QUASITRIANGULAR HOPF GROUP ALGEBRAS AND BRAIDED MONOIDAL CATEGORIES

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Abstract. Let π be a group, and H be a semi-Hopf π-algebra. We first show that the category ${}_H\mathcal{M}$ of left π-modules over H is a monoidal category with a suitably defined tensor product and each element α in π induces a strict monoidal functor F_α from ${}_H\mathcal{M}$ to itself. Then we introduce the concept of quasitriangular semi-Hopf π-algebra, and show that a semi-Hopf π-algebra H is quasitriangular if and only if the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Finally, we give two examples of Hopf π-algebras and describe the categories of modules over them.

Keywords: Hopf π-algebra; H-π-modules; braided monoidal category; braided monoidal functor

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1. Introduction

The notion of a quasitriangular Hopf algebra was introduced by Drinfel'd [4], when he studied the Yang-Baxter equation. The category of modules over a quasitriangular Hopf algebra is a braided monoidal category. Moreover, the braiding structure of a braided monoidal category can supply solutions to the quantum Yang-Baxter equation. Recently, Turaev [9] introduced Hopf π-coalgebra, which generalizes the notion of Hopf algebra. Virelizier also studied algebraic properties of Hopf group-coalgebras and generalized the main properties of quasitriangular Hopf algebras to the setting of quasitriangular Hopf π-coalgebras in [10]. Wang introduced the concept of semi-Hopf group algebra and investigated properties of coquasitriangular Hopf group algebras

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in [11]. Zhu, Chen and Li studied the categories of modules and comodules over a Hopf group coalgebra in [13] and [14], respectively.

In this paper, we first investigate the category ${}_H\mathcal{M}$ of left modules over a semi-Hopf π-algebra H, where π is a group. We define a tensor product module of two modules over H, and show that ${}_H\mathcal{M}$ is a monoidal category with respect to such a tensor product, and each element α in π induces a strict monoidal functor F_α from ${}_H\mathcal{M}$ to itself. Then we introduce the concept of quasitriangular semi-Hopf π-algebra, and show that a semi-Hopf π-algebra H is quasitriangular if and only if the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Finally, we give two examples of Hopf π-algebras and discuss the categories of modules over them.

2. Preliminaries

Throughout the paper, let π be a discrete group (with neutral element 1) and k be a fixed field. All algebras and coalgebras, π-algebras and Hopf π-algebras are defined over k. The definitions and properties of an algebra, coalgebra, Hopf algebra, category and monoidal category can be found in [5]–[7], [12]. We use the standard Sweedler notation for comultiplication. The tensor product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-spaces, $\tau_{U,V}\colon U \otimes V \to V \otimes U$ will denote the twist map defined by $\tau_{U,V}(u \otimes v) = v \otimes u$. The following definitions and notations can be found in [1], [8]–[11].

Definition 2.1. A π-algebra (over k) is a family $A = \{A_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family $m = \{m_{\alpha,\beta}\colon A_\alpha \otimes A_\beta \to A_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ of k-linear maps (the multiplication) and a k-linear map $u\colon k \to A_1$ (the unit) such that m is associative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{A_\gamma}) = m_{\alpha,\beta\gamma}(\mathrm{id}_{A_\alpha} \otimes m_{\beta,\gamma}),$$
$$m_{\alpha,1}(\mathrm{id}_{A_\alpha} \otimes u) = \mathrm{id}_{A_\alpha} = m_{1,\alpha}(u \otimes \mathrm{id}_{A_\alpha}).$$

Note that $(A_1, m_{1,1}, u)$ is an algebra in the usual sense.

Definition 2.2. Let $A = (\{A_\alpha\}_{\alpha\in\pi}, m, u)$ be a π-algebra. A left π-module over A is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family $\eta = \{\eta^M_{\alpha,\beta}\colon A_\alpha \otimes M_\beta \to M_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ of k-linear maps such that for any $\alpha, \beta, \gamma \in \pi$,

(1) $\eta^M_{\alpha,\beta\gamma}(\mathrm{id}_{A_\alpha} \otimes \eta^M_{\beta,\gamma}) = \eta^M_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{M_\gamma})$;
(2) $\eta^M_{1,\alpha}(u \otimes \mathrm{id}_{M_\alpha}) = \mathrm{id}_{M_\alpha}$.

Definition 2.3. Assume that $A = (\{A_\alpha\}_{\alpha\in\pi}, m, u)$ is a π-algebra. Let $M = \{M_\alpha\}_{\alpha\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ be two left π-modules over A. A left A-π-module map from M to N is a family $f = \{f_\alpha\colon M_\alpha \to N_\alpha\}_{\alpha\in\pi}$ of k-linear maps such that

$$\eta^N_{\alpha,\beta}(\mathrm{id}_{A_\alpha} \otimes f_\beta) = f_{\alpha\beta}\eta^M_{\alpha,\beta}, \quad \alpha, \beta \in \pi.$$

Definition 2.4. A semi-Hopf π-algebra is a π-algebra $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ such that:

(1) Each H_α is a k-coalgebra with comultiplication Δ_α and counit ε_α, $\alpha \in \pi$.

(2) $u\colon k \to H_1$ and $m_{\alpha,\beta}\colon H_\alpha \otimes H_\beta \to H_{\alpha\beta}$ are coalgebra maps, $\alpha, \beta \in \pi$.
Furthermore, if there is a family $S = \{S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of k-linear maps (the antipode) such that the following condition (3) is satisfied, then $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is called a Hopf π-algebra.

(3) $m_{\alpha^{-1},\alpha}(S_\alpha \otimes \mathrm{id}_{H_\alpha})\Delta_\alpha = u\varepsilon_\alpha = m_{\alpha,\alpha^{-1}}(\mathrm{id}_{H_\alpha} \otimes S_\alpha)\Delta_\alpha$, $\alpha \in \pi$.

3. Category of modules over a semi-Hopf π-algebra

Throughout this section, assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra. Denote by ${}_H\mathcal{M}$ the category of all left π-modules over H, whose morphisms are left H-π-module maps.

Lemma 3.1. *Suppose that (M, η^M) and (N, η^N) are left π-modules over H. Then the tensor product $M \otimes N = \{(M \otimes N)_\alpha\}_{\alpha\in\pi}$ is also a left π-module over H, where $(M \otimes N)_\alpha = M_\alpha \otimes N_\alpha$, the structure maps $\eta^{M\otimes N} = \{\eta^{M\otimes N}_{\alpha,\beta}\colon H_\alpha \otimes M_\beta \otimes N_\beta \to M_{\alpha\beta} \otimes N_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ are given by*

$$\eta^{M\otimes N}_{\alpha,\beta} := (\eta^M_{\alpha,\beta} \otimes \eta^N_{\alpha,\beta})(\mathrm{id}_{H_\alpha} \otimes \tau_{H_\alpha,M_\beta} \otimes \mathrm{id}_{N_\beta})(\Delta_\alpha \otimes \mathrm{id}_{M_\beta} \otimes \mathrm{id}_{N_\beta}).$$

Proof. On the one hand, for any $h \in H_\alpha$, $l \in H_\beta$, $m \in M_\gamma$ and $n \in N_\gamma$, we have

$$\begin{aligned}
&\eta^{M\otimes N}_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha} \otimes \eta^{M\otimes N}_{\beta,\gamma})(h \otimes l \otimes m \otimes n)\\
&\quad= \eta^{M\otimes N}_{\alpha,\beta\gamma}\Big(\sum h \otimes l_1 \cdot m \otimes l_2 \cdot n\Big)\\
&\quad= \sum h_1 \cdot (l_1 \cdot m) \otimes h_2 \cdot (l_2 \cdot n)\\
&\quad= \sum (h_1 l_1) \cdot m \otimes (h_2 l_2) \cdot n\\
&\quad= \sum (hl)_1 \cdot m \otimes (hl)_2 \cdot n\\
&\quad= \eta^{M\otimes N}_{\alpha\beta,\gamma}(hl \otimes m \otimes n)\\
&\quad= \eta^{M\otimes N}_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{(M\otimes N)_\gamma})(h \otimes l \otimes m \otimes n).
\end{aligned}$$

