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ON THE STRUCTURE OF SEQUENTIALLY COHEN-MACAULAY BIGRADED MODULES

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Abstract. Let K be a field and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over K. In this paper, we explicitly describe the structure of finitely generated bigraded "sequentially Cohen-Macaulay" S-modules with respect to $Q = (y_1, \dots, y_n)$. Next, we give a characterization of sequentially Cohen-Macaulay modules with respect to Q in terms of local cohomology modules. Cohen-Macaulay modules that are sequentially Cohen-Macaulay with respect to Q are considered.

Keywords: dimension filtration; sequentially Cohen-Macaulay filtration; cohomological dimension; bigraded module; Cohen-Macaulay module

MSC 2010: 16W50, 13C14, 13D45, 16W70

Introduction

Let K be a field and $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded K-algebra with $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$ for all i and j. Consider the bigraded irrelevant ideals $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let M be a finitely generated bigraded S-module. The largest integer k for which $H^k_Q(M) \neq 0$ is called the cohomological dimension of M with respect to Q and denoted by $\operatorname{cd}(Q, M)$. A finite filtration $\mathcal{D}\colon\ 0 = D_0 \subsetneq D_1 \subsetneq \dots \subsetneq D_r = M$ of bigraded submodules of M is called the dimension filtration of M with respect to Q if D_{i-1} is the largest bigraded submodule of D_i for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all $i = 1, \dots, r$, see [6]. In Section 1, we explicitly describe the structure of the submodules D_i that extends [8], Proposition 2.2. In fact, it is shown that $D_i = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j$ for $i = 1, \dots, r-1$ where $0 = \bigcap_{j=1}^{s} N_j$ is a reduced primary decomposition of 0 in M where N_j is $\mathfrak{p}_j$-primary for

$j = 1, \ldots, s$ and

$$B_{i,Q} = \{\mathfrak{p} \in \mathrm{Ass}(M)\colon\ \mathrm{cd}(Q, S/\mathfrak{p}) \leqslant \mathrm{cd}(Q, D_i)\}.$$

In [7], we say M is Cohen-Macaulay with respect to Q if $\mathrm{grade}(Q, M) = \mathrm{cd}(Q, M)$. A finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_r = M$ of M by bigraded submodules of M is called a Cohen-Macaulay filtration with respect to Q if each quotient M_i/M_{i-1} is Cohen-Macaulay with respect to Q and

$$0 \leqslant \mathrm{cd}(Q, M_1/M_0) < \mathrm{cd}(Q, M_2/M_1) < \ldots < \mathrm{cd}(Q, M_r/M_{r-1}).$$

If M admits a Cohen-Macaulay filtration with respect to Q, then we say M is sequentially Cohen-Macaulay with respect to Q, see [6]. Note that if M is sequentially Cohen-Macaulay with respect to Q, then the filtration $\mathcal{F}$ is uniquely determined and it is just the dimension filtration of M with respect to Q, that is, $\mathcal{F} = \mathcal{D}$. In Section 2, we give a characterization of sequentially Cohen-Macaulay modules with respect to Q in terms of local cohomology modules which extends [4], Corollary 4.4, and [3], Corollary 3.10. We apply this result and the description of the submodules M_i mentioned earlier, showing that S/I is sequentially Cohen-Macaulay with respect to P and Q where I is the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane $\mathbb{P}^2$. Here $S = K[x_1, x_2, x_3, y_1, y_2, y_3]$, $P = (x_1, x_2, x_3)$ and $Q = (y_1, y_2, y_3)$. Note that S/I is Cohen-Macaulay of dimension 3 if $\mathrm{char}\, K \neq 2$.

