restrictions $L_N|_{CB_+^n(I)} \colon CB_+^n(I) \to CB_+(I)$.

As in [**4**, p. 17], we get that for $f, g \in CB_+(I)$,

$$|L_N(f)(x) - L_N(g)(x)| \le L_N(|f-g|)(x) \qquad \text{for all } x \in I. \tag{21}$$

Furthermore, also from [**4**, p. 17], we have

$$|L_N(f)(x) - f(x)| \le L_N(|f(\cdot) - f(x)|)(x) + |f(x)||L_N(1)(x) - 1| \quad \text{for all } x \in I. \tag{22}$$

Using (14) in (22), we obtain

$$\begin{aligned} &|L_N(f)(x) - f(x)| \\ &\le f(x)|L_N(1)(x) - 1| + \sum_{i=1}^{n} \frac{|f^{(i)}(x)|}{i!} L_N(|\cdot - x|^i)(x) \\ &\quad + \frac{\omega_1(f^{(n)}, \delta)}{n!} \Big[L_N(|\cdot - x|^n)(x) + \frac{L_N(|\cdot - x|^{n+1})(x)}{(n+1)\delta} \Big] \qquad \text{for all } x \in I,\ \delta > 0. \end{aligned} \tag{23}$$

If $L_N(1) = 1$ and $f^{(i)}(x) = 0$, $i = 1, \ldots, n$, x is fixed in I, we derive that

$$|L_N(f)(x) - f(x)| \le \frac{\omega_1(f^{(n)}, \delta)}{n!} \Big[L_N(|\cdot - x|^n)(x) + \frac{L_N(|\cdot - x|^{n+1})(x)}{(n+1)\delta} \Big], \tag{24}$$

$\delta > 0$.

We assume and choose

$$\delta = \big(L_N(|\cdot - x|^{n+1})(x) \big)^{\frac{1}{n+1}} > 0. \tag{25}$$

Therefore, we get

$$\begin{aligned} |L_N(f)(x) - f(x)| \le{}& \frac{\omega_1\big(f^{(n)}, \big(L_N\left(|\cdot - x|^{n+1}\right)(x)\big)^{\frac{1}{n+1}}\big)}{n!} \\ &\times \Big[L_N\left(|\cdot - x|^n\right)(x) + \frac{\big(L_N\left(|\cdot - x|^{n+1}\right)(x)\big)^{\frac{n}{n+1}}}{(n+1)} \Big]. \end{aligned} \tag{26}$$

Using (16) in (22), we also obtain

$$\begin{aligned} |L_N(f)(x) - f(x)| \le{}& f(x)|L_N(1)(x) - 1| + |f'(x)| L_N(|\cdot - x|)(x) \\ &+ \omega_1(f', \delta) \Big[L_N(|\cdot - x|)(x) + \frac{L_N\big((\cdot - x)^2\big)(x)}{2\delta} \Big] \end{aligned} \tag{27}$$

for all $x \in I$, $\delta > 0$.

Assuming $L_N(1) = 1$ and $f'(x) = 0$, for a specific $x \in I$, we get from (27) that ($n = 1$ case)

$$|L_N(f)(x) - f(x)| \le \omega_1(f', \delta) \Big[L_N(|\cdot - x|)(x) + \frac{L_N\big((\cdot - x)^2\big)(x)}{2\delta} \Big], \quad \delta > 0. \tag{28}$$

Assume and choose

$$\delta = \sqrt{L_N\big((\cdot - x)^2\big)(x)} > 0, \tag{29}$$

then it holds

(30)

$$|L_N(f)(x) - f(x)| \le \omega_1\Big(f', \sqrt{L_N\big((\cdot - x)^2\big)(x)}\Big) \times \left[L_N(|\cdot - x|)(x) + \frac{\sqrt{L_N\big((\cdot - x)^2\big)(x)}}{2}\right] \quad \text{for all } N \in \mathbb{N}.$$

We present Hölder's inequality for positive sublinear operators.

