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DEFORMED HEISENBERG ALGEBRA WITH REFLECTION AND d-ORTHOGONAL POLYNOMIALS

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Abstract. This paper is devoted to the study of matrix elements of irreducible representations of the enveloping deformed Heisenberg algebra with reflection, motivated by recurrence relations satisfied by hypergeometric functions. It is shown that the matrix elements of a suitable operator given as a product of exponential functions are expressed in terms of d-orthogonal polynomials, which are reduced to the orthogonal Meixner polynomials when $d = 1$. The underlying algebraic framework allowed a systematic derivation of the recurrence relations, difference equation, lowering and rising operators and generating functions which these polynomials satisfy.

Keywords: d-orthogonal polynomials; matrix element; coherent state; hypergeometric function; Meixner polynomials; d-dimensional linear functional vector

MSC 2010: 33C45, 33D15, 22E47

1. Introduction

Let $(P_n)_{n\geqslant 0}$ be a polynomial sequence with complex coefficients of n-th degree (i.e. $\deg P_n = n$) and $(u_n)_{n\geqslant 0}$ the corresponding dual sequence defined by

$$\langle u_n, P_m\rangle = \delta_{nm}, \quad n, m = 0, 1, \ldots$$

where $\langle u, f\rangle$ is the effect of a linear functional u on a polynomial f and δ_{nm} is the Kronecker symbol.

For a positive integer d, the polynomials $P_n(x)$ are called d-orthogonal with respect to the linear d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \ldots, u_{d-1})$ (see [10], [14])

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if they realize the *vector orthogonality relations*

$$\begin{cases} \langle u_k, P_m P_n \rangle = 0, & n \geqslant md + k + 1, \\ \langle u_k, P_m P_n \rangle \neq 0, & n = md + k, \end{cases}$$

for each integer $k \in \{0, 1, \ldots, d-1\}$.

When $d = 1$, we return to the well known notion of orthogonality.

Recall that the polynomials $P_n(x)$ are d-orthogonal if and only if they satisfy a recurrence relation of order $d+1$ of the type

$$xP_n(x) = \sum_{k=0}^{d+1} \gamma_{k,n} P_{n+1-k}(x), \tag{1.1}$$

where $\gamma_{0,n}\gamma_{d+1,n} \neq 0$, with the convention $P_{-n} = 0,\ n \geqslant 1$. The result for $d = 1$ is reduced to the so-called Favard theorem.

During the last three decades, numerous explicit examples of d-orthogonal polynomials and multiple orthogonal polynomials have been intensively studied and developed by many authors (see [1], [2], [4], [9], [13], [14]). However, only in the past few years, some works dealing with the connection between d-orthogonal polynomials, multiple orthogonal polynomials and Lie algebras appeared. Indeed, by means of an algebraic approach, multivariate Charlier and Meixner polynomials, d-orthogonal Charlier, Al-Salam Carlitz and Krawtchook polynomials appeared as matrix elements of operators in Lie algebras (see [5], [7], [6], [15]). In the present paper, we shall identify and study some d-orthogonal polynomials generalizing the Meixner polynomials, which are presented as matrix elements of a suitable operator of the deformed Heisenberg algebra with reflection.

The paper is structured as follows. In Section 2, we recall basic facts about the deformed Heisenberg algebra with reflection and define the associated coherent states which we need below. Section 3 is devoted to introducing an operator S that shall be studied along with the associated matrix elements which will be expressed in terms of d-orthogonal polynomials. When $d = 1$, the results obtained are reduced to the Meixner polynomials. An algebraic approach allows us to derive a generating function and a recurrence relation. In Section 4, we focus our study on a family of d-orthogonal polynomials of Meixner type that will be expressed in terms of hypergeometric functions and we determine explicitly a linear d-dimensional functional vector ensuring the d-orthogonality of the polynomials involved.

In the following, we need some definitions and results.

The hypergeometric function is denoted and defined by

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!},$$

where we have used the common notation for the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1), \quad (a)_0 = 1.$$

The binomial theorem is

$$(1.2) \qquad (1-t)^\alpha = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} t^n = {}_1F_0\left(\begin{matrix} -\alpha \\ - \end{matrix} \middle| t \right).$$

For every non-negative integers m, r, i we have

$$(1.3) \qquad \begin{cases} (\beta)_{mr+i} = (\beta)_i!\, r^{mr} \displaystyle\prod_{s=0}^{r-1} \Big(\frac{\beta+i+s}{r}\Big)_m, \\ (mr+i)! = i!\, r^{mr} \displaystyle\prod_{s=0}^{r-1} \Big(\frac{s+i+1}{r}\Big)_m. \end{cases}$$

If we denote by Δ_1 the difference operator defined by $\Delta_1 f(x) = f(x+1) - f(x)$, we have for every polynomial f

$$(1.4) \qquad f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Delta_1^n f(0)(-x)_n \quad \text{and} \quad \Delta_1^n f(0) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k),$$

with $\binom{n}{k} = n!/k!(n-k)!$.