Hence $\eta^{M\otimes N}_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha}\otimes\eta^{M\otimes N}_{\beta,\gamma})=\eta^{M\otimes N}_{\alpha\beta,\gamma}(m_{\alpha,\beta}\otimes\mathrm{id}_{(M\otimes N)_\gamma})$. On the other hand, for any $\lambda\in k$, $m\in M_\alpha$ and $n\in N_\alpha$, we have

$$\eta^{M\otimes N}_{1,\alpha}(u\otimes\mathrm{id}_{(M\otimes N)_\alpha})(\lambda\otimes m\otimes n)=\eta^{M\otimes N}_{1,\alpha}(\lambda 1_H\otimes m\otimes n)=\lambda(m\otimes n).$$

Hence $\eta^{M\otimes N}_{1,\alpha}(u\otimes\mathrm{id}_{(M\otimes N)_\alpha})=\mathrm{id}_{(M\otimes N)_\alpha}$. Thus, $M\otimes N=\{(M\otimes N)_\alpha\}_{\alpha\in\pi}$ is a left π-module over H. $\square$

Let $M,N,P\in{}_H\mathcal{M}$. Define $a_{M,N,P}=\{a_\alpha\}_{\alpha\in\pi}\colon (M\otimes N)\otimes P\to M\otimes(N\otimes P)$ by $a_\alpha\colon (M_\alpha\otimes N_\alpha)\otimes P_\alpha\to M_\alpha\otimes(N_\alpha\otimes P_\alpha)$, $(m\otimes n)\otimes p\mapsto m\otimes(n\otimes p)$, where $m\in M_\alpha$, $n\in N_\alpha$, $p\in P_\alpha$. Then we have the following lemma.

Lemma 3.2. *The family $a_{M,N,P}$ is a family of left H-π-module natural isomorphisms, where $M,N,P\in{}_H\mathcal{M}$.*

Proof. For any $\alpha,\beta\in\pi$, $h\in H_\alpha$, $m\in M_\beta$, $n\in N_\beta$ and $p\in P_\beta$, we have

$$\begin{aligned}\eta^{M\otimes(N\otimes P)}_{\alpha,\beta}&(\mathrm{id}_{H_\alpha}\otimes a_\beta)(h\otimes((m\otimes n)\otimes p))\\&=\eta^{M\otimes(N\otimes P)}_{\alpha,\beta}(h\otimes(m\otimes(n\otimes p)))\\&=\sum h_1\cdot m\otimes h_2\cdot(n\otimes p)=\sum h_1\cdot m\otimes(h_2\cdot n\otimes h_3\cdot p)\\&=a_{\alpha\beta}\Big(\sum(h_1\cdot m\otimes h_2\cdot n)\otimes h_3\cdot p\Big)\\&=a_{\alpha\beta}\Big(\sum h_1\cdot(m\otimes n)\otimes h_2\cdot p\Big)\\&=a_{\alpha\beta}\eta^{(M\otimes N)\otimes P}_{\alpha,\beta}(h\otimes((m\otimes n)\otimes p)).\end{aligned}$$

This shows that $\eta^{M\otimes(N\otimes P)}_{\alpha,\beta}(\mathrm{id}_{H_\alpha}\otimes a_\beta)=a_{\alpha\beta}\eta^{(M\otimes N)\otimes P}_{\alpha,\beta}$, and so $a_{M,N,P}$ is a left H-π-module morphism. Consequently, $a_{M,N,P}$ is a left H-π-module isomorphism. Obviously, it is a family of natural isomorphisms of H-π-modules. $\square$

Lemma 3.3. *Let $K=\{K_\alpha\}_{\alpha\in\pi}$ with $K_\alpha=k$. Define $\eta^K_{\alpha,\beta}\colon H_\alpha\otimes K_\beta\to K_{\alpha\beta}$ by $\eta^K_{\alpha,\beta}(h\otimes\lambda)=h\cdot\lambda:=\varepsilon_\alpha(h)\lambda$. Then K is a left π-module over H.*

Proof. For any $h\in H_\alpha$, $l\in H_\beta$, $m\in K_\gamma=k$, $\lambda\in k$, $n\in K_\alpha=k$, we have

$$\begin{aligned}\eta^K_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha}\otimes\eta^K_{\beta,\gamma})(h\otimes l\otimes m)&=\eta^K_{\alpha,\beta\gamma}(h\otimes\varepsilon_\beta(l)m)\\&=\varepsilon_\alpha(h)(\varepsilon_\beta(l)m)=\varepsilon_{\alpha\beta}(hl)m=\eta^K_{\alpha\beta,\gamma}(hl\otimes m)\\&=\eta^K_{\alpha\beta,\gamma}(m_{\alpha,\beta}\otimes\mathrm{id}_{K_\gamma})(h\otimes l\otimes m)\end{aligned}$$

and

$$\eta^K_{1,\alpha}(u\otimes\mathrm{id}_{K_\alpha})(\lambda\otimes n)=\eta^K_{1,\alpha}(\lambda 1_H\otimes n)=\varepsilon_1(\lambda 1_H)n=\lambda n.$$

This shows that $\eta^K_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha}\otimes\eta^K_{\beta,\gamma})=\eta^K_{\alpha\beta,\gamma}(m_{\alpha,\beta}\otimes\mathrm{id}_{K_\gamma})$ and $\eta^K_{1,\alpha}(u\otimes\mathrm{id}_{K_\alpha})=\mathrm{id}_{K_\alpha}$. Thus, K is a left π-module over H. $\square$

For any left π-module M over H, we have $(K \otimes M)_\alpha = K_\alpha \otimes M_\alpha = k \otimes M_\alpha$ and $(M \otimes K)_\alpha = M_\alpha \otimes K_\alpha = M_\alpha \otimes k$, $\alpha \in \pi$. Define $l_M\colon K \otimes M \to M$ and $r_M\colon M \otimes K \to M$ by

$$(l_M)_\alpha\colon k \otimes M_\alpha \to M_\alpha, \quad \lambda \otimes m \mapsto \lambda m,$$
$$(r_M)_\alpha\colon M_\alpha \otimes k \to M_\alpha, \quad m \otimes \lambda \mapsto \lambda m.$$

Then it is easy to see that $l = \{l_M\}$ and $r = \{r_M\}$ are two families of natural isomorphisms of left H-π-modules.

Summarizing the above discussion, one gets the the following theorem.

Theorem 3.4. *$({}_H\mathcal{M}, \otimes, K, a, l, r)$ is a monoidal category, where K is the unit object.*

For any $\alpha \in \pi$, define a functor $F_\alpha\colon {}_H\mathcal{M} \to {}_H\mathcal{M}$ by

$$F_\alpha(M)_\beta = M_{\beta\alpha}, \quad \eta_{\beta,\gamma}^{F_\alpha(M)} = \eta_{\beta,\gamma\alpha}^{M}, \quad F_\alpha(f)_\beta = f_{\beta\alpha},$$

where M is a left π-module over H and f is an H-π-module map. Obviously, $F_\alpha(K) = K$ and $(F_\alpha(M) \otimes F_\alpha(N))_\beta = F_\alpha(M)_\beta \otimes F_\alpha(N)_\beta = M_{\beta\alpha} \otimes N_{\beta\alpha} = (M \otimes N)_{\beta\alpha} = F_\alpha(M \otimes N)_\beta$, where M and N are left π-modules over H. Then by a straightforward verification, one can check the following theorem.

Theorem 3.5. *F_α is a strict monoidal functor from $({}_H\mathcal{M}, \otimes, K, a, l, r)$ to itself, where $\alpha \in \pi$.*

4. Quasitriangular semi-Hopf π-algebras

Throughout this section, assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra, and that ${}_H\mathcal{M}$ is the category of left π-modules over H, which is a monoidal category as stated in the last section.

Definition 4.1. H is called a quasitriangular semi-Hopf π-algebra, if there exists an invertible element $R \in H_1 \otimes H_1$ such that the following conditions are satisfied:

(1) $\Delta_\alpha^{\mathrm{cop}}(h)R = R\Delta_\alpha(h)$;
(2) $(\Delta_1 \otimes \mathrm{id})(R) = R_{13}R_{23}$;
(3) $(\mathrm{id} \otimes \Delta_1)(R) = R_{13}R_{12}$,

where $\alpha \in \pi$, $h \in H_\alpha$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\tau_{H_1,H_1} \otimes \mathrm{id})(1 \otimes R) \in H_1 \otimes H_1 \otimes H_1$ and $\Delta_\alpha^{\mathrm{cop}} = \tau_{H_\alpha,H_\alpha} \circ \Delta_\alpha$. In this case, R is called a quasitriangular structure of H.