Theorem 1. *Let $L\colon C_+(I) \to CB_+(I)$ be a positive sublinear operator and $f, g \in C_+(I)$. Furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L\left((f(\cdot))^p\right)(s_*)$, $L\left((g(\cdot))^q\right)(s_*) > 0$ for some $s_* \in I$. Then*

$$L(f(\cdot)g(\cdot))(s_*) \le \left(L((f(\cdot))^p)(s_*)\right)^{\frac{1}{p}} \left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}. \tag{31}$$

Proof. Let $a, b \ge 0$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. The Young's inequality says

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{32}$$

Then

$$\begin{aligned} &\frac{f(s)}{\left(L\left((f(\cdot))^p\right)(s_*)\right)^{\frac{1}{p}}} \cdot \frac{g(s)}{\left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}} \\ &\quad \le \frac{(f(s))^p}{p\left(L\left((f(\cdot))^p\right)(s_*)\right)} + \frac{(g(s))^q}{q\left(L\left((g(\cdot))^q\right)(s_*)\right)} \qquad \text{for all } s \in I. \end{aligned} \tag{33}$$

Hence it holds

$$\begin{aligned} &\frac{L\left(f(\cdot)g(\cdot)\right)(s_*)}{\left(L\left((f(\cdot))^p\right)(s_*)\right)^{\frac{1}{p}} \left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}} \\ &\quad \le \frac{\left(L\left((f(\cdot))^p\right)\right)(s_*)}{p\left(L\left((f(\cdot))^p\right)(s_*)\right)} + \frac{\left(L\left((g(\cdot))^q\right)\right)(s_*)}{q\left(L\left((g(\cdot))^q\right)(s_*)\right)} = \frac{1}{p} + \frac{1}{q} = 1 \qquad \text{for } s_* \in I, \end{aligned} \tag{34}$$

proving the claim. ∎

By (25), (31), and $L_N(1) = 1$, we obtain

$$L_N\left(|\cdot - x|^n\right)(x) \le \left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}. \tag{35}$$

In case of $n = 1$ we derive

$$L_N(|\cdot - x|)(x) \le \sqrt{\left(L_N((\cdot - x)^2)(x)\right)}. \tag{36}$$

We have proved the following result.

Theorem 2. *Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $C_+(I)$ into $CB_+(I)$, and $f \in CB_+^n(I)$, where $n \in \mathbb{N}$ and $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume $L_N(1) = 1$ for all $N \in \mathbb{N}$, and $f^{(i)}(x) = 0$, $i = 1,\dots,n$, for a fixed $x \in I$. Furthermore, assume that $L_N\big(|\cdot - x|^{n+1}\big)(x) > 0$ for all $N \in \mathbb{N}$.*

Then

$$
\begin{aligned}
|L_N(f)(x) - f(x)| \le{}& \frac{\omega_1\big(f^{(n)}, (L_N(|\cdot - x|^{n+1})(x))^{\frac{1}{n+1}}\big)}{n!} \\
& \times \left[L_N(|\cdot - x|^n)(x) + \frac{(L_N(|\cdot - x|^{n+1})(x))^{\frac{n}{n+1}}}{(n+1)} \right]
\end{aligned} \tag{37}
$$

for all $N \in \mathbb{N}$.

We give ($n = 1$ case).

Corollary 3. *Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $C_+(I)$ into $CB_+(I)$, $f \in CB_+^1(I)$, and $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume $L_N(1) = 1$ for all $N \in \mathbb{N}$, and $f'(x) = 0$ for a fixed $x \in I$. Furthermore, assume that $L_N\left((\cdot - x)^2\right)(x) > 0$ for all $N \in \mathbb{N}$.*

Then

$$
\begin{aligned}
|L_N(f)(x) - f(x)| \le{}& \omega_1\big(f', \sqrt{\big(L_N((\cdot - x)^2)(x)\big)}\big) \\
& \times \left[L_N\left(|\cdot - x|\right)(x) + \frac{\sqrt{\big(L_N((\cdot - x)^2)(x)\big)}}{2} \right]
\end{aligned} \tag{38}
$$

for all $N \in \mathbb{N}$.