In [7] we have shown that if M is a finitely generated bigraded Cohen-Macaulay S-module which is Cohen-Macaulay with respect to P, then M is Cohen-Macaulay with respect to Q. Inspired by this fact and the above example we have the following question: Let $I \subseteq S$ be a monomial ideal. Suppose S/I is Cohen-Macaulay. If S/I is sequentially Cohen-Macauly with respect to P, is S/I sequentially Cohen-Macaulay with respect to Q? We do not know the answer to this question yet, however in the last section, we obtain some properties of a Cohen-Macaulay filtration with respect to Q in general provided that the module itself is Cohen-Macaulay, see Propositions 3.3 and 3.4. Inspired by Proposition 3.4, we pose the following question: Let M be a finitely generated bigraded Cohen-Macaulay S-module such that $H^k_Q(M) \neq 0$ for all $\mathrm{grade}(Q, M) \leqslant k \leqslant \mathrm{cd}(Q, M)$. Is $H^s_P(M) \neq 0$ for all $\mathrm{grade}(P, M) \leqslant s \leqslant \mathrm{cd}(P, M)$? Of course the question has affirmative answer in the case that M has only one (two) non-vanishing local cohomology with respect to Q. The projective plane $\mathbb{P}^2$ would also be the case as module with three non-vanishing local cohomology.

1. The dimension filtration with respect to Q

Let K be a field and $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ the standard bigraded polynomial ring over K. In other words, $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$ for all i and j. Consider the bigraded irrelevant ideals $P = (x_1, \ldots, x_m)$ and $Q = (y_1, \ldots, y_n)$, and let M be a finitely generated bigraded S-module. We denote by $\operatorname{cd}(Q, M)$ the cohomological dimension of M with respect to Q which is the largest integer i for which $H^i_Q(M) \neq 0$. Notice that $0 \leqslant \operatorname{cd}(Q, M) \leqslant n$.

We recall the following facts which will be used in the sequel.

Fact 1.1. If M is Cohen-Macaulay, then

$$\operatorname{grade}(P, M) \leqslant \dim M - \operatorname{cd}(Q, M),$$

and the equality holds, see [7], Formula 5.

Let $q \in \mathbb{Z}$. In [7], we say M is relative Cohen-Macaulay with respect to Q if $H^i_Q(M) = 0$ for all $i \neq q$. In other words, $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M) = q$. From now on, we omit the word "relative" for simplicity and say M is Cohen-Macaulay with respect to Q.

Fact 1.2. If M is Cohen-Macaulay with respect to Q with $|K| = \infty$, then

$$\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M,$$

see [7], Theorem 3.6.

Fact 1.3. The exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated bigraded S-modules yields

$$\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, M'), \operatorname{cd}(Q, M'')\},$$

see the general version of [2], Proposition 4.4.

Fact 1.4.

$$\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, S/\mathfrak{p}) \colon \mathfrak{p} \in \operatorname{Ass}(M)\},$$

see the general version of [2], Corollary 4.6.

For a finitely generated bigraded S-module M, there is a unique largest bigraded submodule N of M for which $\operatorname{cd}(Q, N) < \operatorname{cd}(Q, M)$, see [6], Lemma 1.6. We recall the following definition from [6].

Definition 1.5. We call a filtration $\mathcal{D}\colon\ 0 = D_0 \subsetneq D_1 \subsetneq \ldots \subsetneq D_r = M$ of bigraded submodules of M the dimension filtration of M with respect to Q if D_{i-1} is the largest bigraded submodule of D_i for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all $i = 1, \ldots, r$.

Remark 1.6. Let $\mathcal{D}$ be the dimension filtration of M with respect to Q. For all i, the exact sequence $0 \to D_{i-1} \to D_i \to D_i/D_{i-1} \to 0$ by using Fact 1.3 yields

$$\operatorname{cd}(Q, D_i) = \max\{\operatorname{cd}(Q, D_{i-1}), \operatorname{cd}(Q, D_i/D_{i-1})\} = \operatorname{cd}(Q, D_i/D_{i-1}).$$

Thus, $\operatorname{cd}(Q, D_{i-1}/D_{i-2}) < \operatorname{cd}(Q, D_i/D_{i-1})$ for all i.