2. The deformed Heisenberg algebra with reflection

The deformed Heisenberg algebra with reflection (see [11]) is defined as the associative algebra $\mathcal{R}$ generated by the elements a, a_+, R and 1 subject to the defining relations

$$(2.1) \qquad [a, a_+] = 1 + 2\mu R, \quad \{a, R\} = \{a_+, R\} = 0, \quad R^2 = 1,$$

where

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA,$$

and satisfying the involution relations

$$a_+^* = a_-, \quad R^* = R.$$

2.1. Fock representation. In a Hilbert space $\mathcal{H}$ with orthonormal basis $|n\rangle$, we construct the Fock representation of $\mathcal{R}$ as follows:

$$\begin{cases} a|n\rangle = \sqrt{n+2\mu\varepsilon_n}\,|n-1\rangle, \\ a_+|n\rangle = \sqrt{n+1+2\mu\varepsilon_{n+1}}\,|n+1\rangle, \\ R|n\rangle = (-1)^n|n\rangle, \end{cases} \tag{2.2}$$

where

$$\varepsilon_n = \begin{cases} 0, & \text{if} \quad n = 2p, \\ 1, & \text{if} \quad n = 2p+1. \end{cases}$$

The action by powers on the basis $|n\rangle$ is given by

$$\begin{cases} a^i|n\rangle = \sqrt{\dfrac{\gamma_n}{\gamma_{n-i}}}\,\Big|n-i\Big\rangle, & i \leqslant n, \\ a^i|n\rangle = 0, & i > n, \\ a_+^n|m\rangle = \sqrt{\dfrac{\gamma_{m+n}}{\gamma_m}}\,\Big|m+n\Big\rangle, \end{cases} \tag{2.3}$$

where γ_n is the sequence defined by

$$\gamma_n := \prod_{k=1}^{n}(k+2\mu\varepsilon_k), \quad \gamma_0 = 1. \tag{2.4}$$

It is obvious to see that

$$\frac{a_+^n|0\rangle}{\sqrt{\gamma_n}} = |n\rangle. \tag{2.5}$$

By induction on n and involution we show according to (2.1) that

$$[a^{2n}, a_+] = 2na^{2n-1}, \quad [a_+^{2n}, a] = -2na_+^{2n-1}, \quad n \in \mathbb{N}^*. \tag{2.6}$$

Hence if f is an entire function then

$$[f(a^2), a_+] = 2af'(a^2), \quad [f(a^2), a_+] = -2a_+f'(a_+^2), \tag{2.7}$$

and if f is also invertible (i.e. $f(0) \neq 0$), then

$$\begin{cases} f(a^2)a_+f(a^2)^{-1} = a_+ + 2af'(a^2)f(a^2)^{-1}, \\ f(a_+^2)af(a_+^2)^{-1} = a - 2a_+f'(a_+^2)f(a_+^2)^{-1}. \end{cases} \tag{2.8}$$

2.2. Coherent states. In our work, we introduce the notion of coherent states associated with the algebra $\mathcal{R}$ as an algebraic tool which will be exploited in order to establish the basic properties of some d-orthogonal polynomials. Let z be a complex number. By $|z\rangle$ we denote the coherent state defined as

$$|z\rangle := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\gamma_n}} \Big| n \Big\rangle. \tag{2.9}$$

Its expansion coefficients are $\langle n|z\rangle = z^n/\sqrt{\gamma_n}$.

From (2.5) we get

$$|z\rangle = \sum_{n=0}^{\infty} \frac{(za_+)^n}{\gamma_n} \Big| 0 \Big\rangle = e_\mu(za_+)|0\rangle,$$

where

$$e_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma_n} = {}_0F_1\left(\begin{matrix} - \\ \mu + \frac{1}{2} \end{matrix} \Bigg| -\frac{z^2}{4} \right) + \frac{z}{2\mu+1} {}_0F_1\left(\begin{matrix} - \\ \mu + \frac{3}{2} \end{matrix} \Bigg| -\frac{z^2}{4} \right).$$

Since the generalized Hermite polynomials $H_n^\mu(x)$ (see [12]) are generated by

$$\sum_{n=0}^{\infty} \frac{H_n^\mu(z)}{2^n n!} t^n = \mathrm{e}^{-t^2/4} e_\mu(zt),$$

$|z\rangle$ can be expressed as

$$|z\rangle = \mathrm{e}^{a_+^2/4} \sum_{n=0}^{\infty} \frac{H_n^\mu(z)}{2^n n!} a_+^n |0\rangle. \tag{2.10}$$