Remark 4.

(i) to Theorem 2: Assuming $f^{(n)}$ is uniformly continuous on I, and $L_N(|\cdot - x|^{n+1})(x) \to 0$ as $N \to \infty$, using (35), we get that $(L_N(f))(x) \to f(x)$ as $N \to \infty$.

(ii) to Corollary 3: Assuming f' is uniformly continuous on I, and $L_N((\cdot - x)^2)(x) \to 0$ as $N \to \infty$, using (36), we get that $(L_N(f))(x) \to f(x)$ as $N \to \infty$.

(iii) The right hand sides of (37) and (38) are finite.

We also give the basic result ($n = 0$ case).

Theorem 5. *Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $C_+(I)$ into $CB_+(I)$, and $f \in CB_+(I)$, where $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume that $L_N\left(|\cdot - x|\right)(x) > 0$ for some fixed $x \in I$ and for all $N \in \mathbb{N}$. Then:* 1)

$$
\begin{aligned}
|L_N(f)(x) - f(x)| \le{}& f(x)\left|L_N(1)(x) - 1\right| \\
& + \left[L_N(1)(x) + 1\right]\omega_1\left(f, L_N\left(|\cdot - x|\right)(x)\right)
\end{aligned} \tag{39}
$$

for all $N \in \mathbb{N}$.

2) *When* $L_N(1) = 1$, *we get*

$$|L_N(f)(x) - f(x)| \leq 2\omega_1 \left(f, L_N\left(|\cdot - x|\right)(x)\right) \qquad \text{for all } N \in \mathbb{N}. \tag{40}$$

Proof. From [**4**, p. 17], we get

$$\begin{aligned} |L_N(f)(x) - f(x)| &\leq f(x)\,|L_N(1)(x) - 1| \\ &\quad + \Big[L_N(1)(x) + \frac{1}{\delta} L_N(|\cdot - x|)(x)\Big]\omega_1(f, \delta), \end{aligned} \tag{41}$$

where $\delta > 0$.

In (41), we choose $\delta = L_N\left(|\cdot - x|\right)(x) > 0$. □

Remark 6. (to Theorem 5) Here $x \in I$ is fixed.

(i) Assume $L_N(1)(x) \to 1$ as $N \to \infty$, and $L_N\left(|\cdot - x|\right)(x) \to 0$ as $N \to \infty$, given that f is uniformly continuous we get that $L_n(f)(x) \to f(x)$ as $N \to \infty$ (use of (39)). Notice here that $L_N(1)(x)$ is bounded.

(ii) Assume that $L_N(1) = 1$, $L_N\left(|\cdot - x|\right)(x) \to 0$ as $N \to \infty$, and f is uniformly continuous on I, then $L_n(f)(x) \to f(x)$ as $N \to \infty$ (use of (40)).

(iii) The right hand sides of (39) and (40) are finite.

(iv) Variants of Theorem 5 have been applied extensively in [**4**] and [**3**].

3. Applications

Here we give applications to Theorem 2 and Corollary 3.

Remark 7. We start with the Max-product Bernstein operators

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)} \qquad \text{for all } N \in \mathbb{N}, \tag{42}$$

$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-1}$, $x \in [0,1]$, $\bigvee$ stands for maximum, and $f \in C_+([0,1]) = \{f : [0,1] \to \mathbb{R}_+ \text{ is continuous}\}$.