Remark 4.2. We remark that H_1 is a usual quasitriangular bialgebra if H is quasitriangular, and that H is called an almost cocommutative semi-Hopf π-algebra if only (1) is satisfied.

Let $R = \sum\limits_i s_i \otimes t_i$. Then the three conditions in Definition 4.1 can be formulated as follows:

(1) $\sum\limits_i h_2 s_i \otimes h_1 t_i = \sum\limits_i s_i h_1 \otimes t_i h_2$;
(2) $\sum\limits_i (s_i)_1 \otimes (s_i)_2 \otimes t_i = \sum\limits_{i,j} s_i \otimes s_j \otimes t_i t_j$;
(3) $\sum\limits_i s_i \otimes (t_i)_1 \otimes (t_i)_2 = \sum\limits_{i,j} s_i s_j \otimes t_j \otimes t_i$,

where $\alpha \in \pi$, $h \in H_\alpha$ and $\Delta_\alpha(h) = \sum h_1 \otimes h_2$ as usual.

Lemma 4.3. *If H is almost cocommutative, then there exists a left H-π-module isomorphism $M \otimes N \cong N \otimes M$ for any left π-modules M and N over H.*

Proof. Assume that $R = \sum\limits_i s_i \otimes t_i \in H_1 \otimes H_1$ is an invertible element satisfying condition (1) of Definition 4.1. Let M and N be two left π-modules over H. For any $\alpha \in \pi$, define $(c_{M,N})_\alpha \colon M_\alpha \otimes N_\alpha \to N_\alpha \otimes M_\alpha$ by

$$(c_{M,N})_\alpha(m \otimes n) := \tau_{M_\alpha, N_\alpha}(R \cdot (m \otimes n)) = \sum_i t_i \cdot n \otimes s_i \cdot m,$$

where $m \in M_\alpha$ and $n \in N_\alpha$. Since R is invertible, $(c_{M,N})_\alpha$ is a k-linear isomorphism. Now for any $\alpha, \beta \in \pi$, $m \in M_\beta$, $n \in N_\beta$ and $h \in H_\alpha$, we have

$$\begin{aligned}
&\eta^{N\otimes M}_{\alpha,\beta}(\mathrm{id}_{H_\alpha} \otimes (c_{M,N})_\beta)(h \otimes m \otimes n)\\
&\qquad = \eta^{N\otimes M}_{\alpha,\beta}\Big(\sum_i h \otimes t_i \cdot n \otimes s_i \cdot m\Big)\\
&\qquad = \sum_i h_1 \cdot (t_i \cdot n) \otimes h_2 \cdot (s_i \cdot m) = \sum_i (h_1 t_i) \cdot n \otimes (h_2 s_i) \cdot m\\
&\qquad = \sum_i (t_i h_2) \cdot n \otimes (s_i h_1) \cdot m = \sum_i t_i \cdot (h_2 \cdot n) \otimes s_i \cdot (h_1 \cdot m)\\
&\qquad = (c_{M,N})_{\alpha\beta}\Big(\sum h_1 \cdot m \otimes h_2 \cdot n\Big) = (c_{M,N})_{\alpha\beta}\eta^{M\otimes N}_{\alpha,\beta}(h \otimes m \otimes n).
\end{aligned}$$

Hence $\eta^{N\otimes M}_{\alpha,\beta}(\mathrm{id}_{H_\alpha} \otimes (c_{M,N})_\beta) = (c_{M,N})_{\alpha\beta}\eta^{M\otimes N}_{\alpha,\beta}$. This shows that $c_{M,N}$ is a left H-π-module map, and so

$$c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha\in\pi} \colon M \otimes N \to N \otimes M$$

is a left H-π-module isomorphism. $\square$

Theorem 4.4. *Assume that H is quasitriangular with a quasitriangular structure R. Then the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$.*

Proof. By Theorems 3.4 and 3.5, it follows that ${}_H\mathcal{M}$ is a monoidal category and F_α is a strict monoidal functor for any $\alpha \in \pi$.

For any $M, N \in {}_H\mathcal{M}$, let

$$c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha\in\pi}\colon\ M \otimes N \to N \otimes M$$

be defined as in Lemma 4.3. Then $c_{M,N}$ is a left H-π-module isomorphism. Let $f = \{f_\alpha\}_{\alpha\in\pi}\colon\ M \to M'$ and $g = \{g_\alpha\}_{\alpha\in\pi}\colon\ N \to N'$ be two left H-π-module maps. Then for any $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$, we have

$$\begin{aligned}(g_\alpha \otimes f_\alpha)(c_{M,N})_\alpha(m \otimes n) &= (g_\alpha \otimes f_\alpha)\Big(\sum_i t_i \cdot n \otimes s_i \cdot m\Big)\\ &= \sum_i g_\alpha(t_i \cdot n) \otimes f_\alpha(s_i \cdot m) = \sum_i t_i \cdot g_\alpha(n) \otimes s_i \cdot f_\alpha(m)\\ &= (c_{M',N'})_\alpha(f_\alpha(m) \otimes g_\alpha(n)) = (c_{M',N'})_\alpha(f_\alpha \otimes g_\alpha)(m \otimes n).\end{aligned}$$

Hence $(g \otimes f)c_{M,N} = c_{M',N'}(f \otimes g)$, which shows that $c_{M,N}$ is a family of natural isomorphisms of left H-π-modules.

Now let $M, N, P \in {}_H\mathcal{M}$ and $\alpha \in \pi$. Then for any $m \in M_\alpha$, $n \in N_\alpha$ and $p \in P_\alpha$, we have

$$\begin{aligned}(c_{M,N\otimes P})_\alpha(m \otimes n \otimes p) &= \sum_i t_i \cdot (n \otimes p) \otimes s_i \cdot m = \sum_i (t_i)_1 \cdot n \otimes (t_i)_2 \cdot p \otimes s_i \cdot m\\ &= \sum_{i,j} t_i \cdot n \otimes t_j \cdot p \otimes (s_j s_i) \cdot m = \sum_{i,j} t_i \cdot n \otimes t_j \cdot p \otimes s_j \cdot (s_i \cdot m)\\ &= (\mathrm{id}_{N_\alpha} \otimes (c_{M,P})_\alpha)\Big(\sum_i t_i \cdot n \otimes s_i \cdot m \otimes p\Big)\\ &= (\mathrm{id}_{N_\alpha} \otimes (c_{M,P})_\alpha)((c_{M,N})_\alpha \otimes \mathrm{id}_{P_\alpha})(m \otimes n \otimes p)\end{aligned}$$

and

$$\begin{aligned}(c_{M\otimes N,P})_\alpha(m \otimes n \otimes p) &= \sum_i t_i \cdot p \otimes s_i \cdot (m \otimes n) = \sum_i t_i \cdot p \otimes (s_i)_1 \cdot m \otimes (s_i)_2 \cdot n\\ &= \sum_{i,j} (t_j t_i) \cdot p \otimes s_j \cdot m \otimes s_i \cdot n = \sum_{i,j} t_j \cdot (t_i \cdot p) \otimes s_j \cdot m \otimes s_i \cdot n\\ &= ((c_{M,P})_\alpha \otimes \mathrm{id}_{N_\alpha})\Big(\sum_i m \otimes t_i \cdot p \otimes s_i \cdot n\Big)\\ &= ((c_{M,P})_\alpha \otimes \mathrm{id}_{N_\alpha})(\mathrm{id}_{M_\alpha} \otimes (c_{N,P})_\alpha)(m \otimes n \otimes p).\end{aligned}$$

This shows that $c_{M,N\otimes P} = (\mathrm{id}_N \otimes c_{M,P})(c_{M,N} \otimes \mathrm{id}_P)$ and $c_{M\otimes N,P} = (c_{M,P} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes c_{N,P})$. Therefore, ${}_H\mathcal{M}$ is a braided monoidal category with the braiding c.