In addition, it is easy to see that $|z\rangle$ is an eigenstate of the operator a,

$$a|z\rangle = z|z\rangle.$$

For any entire function f we have

$$f(a)|z\rangle = f(z)|z\rangle. \tag{2.11}$$

For coherent states $|z_1\rangle$ and $|z_2\rangle$, the inner product is

$$\langle z_1|z_2\rangle = e_\mu(\overline{z_1} z_2).$$

3. Matrix elements of an operator and d-orthogonal polynomials

Let r, d be two positive integers such that $d = 2r - 1$ and let Q be a polynomial with complex coefficients of degree r with $Q(0) = 0$. The operator S and its matrix elements which will be the subject of our study in the rest of the paper are defined by

$$S = \mathrm{e}^{-a_+^2/4}\mathrm{e}^{Q(a^2)}, \quad \psi_{n,k} = \langle k|S|n\rangle. \tag{3.1}$$

It is clear that S is invertible and $S^{-1} = \mathrm{e}^{-Q(a^2)}\mathrm{e}^{a_+^2/4}$. According to (2.8)

$$S^{-1}a_+S = a_+ - 2aQ'(a^2) \quad \text{and} \quad S^{-1}aS = a - \tfrac{1}{2}a_+ + aQ'(a^2). \tag{3.2}$$

3.1. Generating function. In order to calculate the (formal) generating function $F(z,k)$ of the matrix elements $\psi_{n,k}$ defined by $F(z,k) := \sum_{n=0}^{\infty} \psi_{n,k} z^n/\sqrt{\gamma_n}$, we consider the expression of $\langle k|S|z\rangle$.

We have from (2.9)

$$\langle k|S|z\rangle = \left\langle k|S\left|\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\gamma_n}}\right|n\right\rangle\Big\rangle = \sum_{n=0}^{\infty} \psi_{n,k}\frac{z^n}{\sqrt{\gamma_n}}. \tag{3.3}$$

On the other hand, taking into account (2.5), (2.10), (2.11) and (3.1), we get successively

$$\begin{aligned}
\langle k|S|z\rangle &= \langle k|\mathrm{e}^{-a_+^2/4}\mathrm{e}^{Q(a^2)}|z\rangle \\
&= \mathrm{e}^{Q(z^2)}\langle k|\mathrm{e}^{-a_+^2/4}e_\mu(za_+)|0\rangle \\
&= \mathrm{e}^{Q(z^2)}\sum_{m=0}^{\infty}\frac{H_m^\mu(z)}{2^m m!}\langle k|a_+^m|0\rangle \\
&= \mathrm{e}^{Q(z^2)}\sum_{m=0}^{\infty}\frac{\sqrt{\gamma_m}}{2^m m!}H_m^\mu(z)\langle k|m\rangle \\
&= \mathrm{e}^{Q(z^2)}\frac{\sqrt{\gamma_k}}{2^k k!}H_k^\mu(z).
\end{aligned}$$

It follows by virtue of (3.3) that the matrix elements $\psi_{n,k}$ are generated by

$$\sum_{n=0}^{\infty}\psi_{n,k}\frac{z^n}{\sqrt{\gamma_n}} = \mathrm{e}^{Q(z^2)}\frac{\sqrt{\gamma_k}}{2^k k!}H_k^\mu(z). \tag{3.4}$$

Replacing in (3.4) k by $2k$ or $2k+1$ and n by $2n$ or $2n+1$, respectively, we get

$$\sum_{n=0}^{\infty}\psi_{2n,2k}\frac{z^{2n}}{\sqrt{\gamma_{2n}}}+z\sum_{n=0}^{\infty}\psi_{2n+1,2k}\frac{z^{2n}}{\sqrt{\gamma_{2n+1}}}=\mathrm{e}^{Q(z^2)}\frac{\sqrt{\gamma_{2k}}}{2^{2k}(2k)!}H_{2k}^{\mu}(z),$$
$$\sum_{n=0}^{\infty}\psi_{2n,2k+1}\frac{z^{2n}}{\sqrt{\gamma_{2n}}}+z\sum_{n=0}^{\infty}\psi_{2n+1,2k+1}\frac{z^{2n}}{\sqrt{\gamma_{2n+1}}}=\mathrm{e}^{Q(z^2)}\frac{\sqrt{\gamma_{2k+1}}}{2^{2k+1}(2k+1)!}H_{2k+1}^{\mu}(z).$$

Since the generalized Hermite polynomials $H_k^{\mu}(x)$ are expressed as

$$\text{(3.5)}\qquad \begin{cases} H_{2k}^{\mu}(z)=(-1)^k\dfrac{(2k)!}{k!}\,{}_1F_1\left(\begin{matrix}-k\\ \mu+\frac{1}{2}\end{matrix}\middle|z^2\right),\\[2ex] H_{2k+1}^{\mu}(z)=(-1)^k\dfrac{(2k+1))!z}{k!(\mu+1/2)}\,{}_1F_1\left(\begin{matrix}-k\\ \mu+\frac{3}{2}\end{matrix}\middle|z^2\right),\end{cases}$$