Let $\mathcal{D}$ be the dimension filtration of M with respect to Q. We set

$$B_{i,Q} = \{\mathfrak{p} \in \operatorname{Ass}(M)\colon\ \operatorname{cd}(Q, S/\mathfrak{p}) \leqslant \operatorname{cd}(Q, D_i)\}, \quad I_{i,Q} = \prod_{\mathfrak{p} \in B_{i,Q}} \mathfrak{p}$$

and

$$A_{i,Q} = \{\mathfrak{p} \in \operatorname{Ass}(M)\colon\ \mathfrak{p} \in V(I_{i,Q})\} \quad \text{for} \quad i = 1, \ldots, r.$$

Lemma 1.7. *Let the notation be as above. Then the following statements hold*

$$A_{i,Q} = B_{i,Q} = \operatorname{Ass}(D_i) \quad \textit{for } i = 1, \ldots, r.$$

Consequently,

$$\operatorname{Supp}(D_i) \subseteq V(I_{i,Q}) \quad \textit{for } i = 1, \ldots, r.$$

Proof. In order to show the first equality, we note that $B_{i,Q} \subseteq A_{i,Q}$ for $i = 1, \ldots, r$. Now let $\mathfrak{p} \in A_{i,Q}$. Then $\mathfrak{p} \in \operatorname{Ass}(M)$ with $I_{i,Q} \subseteq \mathfrak{p}$. Hence $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{q} \in \operatorname{Ass}(M)$ with $\operatorname{cd}(Q, S/\mathfrak{q}) \leqslant \operatorname{cd}(Q, D_i)$. The canonical epimorphism $S/\mathfrak{q} \to S/\mathfrak{p}$ yields $\operatorname{cd}(Q, S/\mathfrak{p}) \leqslant \operatorname{cd}(Q, S/\mathfrak{q})$ by Fact 1.3. It follows that $\mathfrak{p} \in B_{i,Q}$ and hence $A_{i,Q} \subseteq B_{i,Q}$.

To show the second equality, let $\mathfrak{p} \in B_{i,Q}$. Then there is a submodule $N \subseteq M$ such that $N \cong S/\mathfrak{p}$ and $\operatorname{cd}(Q, S/\mathfrak{p}) \leqslant \operatorname{cd}(Q, D_i)$. Using Fact 1.3 we have

$$\operatorname{cd}(Q, N + D_i) = \max\{\operatorname{cd}(Q, D_i), \operatorname{cd}(Q, N/(N \cap D_i))\} = \operatorname{cd}(Q, D_i),$$

and hence $N \subseteq D_i$. This shows that $\mathfrak{p} \in \operatorname{Ass}(D_i)$ and therefore $B_{i,Q} \subseteq \operatorname{Ass}(D_i)$. Now let $\mathfrak{p} \in \operatorname{Ass}(D_i)$. Then $\mathfrak{p} \in \operatorname{Ass}(M)$ and $\operatorname{cd}(Q, S/\mathfrak{p}) \leqslant \operatorname{cd}(Q, D_i)$ by Fact 1.4. This shows that $\mathfrak{p} \in B_{i,Q}$ and hence $\operatorname{Ass}(D_i) \subseteq B_{i,Q}$. $\square$

In the following we describe the structure of the submodules D_i in the dimension filtration of $\mathcal{D}$ with respect to Q which extends [8], Proposition 2.2.

Proposition 1.8. *Let $\mathcal{D}$ be the dimension filtration of M with respect to Q. Then*

$$D_i = H^0_{I_{i,Q}}(M) = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j$$

for $i = 1, \ldots, r-1$ where $0 = \bigcap_{j=1}^{s} N_j$ is a reduced primary decomposition of 0 in M with N_j, $\mathfrak{p}_j$-primary for $j = 1, \ldots, s$.