Clearly $B_N^{(M)}$ is a positive sublinear operator from $C_+([0,1])$ into itself with $B_N^{(M)}(1) = 1$. Furthermore, we notice that

$$B_N^{(M)}\left(|\cdot - x|^m\right)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) \left|\frac{k}{N} - x\right|^m}{\bigvee_{k=0}^{N} p_{N,k}(x)} > 0 \tag{43}$$

for all $x \in (0,1)$ and any $m \in \mathbb{N}$, $N \in \mathbb{N}$.

By [**4**, p. 31], we have

$$B_N^{(M)}\left(|\cdot - x|\right)(x) \leq \frac{6}{\sqrt{N+1}} \qquad \text{for all } x \in [0,1], N \in \mathbb{N}. \tag{44}$$

Notice that $|\cdot - x|^{m-1} \leq 1$, therefore,

$$|\cdot - x|^m = |\cdot - x|\,|\cdot - x|^{m-1} \leq |\cdot - x|, \qquad m \in \mathbb{N},$$

hence by (19),

$$B_N^{(M)}(|\cdot - x|^m)(x) \leq B_N^{(M)}(|\cdot - x|)(x),$$

that is,

$$B_N^{(M)}\left(|\cdot - x|^m\right)(x) \leq \frac{6}{\sqrt{N+1}} \qquad \text{for all } x \in [0,1],\ m, N \in \mathbb{N}. \tag{45}$$

Denote

$$C_+^n([0,1]) = \{f : [0,1] \to \mathbb{R}_+,\ n\text{-times continuously differentiable}\}, \quad n \in \mathbb{N}.$$

We get

Theorem 8. *Let $f \in C_+^n([0,1])$, a fixed $x \in (0,1)$, such that $f^{(i)}(x) = 0$, $i = 1, \ldots, n$. Then*

$$\begin{aligned} \left|B_N^{(M)}(f)(x) - f(x)\right| \leq & \frac{\omega_1\left(f^{(n)}, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}\right)}{n!} \\ & \times \left[\frac{6}{\sqrt{N+1}} + \frac{1}{(n+1)}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{n}{n+1}}\right] \end{aligned} \tag{46}$$

for all $N \in \mathbb{N}$. We get $B_N^{(M)}(f)(x) \to f(x)$ as $N \to \infty$.

Proof. By Theorem 2. □

The case $n = 1$ follows.

Corollary 9. *Let $f \in C_+^1([0,1])$, a fixed $x \in (0,1)$, such that $f'(x) = 0$. Then*

$$\begin{aligned} \left|B_N^{(M)}(f)(x) - f(x)\right| \leq & \ \omega_1\left(f', \frac{\sqrt{6}}{\sqrt[4]{N+1}}\right) \\ & \times \left[\frac{6}{\sqrt{N+1}} + \frac{\sqrt{6}}{2\left(\sqrt[4]{N+1}\right)}\right] \qquad \textit{for all } N \in \mathbb{N}. \end{aligned} \tag{47}$$

Remark 10. Let $f \in C^2([a,b], \mathbb{R}_+)$, then

$$|f(x) - f(y)| \leq \|f'\|_\infty |x - y| \qquad \text{for all } x, y \in [a,b], \tag{48}$$

and

$$|f'(x) - f'(y)| \leq \|f''\|_\infty |x - y| \qquad \text{for all } x, y \in [a,b]. \tag{49}$$

That is, f, f' are Lipschitz type functions.

Next, we provide examples so that

$$\|f''\|_\infty \leq \|f'\|_\infty. \tag{50}$$

i) Let $f(x) = \sin x$, $f'(x) = \cos x$ and $f''(x) = -\sin x$, here $\|f''\|_\infty = \|f't\|_\infty = 1$ for $x \in [0,\pi]$. Notice also that for $x = \frac{\pi}{2}$, we have $f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$.
ii) Let $x \in [0,\pi]$, $f(x) = (x-1)^3 + 1$, $f'(x) = 3(x-1)^2$, $f''(x) = 6(x-1)$ and $f'(1) = 0$. Notice that $\|f'\|_\infty = 3(\pi-1)^2$ and $\|f''\|_\infty = 6(\pi-1)$ by $|x-1| \leq \pi - 1$. Because $6(\pi-1) \leq 3(\pi-1)^2$, we get that $\|f''\|_\infty \leq \|f'\|_\infty$.