$H_{2k}^{\mu}(z)$ is an even function. Consequently $\psi_{2n,2k+1}=\psi_{2n+1,2k}=0$. According to the previous calculations and using the fact that $\gamma_{2n}=2^{2n}n!(\mu+1/2)_n$, $\gamma_{2n+1}=2^{2n+1}n!(\mu+1/2)(\mu+3/2)_n$ we obtain

Proposition 3.1. *The matrix elements $\psi_{2n,2k}$ and $\psi_{2n+1,2k+1}$ are generated by*

$$\sum_{n=0}^{\infty}\psi_{2n,2k}\frac{z^n}{2^n\sqrt{n!(\mu+1/2)_n}}=\frac{(-1)^k}{2^k}\sqrt{\frac{(\mu+1/2)_k}{k!}}\mathrm{e}^{Q(z)}\,{}_1F_1\left(\begin{matrix}-k\\ \mu+\frac{1}{2}\end{matrix}\middle|z\right),$$
$$\sum_{n=0}^{\infty}\psi_{2n+1,2k+1}\frac{z^n}{2^n\sqrt{n!(\mu+3/2)_n}}=\frac{(-1)^k}{2^k}\sqrt{\frac{(\mu+3/2)_k}{k!}}\mathrm{e}^{Q(z)}\,{}_1F_1\left(\begin{matrix}-k\\ \mu+\frac{3}{2}\end{matrix}\middle|z\right).$$

3.2. Recurrence relation. To establish a recurrence relation satisfied by the matrix elements $s_{n,k}=\psi_{2n,2k}$, we start from $\langle 2k|a_+aS|2n\rangle$.

We have according to (2.2)

$$\text{(3.6)}\qquad \langle 2k|a_+aS|2n\rangle=(2k+1+2\mu\varepsilon_{2k+1})s_{n,k}.$$

On the other hand,

$$\text{(3.7)}\qquad \langle 2k|a_+aS|2n\rangle=\langle k|S(S^{-1}a_+S)(S^{-1}aS)|n\rangle.$$

By virtue of (3.2) we have

$$(S^{-1}a_+S)(S^{-1}aS)=a_+a-\tfrac{1}{2}a_+^2+a_+aQ'(a^2)+Q'(a^2)aa_+-2a^2Q'(a^2)(1+Q'(a^2)).$$

Therefore (3.7) becomes

$$\langle 2k|a_+aS|2n\rangle = \langle 2k|Sa_+a|2n\rangle - \tfrac{1}{2}\langle 2k|Sa_+^2|2n\rangle + \langle 2k|Sa_+aQ'(a^2)|2n\rangle + \langle 2k|SQ'(a^2)aa_+|2n\rangle - 2\langle 2k|Sa^2Q'(a^2)(1+Q'(a^2))|2n\rangle. \tag{3.8}$$

From (2.2) we have

$$\langle 2k|Sa_+a|2n\rangle = (2n+2\mu\varepsilon_{2n})s_{n,k}$$

and

$$\langle 2k|Sa_+^2|2n\rangle = \sqrt{(2n+1+2\mu\varepsilon_{2n+1})(2n+2+2\mu\varepsilon_{2n+2})}\,s_{n+1,k}.$$

In addition, after writing the polynomials $Q'(t)$ and $tQ'(t)^2$ in the form,

$$Q'(t) = \sum_{i=0}^{d}\xi_i t^i, \quad tQ'(t)(1+Q'(t)) = \sum_{i=0}^{d}\eta_i t^i,$$

with $\eta_d \neq 0$, $\xi_i = 0$, $i \geqslant r$, we obtain successively according to (2.3)

$$\langle 2k|Sa_+aQ'(a^2)|2n\rangle = \sum_{i=0}^{d}(2n-2i+2\mu\varepsilon_{2n-2i})\xi_i\sqrt{\frac{\gamma_{2n}}{\gamma_{2n-2i}}}s_{n-i,k},$$

$$\langle 2k|SQ'(a^2)aa_+|2n\rangle = (2n+1+2\mu\varepsilon_{2n+1})\sum_{i=0}^{d}\xi_i\sqrt{\frac{\gamma_{2n}}{\gamma_{2n-2i}}}s_{n-i,k},$$

$$\langle 2k|Sa^2Q'(a^2)(1+Q'(a^2))|2n\rangle = \sum_{i=0}^{d}\eta_i\sqrt{\frac{\gamma_{2n}}{\gamma_{2n-2i}}}s_{n-i,k}.$$

Combining (3.6) with the previous calculations, we obtain

Proposition 3.2. *The matrix elements $s_{n,k}$ satisfy the recurrence relation of order $d+1=2r$*

$$s_{n+1,k} = \frac{2(2n+2\mu\varepsilon_{2n}-2k-1-2\mu\varepsilon_{2k+1})}{\sqrt{(2n+1+2\mu\varepsilon_{2n+1})(2n+2+2\mu\varepsilon_{2n+2})}}s_{n,k} + 2\sum_{i=0}^{d}\beta_{n,i}s_{n-i,k}, \tag{3.9}$$

where $\beta_{n,i}$ are complex numbers, with $\beta_{n,d} \neq 0$.