Proof. In order to prove the first equality, we have $V(\mathrm{Ann}(D_i)) = \mathrm{Supp}(D_i) \subseteq V(I_{i,Q})$ for $i = 1, \ldots, r-1$ by Lemma 1.7. Since $I_{i,Q}$ is finitely generated, it follows that $I^{k_i}_{i,Q} \subseteq \mathrm{Ann}(D_i)$ for some integer k_i and hence $I^{k_i}_{i,Q} D_i = 0$ for some k_i. Thus $D_i = H^0_{I_{i,Q}}(D_i) \subseteq H^0_{I_{i,Q}}(M)$ for $i = 1, \ldots, r-1$.

Now we prove the equality by decreasing induction on i. For $i = r-1$, we assume that $D_{r-1} \subsetneq H^0_{I_{r-1,Q}}(M) \subseteq D_r = M$. It follows from the definition of dimension filtration that $\mathrm{cd}(Q, H^0_{I_{r-1,Q}}(M)) = \mathrm{cd}(Q, M)$. Note that

$$\mathrm{Ass}\, H^0_{I_{i,Q}}(M) = A_{i,Q} = \mathrm{Ass}(D_i) \quad \text{for } i = 1, \ldots, r-1$$

by [5], Proposition 3.13, (c) and Lemma 1.7. It follows that $\mathrm{cd}(Q, H^0_{I_{r-1,Q}}(M)) = \mathrm{cd}(Q, D_{r-1,Q})$, and hence $\mathrm{cd}(Q, D_{r-1,Q}) = \mathrm{cd}(Q, M)$, a contradiction. Thus $D_{r-1,Q} = H^0_{I_{r-1,Q}}(M)$. Now let $1 < i < r-1$, and assume that $D_i = H^0_{I_{i,Q}}(M)$. We show that $D_{i-1} = H^0_{I_{i-1,Q}}(M)$. Assume $D_{i-1} \subsetneq H^0_{I_{i-1,Q}}(M)$. As $H^0_{I_{i-1,Q}}(M) \subseteq H^0_{I_{i,Q}}(M) = D_i$, we have $\mathrm{cd}(Q, H^0_{I_{i-1,Q}}(M)) \geqslant \mathrm{cd}(Q, D_i)$. Since $\mathrm{Ass}\, H^0_{I_{i-1,Q}}(M) = \mathrm{Ass}(D_{i-1})$, it follows that $\mathrm{cd}(Q, D_{i-1}) = \mathrm{cd}(Q, H^0_{I_{i-1,Q}}(M)) \geqslant \mathrm{cd}(Q, D_i)$, a contradiction. Therefore, $D_{i-1} = H^0_{I_{i-1,Q}}(M)$. The second equality follows from Lemma 1.7 and [5], Proposition 3.13 (a). □

Remark 1.9. Let $\mathcal{D}$ be the dimension filtration of M with respect to Q with $\mathrm{cd}(Q, M) = q$. We call the submodule

$$D_{r-1} = \bigcap_{\mathfrak{p}_j \notin B_{r-1,Q}} N_j = \bigcap_{\mathrm{cd}(Q, S/\mathfrak{p}_j) = q} N_j$$

the unmixed component of M with respect to Q and denote it by $u_{Q,M}(0)$. Notice that $u_{\mathfrak{m},M}(0) = u_M(0)$ was introduced by Schenzel in [8]. If M is relatively unmixed with respect to Q, that is, $\mathrm{cd}(Q, M) = \mathrm{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$, then by Proposition 1.8 we have

$$D_i = \bigcap_{\mathfrak{p}_j \notin B_{i,Q}} N_j = \bigcap_{j=1}^{s} N_j = 0 \quad \text{for all } i < r.$$

Corollary 1.10. *Let $\mathcal{D}$ be the dimension filtration of M with respect to Q. Then for $i = 1, \ldots, r$ we have*

$$\operatorname{Ass}(M/D_i) = \operatorname{Ass}(M) - \operatorname{Ass}(D_i).$$

P r o o f. The assertion follows from Proposition 1.8, Lemma 1.7 and the fact that $\operatorname{Ass} M/H^0_{I_{i,Q}}(M) = \operatorname{Ass}(M) - A_{i,Q}$, see [5], Proposition 3.13 (c). □

2. Sequentially Cohen-Macaulay with respect to Q

We recall the following definition from [6].