So over Lipschitz classes of functions with Lipschitz derivatives, we would like to compare (6) to (47).

Thus some calculations, we get

$$\frac{\sqrt{6}}{\sqrt[4]{N+1}}\left[\frac{6}{\sqrt{N+1}}+\frac{\sqrt{6}}{2\left(\sqrt[4]{N+1}\right)}\right]\leq\frac{12}{\sqrt{N+1}},\tag{51}$$

true for all $N\in\mathbb{N}$, $N\geq 7$.

Similarly, we get

$$\frac{1}{n!}\Big(\frac{6}{\sqrt{N+1}}\Big)^{\frac{1}{n+1}}\left[\frac{6}{\sqrt{N+1}}+\frac{1}{(n+1)}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{n}{n+1}}\right]\leq\frac{12}{\sqrt{N+1}},\tag{52}$$

for large enough $N\in\mathbb{N}$.

Therefore, (46) and (47), over differentiability, can give better estimates and speeds than (6).

We continue with the following remark.

Remark 11. Now, we focus on the truncated Favard-Szász-Mirakjan operators

$$T_N^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{N}s_{N,k}(x)f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N}s_{N,k}(x)}\qquad\text{for all } x\in[0,1],\ N\in\mathbb{N},\ f\in C_+([0,1]),\tag{53}$$

$s_{N,k}(x)=\frac{(Nx)^k}{k!}$, see also [**4**, p. 11].

By Theorem 3.2.5, [**4**, p. 178], we get

$$\left|T_N^{(M)}(f)(x)-f(x)\right|\leq 6\omega_1\Big(f,\frac{1}{\sqrt{N}}\Big)\qquad\text{for all } N\in\mathbb{N},\ x\in[0,1].\tag{54}$$

Also from [**4**, p. 178–179], we get

$$T_N^{(M)}\left(|\cdot-x|\right)(x)=\frac{\bigvee_{k=0}^{N}\frac{(Nx)^k}{k!}\left|\frac{k}{N}-x\right|}{\bigvee_{k=0}^{N}\frac{(Nx)^k}{k!}}\leq\frac{3}{\sqrt{N}}\qquad\text{for all } x\in[0,1],\ N\in\mathbb{N}.\tag{55}$$

For $m\in\mathbb{N}$, clearly, it holds

$$T_N^{(M)}\left(|\cdot-x|^m\right)(x)\leq T_N^{(M)}\left(|\cdot-x|\right)(x)$$

and

$$T_N^{(M)}\left(|\cdot-x|^m\right)(x)\leq\frac{3}{\sqrt{N}}\qquad\text{for all } x\in[0,1],\ N\in\mathbb{N},\ m\in\mathbb{N}.\tag{56}$$

The operators $T_N^{(M)}$ are positive sublinear from $C_+([0,1])$ into itself with $T_N^{(M)}(1)=1$. Also it holds

$$T_N^{(M)}\left(|\cdot-x|^m\right)(x)=\frac{\bigvee_{k=0}^{N}\frac{(Nx)^k}{k!}\left|\frac{k}{N}-x\right|^m}{\bigvee_{k=0}^{N}\frac{(Nx)^k}{k!}}>0\tag{57}$$

for all $x\in(0,1]$, $m\in\mathbb{N}$, $N\in\mathbb{N}$.

We get the following theorem.

Theorem 12. *Let $f \in C_+^n([0,1])$, x fixed in $(0,1]$, such that $f^{(i)}(x) = 0$, $i = 1, \dots, n$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \le \frac{\omega_1\left(f^{(n)}, \left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{n+1}}\right)}{n!} \cdot \times \left[\frac{3}{\sqrt{N}} + \frac{1}{(n+1)}\left(\frac{3}{\sqrt{N}}\right)^{\frac{n}{n+1}}\right] \quad \text{for all } N \in \mathbb{N}. \tag{58}$$

Proof. By Theorem 2. □

The case $n = 1$ follows.