From this relation one can express $s_{n,k}$ recursively, starting from $s_{0,k}$. Indeed, putting $n=0$ or $n=1$ in (3.9), we get

$$s_{1,k} = 2\left(\frac{2\mu\varepsilon_0 - 2k - 1 - 2\mu\varepsilon_{2k+1}}{\sqrt{(1+2\mu\varepsilon_1)(2+2\mu\varepsilon_2)}} + \beta_{0,0}\right)s_{0,k}.$$

Repeating this process we arrive

Corollary 3.3. *The matrix elements $s_{n,k}$ are expressed in the form*

$$s_{n,k} = s_{0,k} P_n^\mu(k), \tag{3.10}$$

where $P_n\mu(k)$ is a polynomial of degree n in the argument k satisfying the recurrence relation of order $d+1$ given by

$$P_{n+1}^\mu(k) = \frac{2(2n + 2\mu\varepsilon_{2n} - 2k - 1 - 2\mu\varepsilon_{2k+1})}{\sqrt{(2n+1+2\mu\varepsilon_{2n+1})(2n+2+2\mu\varepsilon_{2n+2})}} P_n^\mu(k) + 2\sum_{i=0}^{d} \beta_{n,i} P_{n-i}^\mu(k),$$

with the initial conditions $P_0^\mu(k) = 1,\ P_n^\mu(k) = 0,\ n < 0$.

According to (1.1), *the polynomials $P_n^\mu(k)$ are d-orthogonal.*

By virtue of Proposition 3.1, it is easy to see that $\psi_{0,2k} = (-1)^k 2^{-k}\sqrt{(\mu+\frac{1}{2})_k/k!}$. Then taking into account (3.10) we get

Corollary 3.4. *The d-orthogonal polynomials $P_n^\mu(k)$ are generated by*

$$\sum_{n=0}^{\infty} P_n^\mu(k) \frac{z^n}{2^n\sqrt{n!(\mu+1/2)_n}} = \mathrm{e}^{Q(z)}{}_1F_1\left(\begin{matrix}-k\\ \mu+\frac{1}{2}\end{matrix}\middle| z\right). \tag{3.11}$$

Remark 3.5. If we denote by $t_{n,k}$ the matrix elements defined by $t_{n,k} = \psi_{2n+1,2k+1}$ then due to Proposition 3.1 and (3.10) we get $t_{n,k} = t_{0,k}Q_n^\mu(k)$, where $Q_n^\mu(k)$ are d-orthogonal polynomials generated by

$$\sum_{n=0}^{\infty} Q_n^\mu(k) \frac{z^n}{2^n\sqrt{n!(\mu+3/2)_n}} = \mathrm{e}^{Q(z)}{}_1F_1\left(\begin{matrix}-k\\ \mu+\frac{3}{2}\end{matrix}\middle| z\right).$$

Hence $P_n^\mu(k)$ and $Q_n^\mu(k)$ are related by $Q_n^\mu(k) = P_n^{\mu+1}(k)$.

3.3. Link with the Meixner polynomials of the first kind. The Meixner polynomials $M_n(x,\beta,c)$ of the first kind are generated by (see [8])

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{z^n}{n!} = \mathrm{e}^z{}_1F_1\left(\begin{matrix}-x\\ \beta\end{matrix}\middle| \frac{1-c}{c}z\right), \tag{3.12}$$

where $\beta > 0$ and $0 < c < 1$. $M_n(x;\beta,c)$, $n = 0,1,2,\dots$ satisfy the orthogonality relations

$$\sum_{k=0}^{\infty}(1-c)^\beta(\beta)_k\frac{c^k}{k!}M_n(k;\beta,c)M_m(k;\beta,c) = 0, \quad n \neq m. \tag{3.13}$$

It is clear from (3.11) and (3.12) that when $d = 1$, $Q(z) = cz/(1-c)$ and $\beta = \mu + 1$, the polynomials $M_n(x;\beta,c)$ and $P_n^\mu(x)$ are related by

$$P_n^\mu(x) = \frac{c^n\sqrt{(\beta)_n}}{2^n(1-c)^n} M_n(x;\beta,c).$$

4. d-orthogonal polynomials of Meixner type

In this section, we assume that $Q(z) = (cz/(1-c))^r$ and consider the d-orthogonal polynomials $M_n(x;\beta,c,d) = \bigl(2^n(1-c)^n/c^n\sqrt{(\beta)_n}\bigr)P_n(x)$ generated by

$$\sum_{n=0}^{\infty} M_n(x;\beta,c,d)\frac{z^n}{n!} = \mathrm{e}^{z^r}\,{}_1F_1\left(\begin{matrix}-x\\ \beta\end{matrix}\,\middle|\,\frac{1-c}{c}z\right). \tag{4.1}$$

$M_n(x;\beta,c,d)$ reduced to the Meixner polynomials when $d = r = 1$ are called d-orthogonal polynomials of Meixner type.