Definition 2.1. Let M be a finitely generated bigraded S-module. We call a finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_r = M$ of M by bigraded submodules M a Cohen-Macaulay filtration with respect to Q if

(a) each quotient M_i/M_{i-1} is Cohen-Macaulay with respect to Q;
(b) $0 \leqslant \operatorname{cd}(Q, M_1/M_0) < \operatorname{cd}(Q, M_2/M_1) < \ldots < \operatorname{cd}(Q, M_r/M_{r-1})$.

We call M to be sequentially Cohen-Macaulay with respect to Q if M admits a Cohen-Macaulay filtration with respect to Q.

Note that if M is sequentially Cohen-Macaulay with respect to Q, then the filtration $\mathcal{F}$ in the definition above is uniquely determined and it is just the dimension filtration of M with respect to Q defined in Definition 1.5, see [6], Proposition 1.9.

We have the following characterization of sequentially Cohen-Macaulay modules with respect to Q in terms of local cohomology modules which extends [4], Corollary 4.4, and [3], Corollary 3.10.

Proposition 2.2. *Let $\mathcal{D}$: $0 = D_0 \subsetneq D_1 \subsetneq \ldots \subsetneq D_r = M$ be the dimension filtration of M with respect to Q. Then the following statements are equivalent:*

(a) *M is sequentially Cohen-Macaulay with respect to Q;*
(b) *$H^k_Q(M/D_{i-1}) = 0$ for $i = 1, \ldots, r$ and $k < \operatorname{cd}(Q, D_i)$;*
(c) *$\operatorname{grade}(Q, M/D_{i-1}) = \operatorname{cd}(Q, D_i)$ for $i = 1, \ldots, r$.*

P r o o f. *(a)* $\Rightarrow$ *(b)*: We proceed by decreasing induction on i. As D_i/D_{i-1} is Cohen-Macaulay with respect to Q for all i, for $i = r$ we have $H^k_Q(M/D_{r-1}) = 0$ for $k < \operatorname{cd}(Q, M)$. Now let $1 < i < r$, and assume that $H^k_Q(M/D_{i-1}) = 0$ for $k < \operatorname{cd}(Q, D_i)$. The exact sequence

$$0 \to D_{i-1}/D_{i-2} \to M/D_{i-2} \to M/D_{i-1} \to 0,$$

induces the following long exact sequence

$$(1)\qquad \ldots \to H^k_Q(D_{i-1}/D_{i-2}) \to H^k_Q(M/D_{i-2}) \to H^k_Q(M/D_{i-1}) \to \ldots$$

As D_{i-1}/D_{i-2} is Cohen-Macaulay with respect to Q, we have $H^k_Q(D_{i-1}/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$. By Remark 1.6, we have $\operatorname{cd}(Q, D_{i-1}) = \operatorname{cd}(Q, D_{i-1}/D_{i-2}) < \operatorname{cd}(Q, D_i)$. So, by using (1) and the induction hypothesis, we have $H^k_Q(M/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$, as desired.