Corollary 13. *Let $f \in C_+^1([0,1])$, $x \in (0,1]$, and $f'(x) = 0$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f', \frac{\sqrt{3}}{\sqrt[4]{N}}\right)\left[\frac{3}{\sqrt{N}} + \frac{\sqrt{3}}{2\sqrt[4]{N}}\right] \qquad \text{for all } N \in \mathbb{N}. \tag{59}$$

From (58) and/or (59), we get $T_N^{(M)}(f)(x) \to f(x)$ as $N \to \infty$.
We make some remarks.

Remark 14. We compare (58) and (59) to (54). We have

$$\frac{\sqrt{3}}{\sqrt[4]{N}}\left[\frac{3}{\sqrt{N}} + \frac{\sqrt{3}}{2\sqrt[4]{N}}\right] \le \frac{6}{\sqrt{N}} \iff \frac{1}{\sqrt[4]{N}} \le \frac{3\sqrt{3}}{6}, \tag{60}$$

true for large enough $N \in \mathbb{N}$.
Also we find that

$$\frac{1}{n!}\left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{n+1}}\left[\frac{3}{\sqrt{N}} + \frac{1}{(n+1)}\left(\frac{3}{\sqrt{N}}\right)^{\frac{n}{n+1}}\right] \le \frac{6}{\sqrt{N}} \tag{61}$$

$$\iff$$

$$\frac{1}{\sqrt[2(n+1)]{N}} \le \frac{2n! - \frac{1}{(n+1)}}{\sqrt[n+1]{3}}, \tag{62}$$

true for large enough $N \in \mathbb{N}$.
Therefore, (58) and (59), over differentiability, give better estimates and speeds than (54).

Remark 15. Next, we study the truncated Max-product Baskakov operators (see [**4**, p. 11])

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \qquad x \in [0,1],\ f \in C_+([0,1]),\ N \in \mathbb{N}, \tag{63}$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k}\frac{x^k}{(1+x)^{N+k}}. \tag{64}$$

From [**4**, pp. 217–218], we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x)\left|\frac{k}{N} - x\right|}{\bigvee_{k=0}^{N} b_{N,k}(x)} \leq \frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}}, \qquad N \geq 2,\ N \in \mathbb{N}. \tag{65}$$

Let $f \in C_+([0,1])$, then (by [**4**, p. 217]).

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq 24\omega_1\Big(f, \frac{1}{\sqrt{N+1}}\Big), \qquad N \in \mathbb{N},\ N \geq 2,\ x \in [0,1]. \tag{66}$$

See here that

$$\left|\frac{k}{N} - x\right| \leq 1 \qquad \text{for all } x \in [0,1].$$

Let $m \in \mathbb{N}$, then it holds

$$\left(U_N^{(M)}(|\cdot - x|^m)\right)(x) \leq \frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}}, \qquad N \geq 2,\ N \in \mathbb{N}. \tag{67}$$

Also it holds $U_N^{(M)}(1)(x) = 1$ and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Also it holds

$$U_N^{(M)}(|\cdot - x|^m)(x) > 0$$

for all $x \in (0,1]$, $m \in \mathbb{N}$, $N \in \mathbb{N}$.

We give the following theorem.

Theorem 16. *Let $f \in C_+^n([0,1])$, $x \in (0,1]$ fixed, such that $f^{(i)}(x) = 0$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\Big(f^{(n)}, \big(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\big)^{\frac{1}{n+1}}\Big)}{n!} \times \left[\frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}} + \frac{1}{(n+1)}\Big(\frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}}\Big)^{\frac{n}{n+1}}\right] - \{1\} \tag{68}$$

for all $N \in \mathbb{N}$.