By $\Delta(a;r)$ we denote the array and defined by $\Delta(a;r) = (a/r, (a+1)/r, \dots, (a+r-1)/r)$.

4.1. Explicit expression. To obtain the explicit expression of the polynomials $M_n(x;\beta,c,d)$, we can proceed directly by expanding the generating function (4.1). Indeed, we have

$$\sum_{n=0}^{\infty} M_n(k;\beta,c,d)\frac{z^n}{n!} = \sum_{s=0}^{\infty}\sum_{i=0}^{k}\frac{(1-c)^i(-k)_i}{c^i i!(\beta)_i s!}z^{i+rs}.$$

Then we get

$$M_n(k;\beta,c,d) = n!\sum_{i=0}^{k}\frac{(1-c)^i(-k)_i}{c^i i!\,(\beta)_i((n-i)/r)!}. \tag{4.2}$$

In (4.2) the discrete variable i can take the values such that

$$\frac{n-i}{r} = s = 0, 1, \dots$$

For any non-negative integer we can put $n = mr + j$, $m = 0, 1, \dots$ and $j = 0, 1, \dots, r-1$. Then i can take the values $i = rl + j$, $l = 0, 1, \dots$

Therefore (4.2) can be written in the form

$$M_n(k;\beta,c,d) = n!\sum_{l=0}^{m}\frac{(1-c)^{rl+j}(-k)_{rl+j}}{c^{rl+j}(rl+j)!\,(\beta)_{rl+j}(m-l)!}, \tag{4.3}$$

which becomes after an easy calculation according to (1.3)

$$M_n(k;\beta,c,d)=\frac{n!(-k)_j(1-c)^j}{m!j!(\beta)_jc^j}\sum_{l=0}^{\infty}\frac{(-1)^l l!\,(-m)_l(1-c)^{rl}\prod_{s=0}^{r-1}\big((-k+j+s)/r\big)_l}{c^{rl}r^{rl}l!\prod_{s=0}^{r-1}\big((\beta+j+s)/r\big)_l}.$$

Therefore, the polynomials $M_n(k;\beta,c,d)$ have the following hypergeometric representation:

$$M_n(k;\beta,c,d)=\frac{n!(-k)_j(1-c)^j}{m!j!(\beta)_jc^j}\,{}_{r+2}F_{2r}\left(\begin{matrix}1,-m,\ \Delta(-k+j;r)\\ \Delta(j+1;r),\ \Delta(\beta+j;r)\end{matrix}\,\middle|\,-\Big(\frac{1-c}{cr}\Big)^r\right).$$

In the particular case $d=1$ (then $m=n$ and $j=0$), we get

$$M_n(k;\beta,c,d)={}_3F_2\left(\begin{matrix}1,\ -n,\ -k\\ 1,\ \beta\end{matrix}\,\middle|\,\frac{c-1}{c}\right)={}_2F_1\left(\begin{matrix}-n,\ -k\\ \beta\end{matrix}\,\middle|\,\frac{c-1}{c}\right).$$

Hence we meet again the hypergeometric representation of the Meixner polynomials $M_n(k;\ \beta,c)$.

4.2. d-orthogonality relations. Let us now express explicitly in terms of hypergeometric functions the linear d-dimensional functional vector $\mathcal{U}={}^t(u_0,u_1,\dots,u_{d-1})$ ensuring the d-orthogonality of the polynomials $M_n(x;\beta,c,d)$. The adopted approach is based on obtaining the dual sequence of a polynomial set via inversion coefficients (see [3]).

The main result of this section is the following theorem.

Theorem 4.1. *The polynomials $M_n(x;\beta,c,d)$ generated by (4.1) are d-orthogonal with respect to the linear d-dimensional functional vector $\mathcal{U}={}^t(u_0,u_1,\dots,u_{d-1})$ given for every $0\leqslant i\leqslant d-1$ and a polynomial f by*

$$\langle u_i,f\rangle=\sum_{s=0}^{r-1}\sum_{k=0}^{\infty}\omega_{i,rk+s}f(rk+s)\ , \tag{4.4}$$

where $\omega_{i,rk+s}$ is given by:

(1) *for $rk+s\leqslant i-1$*

$$\omega_{i,rk+s}=\frac{(-1)^{rk+s}(\beta)_ic^i}{(rk+s)!(i-rk-s)!(1-c)^i}\,{}_{2r}F_r\left(\begin{matrix}\Delta(i+1;r),\Delta(\beta+1;r)\\ \Delta(i-rk-s+1;r)\end{matrix}\,\middle|\,-\Big(\frac{c}{1-c}\Big)^r\right),$$