Let $\alpha \in \pi$. Then for any $M, N \in {}_H\mathcal{M}$ and $\beta \in \pi$, it is obvious that $F_\alpha(c_{M,N})_\beta = (c_{M,N})_{\beta\alpha} = (c_{F_\alpha(M),F_\alpha(N)})_\beta$. Hence $F_\alpha(c_{M,N}) = c_{F_\alpha(M),F_\alpha(N)}$, and consequently, F_α is a strict braided monoidal functor for any $\alpha \in \pi$. $\square$

Theorem 4.5. *Suppose that ${}_H\mathcal{M}$ is a braided monoidal category, and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Then H is quasitriangular.*

Proof. Suppose that ${}_H\mathcal{M}$ is a braided monoidal category with a braiding c, and F_α is a strict braided monoidal functor for any $\alpha \in \pi$. Then $c_{H,H}\colon H\otimes H \to H\otimes H$ is a left H-π-module isomorphism, and hence $(c_{H,H})_1\colon H_1\otimes H_1 \to H_1\otimes H_1$ is a k-linear isomorphism. Let $R = \tau_{H_1,H_1}((c_{H,H})_1(1\otimes 1)) \in H_1 \otimes H_1$. Then Lemmas 4.8–4.10 below show that R is a quasitriangular structure of H. $\square$

Throughout the following Lemma 4.6, Corollary 4.7 and Lemmas 4.8–4.10, assume that ${}_H\mathcal{M}$ is a braided monoidal category with a braiding c, F_α is a strict braided monoidal functor for any $\alpha \in \pi$, and let $R = \tau_{H_1,H_1}((c_{H,H})_1(1\otimes 1)) = \sum\limits_i s_i \otimes t_i \in H_1 \otimes H_1$ be given as above. In this case, we have $(c_{H,H})_1(1\otimes 1) = \tau_{H_1,H_1}(R) = \sum\limits_i t_i \otimes s_i$.

Lemma 4.6. *Let $M, N \in {}_H\mathcal{M}$. Then we have*

$$(c_{M,N})_\alpha(m\otimes n) = \tau_{M_\alpha,N_\alpha}(R\cdot(m\otimes n)) = \sum_i t_i\cdot n\otimes s_i\cdot m,$$

where $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$.

Proof. Let $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$. Then one can easily check that the two maps $\overline{m} = \{\overline{m}_\beta\}_{\beta\in\pi}\colon H \to F_\alpha(M)$ and $\overline{n} = \{\overline{n}_\beta\}_{\beta\in\pi}\colon H \to F_\alpha(N)$ defined by $\overline{m}_\beta(h) = h\cdot m$ and $\overline{n}_\beta(h) = h\cdot n$, $\beta \in \pi$, $h \in H_\beta$, are left H-π-module maps. In this case, $\overline{m}_1(1) = m$ and $\overline{n}_1(1) = n$.

Since $c_{M,N}$ is a family of natural isomorphisms of left H-π-modules, we have $c_{F_\alpha(M),F_\alpha(N)}(\overline{m}\otimes\overline{n}) = (\overline{n}\otimes\overline{m})c_{H,H}$. Since F_α is a strict braided monoidal functor, $F_\alpha(c_{M,N}) = c_{F_\alpha(M),F_\alpha(N)}$, and hence $(c_{M,N})_\alpha = F_\alpha(c_{M,N})_1 = (c_{F_\alpha(M),F_\alpha(N)})_1$. Thus, we have

$$\begin{aligned}(c_{M,N})_\alpha(m\otimes n) &= (c_{M,N})_\alpha(\overline{m}_1\otimes\overline{n}_1)(1\otimes 1) = (c_{F_\alpha(M),F_\alpha(N)})_1(\overline{m}_1\otimes\overline{n}_1)(1\otimes 1)\\ &= (c_{F_\alpha(H),F_\alpha(H)}(\overline{m}\otimes\overline{n}))_1(1\otimes 1) = ((\overline{n}\otimes\overline{m})c_{H,H})_1(1\otimes 1)\\ &= (\overline{n}_1\otimes\overline{m}_1)(c_{H,H})_1(1\otimes 1) = (\overline{n}_1\otimes\overline{m}_1)\Big(\sum_i t_i\otimes s_i\Big)\\ &= \sum_i t_i\cdot n\otimes s_i\cdot m = \tau_{M_\alpha,N_\alpha}(R\cdot(m\otimes n)).\end{aligned}$$

$\square$

Corollary 4.7. *For any $\alpha \in \pi$ and $x, y \in H_\alpha$, we have*

$$(c_{H,H})_\alpha(x \otimes y) = \tau_{H_\alpha,H_\alpha}(R(x \otimes y)) = \sum_i t_i y \otimes s_i x.$$

P r o o f. It follows by putting $M = N = H$ in Lemma 4.6. $\square$

Lemma 4.8. *R is an invertible element in $H_1 \otimes H_1$.*

P r o o f. Since $(c_{H,H})_1\colon H_1 \otimes H_1 \to H_1 \otimes H_1$ is a k-linear isomorphism, there exists an element $a \in H_1 \otimes H_1$ such that $(c_{H,H})_1(a) = 1 \otimes 1$. From Corollary 4.7, it follows that $\tau_{H_1,H_1}(Ra) = 1 \otimes 1$, and so $Ra = 1 \otimes 1$. Then $(c_{H,H})_1(aR - 1 \otimes 1) = \tau_{H_1,H_1}(R(aR - 1 \otimes 1)) = \tau_{H_1,H_1}(RaR - R) = \tau_{H_1,H_1}(R - R) = 0$, which implies that $aR - 1 \otimes 1 = 0$, since $(c_{H,H})_1$ is a k-linear automorphism of $H_1 \otimes H_1$, and so $aR = 1 \otimes 1$. Thus, R is an invertible element in $H_1 \otimes H_1$ with $R^{-1} = a$. $\square$

Lemma 4.9. *The following equations hold in $H_1 \otimes H_1 \otimes H_1$:*

(1) $(\mathrm{id} \otimes \Delta_1)(R) = R_{13}R_{12}$;
(2) $(\Delta_1 \otimes \mathrm{id})(R) = R_{13}R_{23}$.

P r o o f. Since c is a braiding and $H \in {}_H\mathcal{M}$, we have

$$c_{H,H\otimes H} = (\mathrm{id}_H \otimes c_{H,H})(c_{H,H} \otimes \mathrm{id}_H), \quad c_{H\otimes H,H} = (c_{H,H} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes c_{H,H}),$$

and hence

$$(c_{H,H\otimes H})_1 = (\mathrm{id}_{H_1} \otimes (c_{H,H})_1)((c_{H,H})_1 \otimes \mathrm{id}_{H_1}),$$
$$(c_{H\otimes H,H})_1 = ((c_{H,H})_1 \otimes \mathrm{id}_{H_1})(\mathrm{id}_{H_1} \otimes (c_{H,H})_1).$$

By Lemma 4.6 (and Corollary 4.7), we have

$$(c_{H,H\otimes H})_1(1 \otimes 1 \otimes 1) = \sum_i t_i \cdot (1 \otimes 1) \otimes s_i = \sum_i \Delta(t_i) \otimes s_i$$

and

$$\begin{aligned}(\mathrm{id}_{H_1} &\otimes (c_{H,H})_1)((c_{H,H})_1 \otimes \mathrm{id}_{H_1})(1 \otimes 1 \otimes 1)\\ &= (\mathrm{id}_{H_1} \otimes (c_{H,H})_1)\Big(\sum_i t_i \otimes s_i \otimes 1\Big) = \sum_{i,j} t_i \otimes t_j \otimes s_j s_i.\end{aligned}$$

Hence $\sum\limits_i \Delta(t_i) \otimes s_i = \sum\limits_{i,j} t_i \otimes t_j \otimes s_j s_i$, and so $\sum\limits_i s_i \otimes \Delta(t_i) = \sum\limits_{i,j} s_j s_i \otimes t_i \otimes t_j$. This shows equation (1). Equation (2) can be proved similarly. $\square$

Lemma 4.10. *Let $\alpha \in \pi$ and $h \in H_\alpha$. Then we have*

$$\Delta_\alpha^{\mathrm{cop}}(h)R = R\Delta_\alpha(h).$$

Proof. Since $c_{H,H}$ is a left H-π-module map, we have

$$\eta_{\alpha,1}^{H\otimes H}(\mathrm{id}_{H_\alpha} \otimes (c_{H,H})_1) = (c_{H,H})_\alpha \eta_{\alpha,1}^{H\otimes H}, \quad \forall \alpha \in \pi.$$