Proof. By Theorem 2. □

The case $n = 1$ follows.

Corollary 17. *Let $f \in C_+^1([0,1])$, $x \in (0,1]$ fixed $f'(x) = 0$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \omega_1\Big(f', \Big(\frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}}\Big)^{\frac{1}{2}}\Big) \times \left[\frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}} + \frac{1}{2}\Big(\frac{2\sqrt{3}\,(\sqrt{2}+2)}{\sqrt{N+1}}\Big)^{\frac{1}{2}}\right] \tag{69}$$

for all $N \in \mathbb{N} - \{1\}$.

From (68) and/or (69), we get $U_N^{(M)}(f)(x) \to f(x)$ as $N \to \infty$.

Remark 18. Next we compare (68) and (69) to (66). We notice that

$$\left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{1}{2}}\left[\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}+\frac{1}{2}\left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{1}{2}}\right]\leq\frac{24}{\sqrt{N+1}}$$

$$\Longleftrightarrow \tag{70}$$

$$\frac{1}{\sqrt[4]{N+1}}\leq\frac{24-\sqrt{3}(\sqrt{2}+2)}{\sqrt{2\sqrt{3}(\sqrt{2}+2)}\left(2\sqrt{3}\left(\sqrt{2}+2\right)\right)},$$

true for large enough $N\in\mathbb{N}-\{1\}$.

We also observe that

$$\frac{1}{n!}\left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}\left[\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}+\frac{\left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{n}{n+1}}}{(n+1)}\right]\leq\frac{24}{\sqrt{N+1}} \tag{71}$$

$$\Longleftrightarrow$$

$$\frac{1}{\sqrt[2(n+1)]{N+1}}\leq\frac{n!}{\sqrt[n+1]{2\sqrt{3}(\sqrt{2}+2)}}\left[\frac{12-\frac{\sqrt{3}(\sqrt{2}+2)}{(n+1)!}}{\sqrt{3}(\sqrt{2}+2)}\right], \tag{72}$$

true for large enough $N\in\mathbb{N}-\{1\}$.

Therefore, (68) and (69), over differentiability, give better estimates and speeds than (66).

We continue with the following remarks.

Remark 19. Here we study Max-product Meyer-Köning and Zeller operators (see [**4**, p. 11)] defined by

$$Z_N^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{\infty}s_{N,k}(x)f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty}s_{N,k}(x)}\qquad\text{for all } N\in\mathbb{N},\ f\in C_+([0,1]), \tag{73}$$

$s_{N,k}(x)=\binom{N+k}{k}x^k$, $x\in[0,1]$.

By [**4**, p. 248], we obtain

$$\left|Z_N^{(M)}(f)(x)-f(x)\right|\leq 18\omega_1\left(f,\frac{(1-x)\sqrt{x}}{\sqrt{N}}\right),\qquad N\geq 4,\ x\in[0,1]. \tag{74}$$

By [**4**, p. 253], we get

$$Z_N^{(M)}\left(|\cdot-x|\right)(x)\leq\frac{8(1+\sqrt{5})}{3}\frac{\sqrt{x}(1-x)}{\sqrt{N}}\qquad\text{for all } x\in[0,1]. \tag{75}$$

Let $m\in\mathbb{N}$, then

$$Z_N^{(M)}\left(|\cdot-x|^m\right)(x)=\frac{\bigvee_{k=0}^{N}s_{N,k}(x)\left|\frac{k}{N+k}-x\right|^m}{\bigvee_{k=0}^{N}s_{N,k}(x)}\leq Z_N^{(M)}\left(|\cdot-x|\right)(x), \tag{76}$$

so that

$$Z_N^{(M)}\left(|\cdot - x|^m\right)(x) \leq \frac{8(1+\sqrt{5})}{3}\frac{\sqrt{x}(1-x)}{\sqrt{N}} =: \rho(x) \tag{76}$$

for all $x \in [0,1]$, $N \geq 4$, $m \in \mathbb{N}$.