(b) ⇒ (a): By Remark 1.6 we have $\operatorname{cd}(Q, D_i/D_{i-1}) < \operatorname{cd}(Q, D_{i+1}/D_i)$ for all i. Thus it suffices to show that D_i/D_{i-1} is Cohen-Macaulay with respect to Q for all i. We prove this statement by decreasing induction on i. In condition (b), we first assume $i = r$. It follows that M/D_{r-1} is Cohen-Macaulay with respect to Q. Now let $1 < i < r$, and assume that D_i/D_{i-1} is Cohen-Macaulay with respect to Q. The exact sequence

$$0 \to D_i/D_{i-1} \to M/D_{i-1} \to M/D_i \to 0,$$

induces the following long exact sequence

$$(2)\qquad \ldots \to H^{k-1}_Q(D_i/D_{i-1}) \to H^{k-1}_Q(M/D_{i-1}) \to H^{k-1}_Q(M/D_i) \to \ldots$$

Suppose $k < \operatorname{cd}(Q, D_{i-1})$. Induction hypothesis and our assumption say that $H^{k-1}_Q(D_i/D_{i-1}) = H^{k-1}_Q(M/D_i) = 0$. Hence $H^{k-1}_Q(M/D_{i-1}) = 0$ by (2). We have $H^k_Q(M/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$ because of our assumption again. Thus $H^k_Q(D_{i-1}/D_{i-2}) = 0$ for $k < \operatorname{cd}(Q, D_{i-1})$ by (1). Therefore D_{i-1}/D_{i-2} is Cohen-Macaulay with respect to Q, as desired.

(b) ⇒ (c): We set $\operatorname{cd}(Q, D_i) = \operatorname{cd}(Q, D_i/D_{i-1}) = q_i$ for $i = 1, \ldots, r$. Our assumption says that $\operatorname{grade}(Q, M/D_{i-1}) \geqslant q_i$ for $i = 1, \ldots, r$. We only need to show that $H^{q_i}_Q(M/D_{i-1}) \neq 0$. Consider the long exact sequence

$$(3)\qquad \ldots \to H^{q_i-1}_Q(M/D_i) \to H^{q_i}_Q(D_i/D_{i-1}) \to H^{q_i}_Q(M/D_{i-1}) \to \ldots$$

Since $q_i - 1 < q_i < q_{i+1}$, it follows from our assumption that $H^{q_i-1}_Q(M/D_i) = 0$. If $H^{q_i}_Q(M/D_{i-1}) = 0$, then by (3) we have $H^{q_i}_Q(D_i/D_{i-1}) = 0$, a contradiction. The implication (c) ⇒ (b) is obvious. □

As an application of Proposition 1.8 and Proposition 2.2 we have

Example 2.3. Let I be the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane $\mathbb{P}^2$. Then

$$I = (x_1x_2x_3, x_1x_2y_1, x_1x_3y_2, x_1y_1y_3, x_1y_2y_3, x_2x_3y_3, x_2y_1y_2, x_2y_2y_3, x_3y_1y_2, x_3y_1y_3).$$

We set $R = S/I$ where $S = K[x_1, x_2, x_3, y_1, y_2, y_3]$, $P = (x_1, x_2, x_3)$ and $Q = (y_1, y_2, y_3)$. Our aim is to show that R is sequentially Cohen-Macaulay with respect to P and Q. Note that R is Cohen-Macaulay of dimension 3 if char $K \neq 2$. The ideal I has the minimal primary decomposition $I = \bigcap_{i=1}^{10} \mathfrak{p}_i$ where $\mathfrak{p}_1 = (x_3, y_1, y_3)$, $\mathfrak{p}_2 = (x_1, y_1, y_3)$, $\mathfrak{p}_3 = (x_2, y_1, y_2)$, $\mathfrak{p}_4 = (x_3, y_1, y_2)$, $\mathfrak{p}_5 = (x_1, y_2, y_3)$, $\mathfrak{p}_6 = (x_2, y_2, y_3)$, $\mathfrak{p}_7 = (x_2, x_3, y_3)$, $\mathfrak{p}_8 = (x_1, x_2, y_1)$, $\mathfrak{p}_9 = (x_1, x_3, y_2)$, $\mathfrak{p}_{10} = (x_1, x_2, x_3)$. Since $P = \mathfrak{p}_{10} \in \operatorname{Ass}(R)$, we have $\operatorname{grade}(P, R) = 0$. By Fact 1.4 we have $\operatorname{cd}(P, R) = 2$ and $\operatorname{cd}(Q, R) = 3$. As R is Cohen-Macaulay, it follows from Fact 1.1 that $\operatorname{grade}(Q, R) = 1$. We first show that R is sequentially Cohen-Macaulay with respect to P. By Proposition 1.8, R has the dimension filtration