Let $\alpha \in \pi$ and $h \in H_\alpha$. By Lemma 4.6 or Corollary 4.7, we have

$$\eta_{\alpha,1}^{H\otimes H}(\mathrm{id}_{H_\alpha} \otimes (c_{H,H})_1)(h \otimes 1 \otimes 1) = \eta_{\alpha,1}^{H\otimes H}\Big(h \otimes \sum_i t_i \otimes s_i\Big) = \sum_i h_1 t_i \otimes h_2 s_i$$

and

$$(c_{H,H})_\alpha \eta_{\alpha,1}^{H\otimes H}(h \otimes 1 \otimes 1) = (c_{H,H})_\alpha\Big(\sum h_1 \otimes h_2\Big) = \sum_i t_i h_2 \otimes s_i h_1.$$

Hence $\sum\limits_i h_1 t_i \otimes h_2 s_i = \sum\limits_i t_i h_2 \otimes s_i h_1$, and so $\sum\limits_i h_2 s_i \otimes h_1 t_i = \sum\limits_i s_i h_1 \otimes t_i h_2$. That is, $\Delta_\alpha^{\mathrm{cop}}(h)R = R\Delta_\alpha(h)$. □

Combining Theorems 4.4 and 4.5, one gets the following theorem.

Theorem 4.11. *Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ be a semi-Hopf π-algebra. Then H is a quasitriangular semi-Hopf π-algebra if and only if the category ${}_H\mathcal{M}$ is a braided monoidal category and F_α is a strict braided monoidal functor for any $\alpha \in \pi$.*

5. Examples

In this section, we will give two examples of Hopf π-algebras, and consider the category of modules over them.

Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ be a semi-Hopf π-algebra. Then H_1 is a usual bialgebra, and hence the category ${}_{H_1}\mathcal{M}$ of the left H_1-modules is a monoidal category as usual. Let $V \in {}_{H_1}\mathcal{M}$. For any $\alpha, \beta \in \pi$, let $M_\alpha = H_\alpha \otimes_{H_1} V$ and $\eta_{\alpha,\beta}^M = m_{\alpha,\beta} \otimes_{H_1} \mathrm{id}_V\colon H_\alpha \otimes H_\beta \otimes_{H_1} V \to H_{\alpha\beta} \otimes_{H_1} V$. Then it is easy to see that $M = \{M_\alpha\}_{\alpha\in\pi}$ is a left π-module over H with the module structure map $\eta = \{\eta_{\alpha,\beta}^M\}_{\alpha,\beta\in\pi}$. Denote M by $H \otimes_{H_1} V$. Let $f\colon U \to V$ be a left H_1-module map. Then $\mathrm{id}_H \otimes_{H_1} f = \{\mathrm{id}_{H_\alpha} \otimes_{H_1} f\colon H_\alpha \otimes_{H_1} U \to H_\alpha \otimes_{H_1} V\}_{\alpha\in\pi}$ is a left H-π-module map. Thus, we have a functor F from ${}_{H_1}\mathcal{M}$ to ${}_H\mathcal{M}$ as follows:

$$F\colon {}_{H_1}\mathcal{M} \to {}_H\mathcal{M}, \quad F(V) = H \otimes_{H_1} V, \quad F(f) = \mathrm{id}_H \otimes_{H_1} f,$$

where V is an object of ${}_{H_1}\mathcal{M}$ and f is a morphism of ${}_{H_1}\mathcal{M}$. We have another functor G from ${}_H\mathcal{M}$ to ${}_{H_1}\mathcal{M}$ as follows:

$$G\colon {}_H\mathcal{M} \to {}_{H_1}\mathcal{M}, \quad G(M) = M_1, \quad F(f) = f_1,$$

where $M = \{M_\alpha\}_{\alpha\in\pi}$ is an object of ${}_H\mathcal{M}$ and $f = \{f_\alpha\}_{\alpha\in\pi}$ is a morphism of ${}_H\mathcal{M}$. For the unit object K of the monoidal category ${}_H\mathcal{M}$ as stated in the last two sections, $G(K) = K_1 = k$ is exactly the unit object k of the monoidal category ${}_{H_1}\mathcal{M}$. For any $M, N \in {}_H\mathcal{M}$, $G(M \otimes N) = (M \otimes N)_1 = M_1 \otimes N_1 = G(M) \otimes G(N)$. Then one can easily check that G is a strict monoidal functor from ${}_H\mathcal{M}$ to ${}_{H_1}\mathcal{M}$.

For any H_1-module V, let $\theta(V)\colon GF(V) \to V$ be the canonical H_1-module isomorphism $H_1 \otimes_{H_1} V \to V$, $h\otimes v \mapsto h\cdot v$. Then one can easily check that θ is a natural isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$.

Example 5.1. Let π be a cyclic group of order 2 generated by α. Then, $\pi = \{1, \alpha\}$ with $\alpha^2 = 1$. Let H_1 be a 2-dimensional k-space with a k-basis $\{h_0, h_2\}$, and H_α a 2-dimensional k-space with a k-basis $\{h_1, h_3\}$. Define k-linear maps $m_{1,1}\colon H_1 \otimes H_1 \to H_1$ by $m_{1,1}(h_0 \otimes h_0) = m_{1,1}(h_2 \otimes h_2) = h_0$ and $m_{1,1}(h_0 \otimes h_2) = m_{1,1}(h_2 \otimes h_0) = h_2$; $m_{\alpha,\alpha}\colon H_\alpha \otimes H_\alpha \to H_1$ by $m_{\alpha,\alpha}(h_1 \otimes h_3) = m_{\alpha,\alpha}(h_3 \otimes h_1) = h_0$ and $m_{\alpha,\alpha}(h_1 \otimes h_1) = m_{\alpha,\alpha}(h_3 \otimes h_3) = h_2$; $m_{1,\alpha}\colon H_1 \otimes H_\alpha \to H_\alpha$ by $m_{1,\alpha}(h_0 \otimes h_1) = m_{1,\alpha}(h_2 \otimes h_3) = h_1$ and $m_{1,\alpha}(h_0 \otimes h_3) = m_{1,\alpha}(h_2 \otimes h_1) = h_3$; and $m_{\alpha,1}\colon H_\alpha \otimes H_1 \to H_\alpha$ by $m_{\alpha,1} = m_{1,\alpha}\tau_{H_\alpha,H_1}$. Define a k-linear map $u \to H_1$ by $u(\lambda) = \lambda h_0$, $\lambda \in k$. Then one can check that $H = (\{H_1, H_\alpha\}, m, u)$ is a π-algebra with $h_0 = 1$.

Define k-linear maps $\Delta_1\colon H_1 \to H_1 \otimes H_1$ by $\Delta(h_i) = h_i \otimes h_i$, and $\varepsilon_1\colon H_1 \to k$ by $\varepsilon_1(h_i) = 1$, $i = 0, 2$. Then one can see that H_1 is a coalgebra. Similarly, H_α is also a coalgebra with comultiplication and counit given by $\Delta_\alpha\colon H_\alpha \to H_\alpha \otimes H_\alpha$, $\Delta(h_i) = h_i \otimes h_i$, and $\varepsilon_\alpha\colon H_\alpha \to k$, $\varepsilon_\alpha(h_i) = 1$, $i = 1, 3$.