Also it holds that $Z_N^{(M)}(1) = 1$, and $Z_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Also it holds that

$$Z_N^{(M)}\left(|\cdot - x|^m\right)(x) > 0 \tag{77}$$

for all $x \in (0,1)$, $m \in \mathbb{N}$, $N \in \mathbb{N}$.

Theorem 20. *Let $f \in C_+^n([0,1])$, $n \in \mathbb{N}$, $x \in (0,1)$, and $f^{(i)}(x) = 0$, $i = 1,\ldots,n$. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, (\rho(x))^{\frac{1}{n+1}}\right)}{n!}\left[\rho(x) + \frac{(\rho(x))^{\frac{n}{n+1}}}{(n+1)}\right] \tag{78}$$

for all $N \geq 4$, $N \in \mathbb{N}$.

Proof. By Theorem 2. □

The case $n = 1$ follows.

Corollary 21. *Let $f \in C_+^1([0,1])$, $x \in (0,1)$, and $f'(x) = 0$. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f', \sqrt{\rho(x)}\right)\left[\rho(x) + \frac{\sqrt{\rho(x)}}{2}\right] \tag{79}$$

for all $N \geq 4$, $N \in \mathbb{N}$.

From (78) and (79), we get that $Z_N^{(M)}(f)(x) \to f(x)$ as $N \to \infty$.

We finish with the remark.

Remark 22. Next we compare (78) and (79) to (74).
We notice that

$$\sqrt{\rho(x)}\left[\rho(x) + \frac{\sqrt{\rho(x)}}{2}\right] \leq \frac{18(1-x)\sqrt{x}}{\sqrt{N}}$$

$$\Longleftrightarrow \tag{80}$$

$$\frac{1}{\sqrt{N}} \leq \frac{3}{8(1+\sqrt{5})\sqrt{x}(1-x)}\left(\frac{27 - 2(1+\sqrt{5})}{4(1+\sqrt{5})}\right)^2,$$

true for large enough $N \geq 4$, $N \in \mathbb{N}$, $x \in (0,1)$.

We also observe that

$$\frac{(\rho(x))^{\frac{1}{n+1}}}{n!}\left[\rho(x) + \frac{(\rho(x))^{\frac{n}{n+1}}}{n+1}\right] \leq \frac{18(1-x)\sqrt{x}}{\sqrt{N}} \tag{81}$$

$$\Longleftrightarrow$$

$$\frac{1}{\sqrt{N}} \leq \frac{3}{8\left(1+\sqrt{5}\right)\sqrt{x}(1-x)}\left(\frac{27n!}{4\left(1+\sqrt{5}\right)} - \frac{1}{n+1}\right)^{n+1}, \tag{82}$$

true for large enough $N \geq 4$, $N \in \mathbb{N}$, $x \in (0,1)$.

Therefore, (78) and (79), under differentiability, perform better than (74).

References

1. Anastassiou G., *Moments in probability and approximation theory*, Pitman Research Notes in Mathematics Series, Longman Group UK, New York, NY, 1993.
2. Anastassiou G., *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.
3. Anastassiou G., Coroianu L. and Gal S., *Approximation by a nonlinear Cardaliagnet-Euvrard neural network operator of max-product kind*, J. Comput. Anal. Appl. **12(2)** (2010), 396–406.
4. Bede B., Coroianu L. and Gal S., *Approximation by Max-Product type Operators*, Springer, Heidelberg, New York, 2016.
5. Lorentz G. G., *Bernstein Polynomials*, Chelsea Publishing Company, New ork, NY, 1986, 2nd edition.
6. Popoviciu T., *Sur l'approximation de fonctions convexes d'order superieur*, Mathematica (Cluj), **10** (1935), 49–54.

G. A. Anastassiou, Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A., *e-mail*: `ganastss@memphis.edu`