$$0 = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = R,$$

with respect to P where

$$R_1 = \bigcap_{i=1}^{9} \mathfrak{p}_i/I \quad \text{and} \quad R_2 = \bigcap_{i=1}^{6} \mathfrak{p}_i/I.$$

By Corollary 1.10 we have

$$\operatorname{Ass}(R_1) = \operatorname{Ass}(R) - \operatorname{Ass}(R/R_1) = \{\mathfrak{p}_{10}\}$$

and

$$\operatorname{Ass}(R_2) = \operatorname{Ass}(R) - \operatorname{Ass}(R/R_2) = \{\mathfrak{p}_7, \mathfrak{p}_8, \mathfrak{p}_9, \mathfrak{p}_{10}\}.$$

It follows that $\operatorname{cd}(P, R_1) = 0$ and $\operatorname{cd}(P, R_2) = 1$. We set $I_1 = \bigcap_{i=1}^{9} \mathfrak{p}_i$ and $I_2 = \bigcap_{i=1}^{6} \mathfrak{p}_i$. In view of Proposition 2.2, we need to show that

$$\operatorname{grade}(P, R_3/R_0) = \operatorname{grade}(P, R) = \operatorname{cd}(P, R_1) = 0,$$
$$\operatorname{grade}(P, R_3/R_1) = \operatorname{grade}(P, S/I_1) = \operatorname{cd}(P, R_2) = 1$$

and

$$\operatorname{grade}(P, R_3/R_2) = \operatorname{grade}(P, S/I_2) = \operatorname{cd}(P, R) = 2.$$

The first equality is obvious. As $P \nsubseteq \mathfrak{p}_i$ for $i = 1, \ldots, 9$, we have $\operatorname{grade}(P, S/I_1) \geqslant 1$. On the other hand, $\operatorname{grade}(P, S/I_1) \leqslant \dim S/I_1 - \operatorname{cd}(Q, S/I_1) = 3 - 2 = 1$. Thus the second equality holds. In order to show the third equality, we note that S/I_2 has dimension 3 and, by using CoCoA [1], depth 2. Thus Fact 1.1 can not be used to compute $\operatorname{grade}(P, S/I_2)$. We set $\mathfrak{q}_1 = \mathfrak{p}_1 \cap \mathfrak{p}_2 = (x_1x_3, y_1, y_3)$, $\mathfrak{q}_2 = \mathfrak{p}_3 \cap \mathfrak{p}_4 = (x_2x_3, y_1, y_2)$ and $\mathfrak{q}_3 = \mathfrak{p}_5 \cap \mathfrak{p}_6 = (x_1x_2, y_2, y_3)$. Consider the exact sequence

$$0 \to S/\mathfrak{q}_1 \cap \mathfrak{q}_2 \to S/\mathfrak{q}_1 \oplus S/\mathfrak{q}_2 \to S/(\mathfrak{q}_1 + \mathfrak{q}_2) \to 0.$$

Since $\operatorname{grade}(P, S/\mathfrak{q}_1 \oplus S/\mathfrak{q}_2) = 2$ and $\operatorname{grade}(P, S/(\mathfrak{q}_1+\mathfrak{q}_2)) = 1$, it follows that $\operatorname{grade}(P, S/(\mathfrak{q}_1 \cap \mathfrak{q}_2)) \geqslant 2$. Since $\operatorname{cd}(P, S/(\mathfrak{q}_1 \cap \mathfrak{q}_2)) = 2$, we have $\operatorname{grade}(P, S/(\mathfrak{q}_1 \cap \mathfrak{q}_2)) = 2$. Consider the exact sequence

$$0 \to S/I_2 \to S/\mathfrak{q}_1 \cap \mathfrak{q}_2 \oplus S/\mathfrak{q}_3 \to S/(\mathfrak{q}_1 + \mathfrak{q}_3) \cap (\mathfrak{q}_2 + \mathfrak{q}_3) \to 0. \tag{4}$$