With the above structure, a straightforward verification shows that H is a semi-Hopf π-algebra. Moreover, H is a Hopf π-algebra with the antipode $S = \{S_1, S_\alpha\}$ given by

$$S_1\colon H_1 \to H_1, \quad h_0 \mapsto h_0, \quad h_2 \mapsto h_2;$$
$$S_\alpha\colon H_\alpha \to H_\alpha, \quad h_1 \mapsto h_3, \quad h_3 \mapsto h_1.$$

It is easy to see that $R = 1 \otimes 1$ is a (trivial) quasitriangular structure of H. If $\mathrm{Char}(k) \neq 2$, then H has a nontrivial quasitriangular structure as follows:

$$R = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes h_2 + h_2 \otimes 1 - h_2 \otimes h_2).$$

Now we consider the functors $F\colon {}_{H_1}\mathcal{M} \to {}_H\mathcal{M}$ and $G\colon {}_H\mathcal{M} \to {}_{H_1}\mathcal{M}$ given as above. We have already shown that G is a strict monoidal functor. Let $(\varphi_0)_1\colon$

$K_1 = k \to F(k)_1 = H_1 \otimes_{H_1} k$, $\lambda \mapsto \lambda h_0 \otimes_{H_1} 1 = 1 \otimes_{H_1} \lambda$ be the canonical k-linear isomorphism, and let $(\varphi_0)_\alpha \colon K_\alpha = k \to F(k)_\alpha = H_\alpha \otimes_{H_1} k$ be the k-linear map defined by $(\varphi_0)_\alpha(\lambda) = \lambda h_1 \otimes_{H_1} 1 = h_1 \otimes_{H_1} \lambda$. Then one can easily check that $\varphi_0 = \{(\varphi_0)_1, (\varphi_0)_\alpha\}$ is a left H-π-module isomorphism from K to $F(k)$. Let $V, W \in {}_{H_1}\mathcal{M}$. Define $\varphi_2(V,W)_1 \colon (F(V) \otimes F(W))_1 \to F(V \otimes W)_1$ by

$$\varphi_2(V,W)_1((h \otimes_{H_1} v) \otimes (l \otimes_{H_1} w)) = 1 \otimes_{H_1} (h \cdot v \otimes l \cdot w),$$
$$h, l \in H_1,\ v \in V,\ w \in W;$$

and $\varphi_2(V,W)_\alpha \colon (F(V) \otimes F(W))_\alpha \to F(V \otimes W)_\alpha$ by

$$\varphi_2(V,W)_\alpha((h \otimes_{H_1} v) \otimes (l \otimes_{H_1} w)) = h_1 \otimes_{H_1} ((h_3 h) \cdot v \otimes (h_3 l) \cdot w),$$
$$h, l \in H_\alpha,\ v \in V,\ w \in W.$$

Then a straightforward verification shows that $\varphi_2(V,W) = \{\varphi_2(V,W)_1, \varphi_2(V,W)_\alpha\}$ is a left H-π-module isomorphism from $F(V) \otimes F(W)$ to $F(V \otimes W)$. Moreover, one can easily check that $\varphi_2(V,W)$ is a family of natural isomorphisms of left π-modules over H indexed by all couples (V,W) of objects of ${}_{H_1}\mathcal{M}$. Now by a standard verification, one can check that $(F, \varphi_0, \varphi_2)$ is a monoidal functor from ${}_{H_1}\mathcal{M}$ to ${}_H\mathcal{M}$.

We have already seen that there is a natural isomorphism $\theta \colon GF \to \mathrm{id}_{{}_{H_1}\mathcal{M}}$ as given before. It is easy to check that θ is a natural monoidal isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$.

Let $M = \{M_1, M_\alpha\} \in {}_H\mathcal{M}$. Let $\sigma(M)_1 \colon M_1 \to FG(M)_1 = H_1 \otimes_{H_1} M_1$ be the canonical left H_1-module isomorphism, and let $\sigma(M)_\alpha \colon M_\alpha \to FG(M)_\alpha = H_\alpha \otimes_{H_1} M_1$ be the k-linear map defined by $\sigma(M)_\alpha(m) = h_1 \otimes_{H_1} h_3 \cdot m$, $m \in M_\alpha$. Then one can check that $\sigma(M)_\alpha$ is a bijection with the inverse given by $(\sigma(M)_\alpha)^{-1}(h \otimes m) = h \cdot m$, where $h \in H_\alpha$ and $m \in M_1$. Now by a straightforward verification, one can check that $\sigma(M) = \{\sigma(M)_\alpha\}_{\alpha \in \pi}$ is a left H-π-module map, and so it is an H-π-module isomorphism. Moreover, σ is a natural isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. Then a standard verification shows that σ is a natural monoidal isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. This shows that ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories.

Finally, since H_1 is the group algebra of the cyclic group $\{1, h_2\}$ of order 2, the category ${}_{H_1}\mathcal{M}$ can be well described. When $\mathrm{Char}(k) \neq 2$, H_1 is semisimple. There are only two simple H_1-modules V_0 and V_1 in this case. V_0 and V_1 are both one-dimensional with the actions given by $h_2 \cdot v = v$ for $v \in V_0$ and $h_2 \cdot v = -v$ for $v \in V_1$. When $\mathrm{Char}(k) = 2$, there is a unique simple H_1-module V_0 as given above, and the regular module H_1 is the unique non-simple indecomposable H_1-module, which is projective and uniserial.

In order to give another example, we first give some properties of a semi-Hopf π-algebra.

Definition 5.1. Let $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ be a semi-Hopf π-algebra. A family $e = \{e_\alpha\}_{\alpha\in\pi}$ of nonzero elements with $e_\alpha \in H_\alpha$ is called a generalized idempotent if $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in \pi$. Furthermore,

(1) if $e_1 = 1$, then e is called a strong generalized idempotent;
(2) if $\Delta_\alpha(e_\alpha) = e_\alpha \otimes e_\alpha$ for all $\alpha \in \pi$, then e is called a group-like generalized idempotent;
(3) if π is abelian and $e_\alpha h = h e_\alpha$ for all $\alpha, \beta \in \pi$ and $h \in H_\beta$, then e is called a central generalized idempotent.

Remark 5.2. Assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra and $e = \{e_\alpha\}_{\alpha\in\pi}$ is a generalized idempotent in H. Then the set $\{e_\alpha;\ \alpha \in \pi\}$ forms a group, which is isomorphic to π. If e is strong, then $e_\alpha e_{\alpha^{-1}} = e_{\alpha^{-1}} e_\alpha = e_1 = 1$ for all $\alpha \in \pi$. If e is group-like, then $\varepsilon_\alpha(e_\alpha) = 1$ for all $\alpha \in \pi$.

Lemma 5.3. *Assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra and that H has a strong generalized idempotent $e = \{e_\alpha\}_{\alpha\in\pi}$. Then ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent categories.*

Proof. We use the functors F and G given before. We have already seen that θ is a natural isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$.

For any $M = \{M_\alpha\}_{\alpha\in\pi} \in {}_H\mathcal{M}$ and $\alpha \in \pi$, let $\sigma(M)_\alpha \colon M_\alpha \to FG(M)_\alpha = H_\alpha \otimes_{H_1} M_1$ be defined by $\sigma(M)_\alpha(m) = e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot m)$, $m \in M_\alpha$. Then it is obvious that $\sigma(M)_\alpha$ is a k-linear map. Let $\tau(M)_\alpha \colon H_\alpha \otimes_{H_1} M_1 \to M_\alpha$ be the k-linear map defined by $\tau(M)_\alpha(h \otimes_{H_1} m) = h \cdot m$, where $h \in H_\alpha$ and $m \in M_1$. Then for any $\alpha \in \pi$, $m \in M_\alpha$, $h \in H_\alpha$ and $m' \in M_1$, we have $(\tau(M)_\alpha \sigma(M)_\alpha)(m) = \tau(M)_\alpha(e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot m)) = e_\alpha \cdot (e_{\alpha^{-1}} \cdot m) = (e_\alpha e_{\alpha^{-1}}) \cdot m = 1 \cdot m = m$ and $(\sigma(M)_\alpha \tau(M)_\alpha)(h \otimes_{H_1} m') = e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot (h \cdot m')) = e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}} h) \cdot m') = e_\alpha e_{\alpha^{-1}} h \otimes_{H_1} m' = h \otimes_{H_1} m'$. This shows that $\sigma(M)_\alpha$ is a k-linear isomorphism with $(\sigma(M)_\alpha)^{-1} = \tau(M)_\alpha$, $\alpha \in \pi$. Now it is easy to see that $\tau(M) = \{\tau(M)_\alpha\}_{\alpha\in\pi}$ is a left H-π-module map, and so it is an isomorphism. It follows that $\sigma(M) = \{\sigma(M)_\alpha\}_{\alpha\in\pi}$ is a left H-π-module isomorphism from M to $FG(M)$. Then it is easy to check that $\sigma(M)$ is a family of natural morphisms indexed by all objects M of ${}_H\mathcal{M}$. Therefore, σ is a natural isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. $\square$

Proposition 5.4. *Assume that π is abelian and that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a semi-Hopf π-algebra with a generalized idempotent $e = \{e_\alpha\}_{\alpha\in\pi}$. If e is a central, strong and group-like generalized idempotent, then ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories.*

Proof. Suppose that e is a central, strong and group-like generalized idempotent. We use the notations introduced in the proof of Lemma 5.3.