(2) *for* $rk+s \geqslant i$ *and* $1-r \leqslant s-i \leqslant 0$

$$\omega_{i,rk+s} = \frac{(-1)^{(r+1)k+s}(rk+i)!(\beta)_{rk+i}c^{rk+i}}{(rk+s)!i!k!((r-1)k+i)!(1-c)^{rk+i}} \times {}_{2r+1}F_{r+1}\left(\begin{matrix}1, \Delta(rk+i+1;r), \Delta(\beta+rk+i;r)\\ k+1, \Delta((r-1)k+i+1;r)\end{matrix}\,\middle|\, -\left(\frac{cr}{1-c}\right)^r\right),$$

(3) *for* $rk+s \geqslant i$ *and* $1 \leqslant s-i \leqslant r-1$

$$\omega_{i,rk+s} = \frac{(-1)^{(r+1)k+s+1}(r(k+1)+i)!(\beta)_{r(k+1)+i}c^{r(k+1)+i}}{(rk+s)!i!(r(k+1)+i-k)!(1-c)^{r(k+1)+i}} \times {}_{2r+1}F_{r+1}\left(\begin{matrix}1, \Delta(r(k+1)+i+1;r), \Delta(\beta+r(k+1)+i;r))\\ k+2, \Delta((r(k+1)+i-k+1;r)\end{matrix}\,\middle|\, -\left(\frac{cr}{1-c}\right)^r\right),$$

(4) *for* $rk+s \geqslant i$ *and* $2(1-r) \leqslant d-i \leqslant -r$

$$\omega_{i,rk+s} = \frac{(-1)^{(r+1)k+s-1}(r(k-1)+i)!(\beta)_{r(k-1)+i}c^{r(k-1)+i}}{(rk+s)!i!(r(k-1)+i-k)!(1-c)^{r(k-1)+i}} \times {}_{2r+1}F_{r+1}\left(\begin{matrix}1, \Delta(r(k-1)+i+1;r), \Delta(\beta+r(k-1)+i;r))\\ k, \Delta((r(k-1)+i-k+1;r)\end{matrix}\,\middle|\, -\left(\frac{cr}{1-c}\right)^r\right).$$

Example. Let $d=1$ (then $i=s=0$). We get from Theorem 4.1, Case (2), and with help of the binomial theorem

$$\omega_{0,k} = \frac{c^k(\beta)_k}{k!(1-c)^k}\,{}_3F_2\left(\begin{matrix}1, k+1, \beta+k\\ k+1, 1\end{matrix}\,\middle|\, -\frac{c}{1-c}\right) = \frac{c^k(\beta)_k}{k!(1-c)^k}\,{}_1F_0\left(\begin{matrix}\beta+k\\ -\end{matrix}\,\middle|\, -\frac{c}{1-c}\right) = \frac{(1-c)^\beta c^k(\beta)_k}{k!}.$$

Then we obtain

$$\langle u_0, f\rangle = (1-c)^\beta \sum_{k=0}^{\infty} \frac{c^k(\beta)_k}{k!} f(k).$$

Hence we conclude the orthogonality of Meixner polynomials $M_n(x;\beta,c)$.

Proof of Theorem 4.1. We have from (4.1)

$${}_1F_1\left(\begin{matrix}-x\\ \beta\end{matrix}\,\middle|\, \frac{1-c}{c}z\right) = \mathrm{e}^{-z^r}\sum_{n=0}^{\infty} M_n(x;\beta,c,d)\frac{z^n}{n!}.$$

Then by equalizing the coefficients of z^n, we get

$$(4.5) \qquad (-x)_n = \frac{n!(\beta)_n c^n}{(1-c)^n}\sum_{m=0}^{[n/r]} \frac{(-1)^m}{m!(n-mr)!} M_{n-mr}(x;\beta,c,d)$$

(where $[a]$ is the integer part of a). Applying the dual sequence $(u_i)_{i\geqslant 0}$ of $M_n(x;\beta,c)$ to each member of (4.5) we obtain

$$\begin{cases} \langle u_i,(-x)_n\rangle = \dfrac{(-1)^m n!(\beta)_n c^n}{m!i!(1-c)^n} & \text{if } \ n=mr+i, \\ \langle u_i,(-x)_n\rangle = 0, & \text{otherwise.} \end{cases}$$