Note that the unit object of the monoidal category ${}_{H_1}\mathcal{M}$ is the trivial H_1-module k with the action given by $h \cdot 1 = \varepsilon_1(h)$, where $h \in H_1$. Hence $F(k) = H \otimes_{H_1} k = \{H_\alpha \otimes_{H_1} k\}_{\alpha\in\pi}$. For any $\alpha \in \pi$, $H_\alpha = (e_\alpha e_{\alpha^{-1}})H_\alpha = e_\alpha(e_{\alpha^{-1}}H_\alpha) \subseteq e_\alpha H_1 \subseteq H_\alpha$, and hence $H_\alpha = e_\alpha H_1$. It follows that H_α is a free right H_1-module of rank one with an H_1-basis e_α, since $e_{\alpha^{-1}}e_\alpha = 1$. Therefore, $H_\alpha \otimes_{H_1} k$ is a one-dimensional k-vector space with the k-basis $e_\alpha \otimes_{H_1} 1$. Thus, there is a k-linear isomorphism $(\varphi_0)_\alpha \colon K_\alpha = k \to H_\alpha \otimes_{H_1} k$, $\lambda \mapsto \lambda e_\alpha \otimes_{H_1} 1 = e_\alpha \otimes_{H_1} \lambda$ for any $\alpha \in \pi$. Now let $\alpha, \beta \in \pi$, $h \in H_\alpha$ and $\lambda \in K_\beta = k$. Then $h \cdot (\varphi_0)_\beta(\lambda) = h \cdot (e_\beta \otimes_{H_1} \lambda) = (e_\beta h) \otimes_{H_1} \lambda = (e_{\alpha\beta}e_{\alpha^{-1}}h) \otimes_{H_1} \lambda = e_{\alpha\beta} \otimes_{H_1} (e_{\alpha^{-1}}h)\cdot\lambda = e_{\alpha\beta} \otimes_{H_1} \varepsilon_1(e_{\alpha^{-1}}h)\lambda = e_{\alpha\beta} \otimes_{H_1} \varepsilon_{\alpha^{-1}}(e_{\alpha^{-1}})\varepsilon_\alpha(h)\lambda = e_{\alpha\beta} \otimes_{H_1} \varepsilon_\alpha(h)\lambda = (\varphi_0)_{\alpha\beta}(\varepsilon_\alpha(h)\lambda) = (\varphi_0)_{\alpha\beta}(h\cdot\lambda)$. Thus, φ_0 is a left H-π-module isomorphism from K to $F(k)$.

Let $U, V \in {}_{H_1}\mathcal{M}$ and $\alpha \in \pi$. Define $\varphi_2(U,V)_\alpha \colon (F(U) \otimes F(V))_\alpha \to F(U \otimes V)_\alpha$ by

$$\varphi_2(U,V)_\alpha((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v)) = e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x \otimes (e_{\alpha^{-1}}l)\cdot v),$$

where $h, l \in H_\alpha$, $x \in U$ and $v \in V$. Since H_α is a free right H_1-module of rank one with an H_1-basis e_α as stated before, it is easy to check that $\varphi_2(U,V)_\alpha$ is a k-linear isomorphism. Let $h, l \in H_\alpha$, $y \in H_\beta$ with $\alpha, \beta \in \pi$, $x \in U$ and $v \in V$. Then

$$\begin{aligned}
y \cdot \varphi_2(U,V)_\alpha&((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v)) \\
&= ye_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x \otimes (e_{\alpha^{-1}}l)\cdot v) \\
&= e_{\beta\alpha}e_{\beta^{-1}}y \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x \otimes (e_{\alpha^{-1}}l)\cdot v) \\
&= e_{\beta\alpha} \otimes_{H_1} (e_{\beta^{-1}}y)\cdot((e_{\alpha^{-1}}h)\cdot x \otimes (e_{\alpha^{-1}}l)\cdot v) \\
&= \sum e_{\beta\alpha} \otimes_{H_1} (((e_{\beta^{-1}}y)_1 e_{\alpha^{-1}}h)\cdot x \otimes ((e_{\beta^{-1}}y)_2 e_{\alpha^{-1}}l)\cdot v) \\
&= \sum e_{\beta\alpha} \otimes_{H_1} ((e_{\beta^{-1}}y_1 e_{\alpha^{-1}}h)\cdot x \otimes (e_{\beta^{-1}}y_2 e_{\alpha^{-1}}l)\cdot v) \\
&= \sum e_{\beta\alpha} \otimes_{H_1} ((e_{(\beta\alpha)^{-1}}y_1 h)\cdot x \otimes (e_{(\beta\alpha)^{-1}}y_2 l)\cdot v) \\
&= \varphi_2(U,V)_{\beta\alpha}\Big(\sum (y_1 h \otimes_{H_1} x) \otimes (y_2 l \otimes_{H_1} v)\Big) \\
&= \varphi_2(U,V)_{\beta\alpha}(y\cdot((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v))).
\end{aligned}$$

It follows that $\varphi_2(U,V)$ is a left H-π-module isomorphism. A straightforward verification shows that $\varphi_2(U,V)$ is a family of natural isomorphisms of left H-π-modules indexed by all couples (U,V) of objects of ${}_{H_1}\mathcal{M}$.

Let $U, V, W \in {}_{H_1}\mathcal{M}$ and $\alpha \in \pi$. For any $h, l, s \in H_\alpha$, $x \in U$, $v \in V$ and $w \in W$, we have

$$\begin{aligned}
(\varphi_2&(U, V \otimes W)_\alpha(\mathrm{id}_{F(U)_\alpha} \otimes \varphi_2(V,W)_\alpha)a_\alpha)(((h \otimes_{H_1} x) \otimes (l \otimes_{H_1} v)) \otimes (s \otimes_{H_1} w)) \\
&= (\varphi_2(U, V \otimes W)_\alpha(\mathrm{id}_{F(U)_\alpha} \otimes \varphi_2(V,W)_\alpha))((h \otimes_{H_1} x) \otimes ((l \otimes_{H_1} v) \otimes (s \otimes_{H_1} w))) \\
&= \varphi_2(U, V \otimes W)_\alpha((h \otimes_{H_1} x) \otimes (e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}l)\cdot v \otimes (e_{\alpha^{-1}}s)\cdot w)))
\end{aligned}$$

$$= e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x \otimes ((e_{\alpha^{-1}}e_\alpha)\cdot((e_{\alpha^{-1}}l)\cdot v \otimes (e_{\alpha^{-1}}s)\cdot w)))$$
$$= e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x \otimes ((e_{\alpha^{-1}}l)\cdot v \otimes (e_{\alpha^{-1}}s)\cdot w))$$

and

$$\begin{aligned}
&(F(a)_\alpha \varphi_2(U\otimes V, W)_\alpha(\varphi_2(U,V)_\alpha \otimes \mathrm{id}_{F(W)_\alpha}))(((h\otimes_{H_1} x)\otimes(l\otimes_{H_1} v))\otimes(s\otimes_{H_1} w))\\
&\quad = (F(a)_\alpha \varphi_2(U\otimes V, W)_\alpha)((e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x\otimes(e_{\alpha^{-1}}l)\cdot v))\otimes(s\otimes_{H_1} w))\\
&\quad = F(a)_\alpha(e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}e_\alpha)\cdot((e_{\alpha^{-1}}h)\cdot x\otimes(e_{\alpha^{-1}}l)\cdot v)\otimes(e_{\alpha^{-1}}s)\cdot w))\\
&\quad = F(a)_\alpha(e_\alpha \otimes_{H_1} (((e_{\alpha^{-1}}h)\cdot x\otimes(e_{\alpha^{-1}}l)\cdot v)\otimes(e_{\alpha^{-1}}s)\cdot w))\\
&\quad = e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}}h)\cdot x\otimes((e_{\alpha^{-1}}l)\cdot v\otimes(e_{\alpha^{-1}}s)\cdot w)).
\end{aligned}$$