With help of (1.4) and (4.6), we get successively for every polynomial f

$$\begin{aligned}
\langle u_i,f\rangle &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Delta_1^n f(0)\langle u_i,(-x)_n\rangle \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{mr+i}}{(mr+i)!}\Delta_1^{mr+i} f(0)\langle u_i,(-x)_{md+i}\rangle \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{mr+i}\frac{(-1)^{m+k}(mr+i)!(\beta)_{mr+i}c^{mr+i}}{i!k!m!(mr+i-k)!(1-c)^{mr+i}}f(k) \\
&= \underbrace{\sum_{m=0}^{\infty}\sum_{k=0}^{i-1}\frac{(-1)^{m+k}(mr+i)!(\beta)_{mr+i}c^{mr+i}}{i!k!m!(mr+i-k)!(1-c)^{mr+i}}f(k)}_{\mathcal{A}_i(f)} \\
&\quad + \underbrace{\sum_{m=0}^{\infty}\sum_{k=0}^{i-1}\frac{(-1)^{m+k}(mr+i)!(\beta)_{mr+i}c^{mr+i}}{i!k!m!(mr+i-k)!(1-c)^{mr+i}}f(k)}_{\mathcal{B}_i(f)},
\end{aligned}$$

with $\mathcal{A}_0(f)=0$. Using the identities (1.3), we obtain

$$\mathcal{A}_i(f)=\sum_{k=0}^{i-1}\omega_{i,k}^{(1)}f(k),$$

where

$$\begin{aligned}
\omega_{i,k}^{(1)} &= \frac{(-1)^k(\beta)_i c^i}{k!(i-k)!(1-c)^i}\sum_{m=0}^{\infty}\frac{(-1)^m r^{mr}\prod_{s=0}^{r-1}\big((i+s+1)/r\big)_m\big((\beta+i+s)/r\big)_m c^{mr}}{\prod_{s=0}^{r-1}\big((i-k+s+1)/r\big)_m(1-c)^{mr}m!} \\
&= \frac{(-1)^k(\beta)_i c^i}{k!(i-k)!(1-c)^i}\,{}_{2r}F_r\left(\begin{matrix}\Delta(i+1;r),\Delta(\beta+i;r)\\ \Delta(i-k+1;r)\end{matrix}\,\middle|\,-\left(\frac{cr}{1-c}\right)^r\right).
\end{aligned}$$

By virtue of the transformation

$$\sum_{m=0}^{\infty}\sum_{k=i}^{mr+i}H(m,k)=\sum_{k=i}^{\infty}\sum_{m=0}^{\infty}H(m+\eta_{i,k},k),\quad \eta_{i,k}=1+\left[\frac{k-i-1}{r}\right]$$

and (1.3) we get

$$\mathcal{B}_i(f) = \sum_{k=0}^{i-1} \omega_{i,k}^{(2)} f(k),$$

where

$$\begin{aligned}
\omega_{i,k}^{(2)} &= \frac{(-1)^{k+\eta_{i,k}} (\beta)_{r\eta_{i,k}+i} c^{r\eta_{i,k}+i}}{k!i!\eta_{i,k}!(r\eta_{i,k}+i-k)!(1-c)^{r\eta_{i,k}+i}} \\
&\quad\times \sum_{m=0}^{\infty} \frac{(-1)^m r^{mr} \prod\limits_{s=0}^{r-1} \big((r\eta_{i,k}+i+s)/r\big)_m \big((\beta+r\eta_{i,k}+i+s)/r\big)_m c^{mr}}{\prod\limits_{s=0}^{r-1} \big((i-k+s+1)/r\big)_m (1-c)^{mr} m!} \\
&= \frac{(-1)^{k+\eta_{i,k}} (\beta)_{r\eta_{i,k}+i} c^{r\eta_{i,k}+i}}{k!i!\eta_{i,k}!(r\eta_{i,k}+i-k)!(1-c)^{r\eta_{i,k}+i}} \\
&\quad\times {}_{2r+1}F_{r+1}\left(\begin{matrix} 1, \Delta(r\eta_{i,k}+i+1;r), \Delta(\beta+r\eta_{i,k}+i;r) \\ 1+\eta_{i,k}, \Delta(r\eta_{i,k}+i-k+1;r) \end{matrix} \,\middle|\, -\Big(\frac{cr}{1-c}\Big)^r\right).
\end{aligned}$$

Then we get

$$\langle u_i, f\rangle = \sum_{k=0}^{\infty} \omega_{i,k} f(k),$$

with

$$\begin{cases} \omega_{i,k} = \omega_{i,k}^{(1)}, & \text{if } 0 \leqslant k \leqslant i-1, \\ \omega_{i,k} = \omega_{i,k}^{(2)}, & \text{if } i \leqslant k. \end{cases}$$

Since $0 \leqslant i \leqslant 2r-2$ and $0 \leqslant s \leqslant r-1$, the result follows from the following three cases:

$$\eta_{i,dk+s} = \begin{cases} k, & \text{if } 1-r \leqslant s-i \leqslant 0, \\ k+1, & \text{if } 1 \leqslant s-i \leqslant r-1, \\ k-1, & \text{if } 2(1-r) \leqslant s-i \leqslant -r. \end{cases}$$

□

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