Therefore, for any objects U, V, W of ${}_{H_1}\mathcal{M}$, we have

$$\begin{aligned}
&\varphi_2(U, V\otimes W)(\mathrm{id}_{F(U)}\otimes\varphi_2(V,W))a_{F(U),F(V),F(W)}\\
&\qquad\qquad = F(a_{U,V,W})\varphi_2(U\otimes V, W)(\varphi_2(U,V)\otimes \mathrm{id}_{F(W)}).
\end{aligned}$$

For any $h \in H_\alpha$, $v \in V$ and $\lambda \in K_\alpha = k$ with $\alpha \in \pi$, we have

$$\begin{aligned}
(F(l_V)_\alpha \varphi_2(k,V)_\alpha((\varphi_0)_\alpha \otimes \mathrm{id}_{F(V)_\alpha}))&(\lambda\otimes(h\otimes_{H_1} v))\\
&= (F(l_V)_\alpha\varphi_2(k,V)_\alpha)((e_\alpha\otimes_{H_1}\lambda)\otimes(h\otimes_{H_1} v))\\
&= F(l_V)_\alpha(e_\alpha\otimes_{H_1}((e_{\alpha^{-1}}e_\alpha)\cdot\lambda\otimes(e_{\alpha^{-1}}h)\cdot v))\\
&= F(l_V)_\alpha(e_\alpha\otimes_{H_1}(\lambda\otimes(e_{\alpha^{-1}}h)\cdot v))\\
&= e_\alpha\otimes_{H_1}(\lambda(e_{\alpha^{-1}}h)\cdot v)\\
&= e_\alpha\lambda e_{\alpha^{-1}}h\otimes_{H_1} v\\
&= \lambda(h\otimes_{H_1} v)\\
&= (l_{F(V)})_\alpha(\lambda\otimes(h\otimes_{H_1} v)).
\end{aligned}$$

Hence $F(l_V)\varphi_2(k,V)(\varphi_0\otimes\mathrm{id}_{F(V)}) = l_{F(V)}$ for any object V of ${}_{H_1}\mathcal{M}$. Similarly, one can show that $F(r_V)\varphi_2(V,k)(\mathrm{id}_{F(V)}\otimes\varphi_0) = r_{F(V)}$ for any object V of ${}_{H_1}\mathcal{M}$. Thus, we have proved that $(F, \varphi_0, \varphi_2)$ is a monoidal functor.

Note that G is a strict monoidal functor from ${}_H\mathcal{M}$ to ${}_{H_1}\mathcal{M}$ as stated before.

Finally, a straightforward verification shows that θ is a natural monoidal isomorphism from GF to $\mathrm{id}_{{}_{H_1}\mathcal{M}}$, and σ is a natural monoidal isomorphism from $\mathrm{id}_{{}_H\mathcal{M}}$ to FG. Hence ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories. □

Example 5.2. Assume that $\mathrm{Char}(k) \neq 2$. Let π be any group. For any $\alpha \in \pi$, let H_α be a 4-dimensional vector space with a k-basis $\{e_\alpha, g_\alpha, h_\alpha, x_\alpha\}$. Define k-linear maps $\Delta_\alpha\colon H_\alpha \to H_\alpha\otimes H_\alpha$ and $\varepsilon_\alpha\colon H_\alpha \to k$ by

$$\begin{gathered}
\Delta_\alpha(e_\alpha) = e_\alpha\otimes e_\alpha,\qquad \Delta_\alpha(h_\alpha) = h_\alpha\otimes g_\alpha + e_\alpha\otimes h_\alpha,\\
\Delta_\alpha(g_\alpha) = g_\alpha\otimes g_\alpha,\qquad \Delta_\alpha(x_\alpha) = x_\alpha\otimes e_\alpha + g_\alpha\otimes x_\alpha,\\
\varepsilon_\alpha(e_\alpha) = \varepsilon_\alpha(g_\alpha) = 1,\qquad \varepsilon_\alpha(h_\alpha) = \varepsilon_\alpha(x_\alpha) = 0.
\end{gathered}$$

Then a straightforward verification shows that $(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)$ is a coalgebra over k for any $\alpha \in \pi$.

For any $\alpha, \beta \in \pi$, define a k-linear map $m_{\alpha,\beta}\colon H_\alpha \otimes H_\alpha \to H_{\alpha\beta}$ by

$$\begin{aligned}
e_\alpha e_\beta &= e_{\alpha\beta}, & e_\alpha g_\beta &= g_{\alpha\beta}, & e_\alpha h_\beta &= h_{\alpha\beta}, & e_\alpha x_\beta &= x_{\alpha\beta},\\
g_\alpha e_\beta &= g_{\alpha\beta}, & g_\alpha g_\beta &= e_{\alpha\beta}, & g_\alpha h_\beta &= x_{\alpha\beta}, & g_\alpha x_\beta &= h_{\alpha\beta},\\
h_\alpha e_\beta &= h_{\alpha\beta}, & h_\alpha g_\beta &= -x_{\alpha\beta}, & h_\alpha h_\beta &= 0, & h_\alpha x_\beta &= 0,\\
x_\alpha e_\beta &= x_{\alpha\beta}, & x_\alpha g_\beta &= -h_{\alpha\beta}, & x_\alpha h_\beta &= 0, & x_\alpha x_\beta &= 0,
\end{aligned}$$

where we denote $m_{\alpha,\beta}(y \otimes z)$ by yz for any $y \in H_\alpha$ and $z \in H_\beta$. Then define a k-linear map $u\colon k \to H_1$ by $u(1) = e_1$. A tedious but standard verification shows that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u)$ is a π-algebra with $e_1 = 1$. Moreover, one can check that H is a semi-Hopf π-algebra.

For any $\alpha \in \pi$, define a k-linear map $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ by $S_\alpha(e_\alpha) = e_{\alpha^{-1}}$, $S_\alpha(g_\alpha) = g_{\alpha^{-1}}$, $S_\alpha(h_\alpha) = x_{\alpha^{-1}}$ and $S_\alpha(x_\alpha) = -h_{\alpha^{-1}}$. Then one can check that $H = (\{H_\alpha\}_{\alpha\in\pi}, m, u, S)$ is a Hopf π-algebra.

For any $\lambda \in k$, let

$$\begin{aligned}
R_\lambda = {} & \tfrac{1}{2}(1 \otimes 1 + 1 \otimes g_1 + g_1 \otimes 1 - g_1 \otimes g_1)\\
& + \tfrac{1}{2}\lambda(x_1 \otimes x_1 - x_1 \otimes h_1 + h_1 \otimes x_1 + h_1 \otimes h_1).
\end{aligned}$$

Then one can check that R_λ is a quasitriangular structure of H for any $\lambda \in k$.

Let $e = \{e_\alpha\}_{\alpha\in\pi}$. Then e is a strong group-like generalized idempotent. Now assume that π is abelian. Then e is central. It follows from Proposition 5.4 that ${}_H\mathcal{M}$ and ${}_{H_1}\mathcal{M}$ are equivalent monoidal categories. Thus, in order to describe the left π-modules over H, we only need to describe the left H_1-modules.

Note that H_1 is a usual Hopf algebra, which is generated, as an algebra, by g_1 and h_1. Algebra H_1 is isomorphic, as a Hopf algebra, to Sweedler's 4-dimensional Hopf algebra. Hence there are only 4 non-isomorphic finite-dimensional indecomposable modules V_0, V_1, U_0 and U_1. Modules V_0 and V_1 are both one-dimensional with the actions given by $g_1 \cdot v = (-1)^i v$ and $h_1 \cdot v = 0$ for all $v \in V_i$, where $i = 0, 1$. Modules U_0 and U_1 are both 2-dimensional. The matrix representation $\varrho_i\colon H_1 \to M_2(k)$ corresponding to U_i is given by

$$\varrho_i(g_1) = \begin{pmatrix} (-1)^i & 0 \\ 0 & (-1)^{i-1} \end{pmatrix}, \quad \varrho_i(h_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where $i = 0, 1$. Moreover, U_0 and U_1 are both projective and uniserial. For details, one can see [2] and [3].

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