The exact sequence

$$0 \to S/(\mathfrak{q}_1 + \mathfrak{q}_3) \cap (\mathfrak{q}_2 + \mathfrak{q}_3) \to S/(\mathfrak{q}_1 + \mathfrak{q}_3) \oplus S/(\mathfrak{q}_2 + \mathfrak{q}_3) \to S/(\mathfrak{q}_1 + \mathfrak{q}_2 + \mathfrak{q}_3) \to 0$$

yields that $\operatorname{grade}(P, S/(\mathfrak{q}_1+\mathfrak{q}_3)\cap(\mathfrak{q}_2+\mathfrak{q}_3)) \geqslant 1$. So, by (4) we have $\operatorname{grade}(P, S/I_2) \geqslant 2$. As $\operatorname{cd}(P, S/I_2) = 2$, we conclude that $\operatorname{grade}(P, S/I_2) = 2$, as desired.

Next, we show that R is sequentially Cohen-Macaulay with respect to Q. By Proposition 1.8, R has the dimension filtration $0 = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = R$ with respect to Q where $R_1 = \bigcap_{i=7}^{10} \mathfrak{p}_i/I$ and $R_2 = \mathfrak{p}_{10}/I$. By Corollary 1.10 we have $\operatorname{cd}(Q, R_1) = 1$ and $\operatorname{cd}(Q, R_2) = 2$. We set $J = \bigcap_{i=7}^{10} \mathfrak{p}_i$. In view of Proposition 2.2, we need to show that

$$\begin{aligned} \operatorname{grade}(Q, R_3/R_0) &= \operatorname{grade}(Q, R) = \operatorname{cd}(Q, R_1) = 1, \\ \operatorname{grade}(Q, R_3/R_1) &= \operatorname{grade}(Q, S/J) = \operatorname{cd}(Q, R_2) = 2 \end{aligned}$$

and

$$\operatorname{grade}(Q, R_3/R_2) = \operatorname{grade}(Q, S/\mathfrak{p}_{10}) = \operatorname{cd}(Q, R) = 3.$$

The first and the third statements are obvious. In order to prove the second equality, consider the exact sequence

$$0 \to S/J \to S\Big/\bigcap_{i=7}^{9} \mathfrak{p}_i \oplus S/\mathfrak{p}_{10} \to S\Big/\bigcap_{i=7}^{9}(\mathfrak{p}_i + \mathfrak{p}_{10}) \to 0. \tag{5}$$

An exact sequence argument shows that

$$\operatorname{grade}\Big(Q, S\Big/\bigcap_{i=7}^{9} \mathfrak{p}_i\Big) = \operatorname{grade}\Big(Q, S\Big/\bigcap_{i=7}^{9}(\mathfrak{p}_i + \mathfrak{p}_{10})\Big) = 2.$$

Thus it follows from (5) that $\operatorname{grade}(Q, S/J) \geqslant 2$. On the other hand,

$$\operatorname{grade}(Q, S/J) \leqslant \dim S/J - \operatorname{cd}(P, S/J) = 3 - 1 = 2.$$

Therefore, $\operatorname{grade}(Q, S/J) = 2$, as desired.

3. Cohen-Macaulay modules that are sequentially Cohen-Macaulay with respect to Q

In [7] we have shown that if M is a finitely generated bigraded Cohen-Macaulay S-module which is Cohen-Macaulay with respect to P, then M is Cohen-Macaulay with respect to Q. Inspired by this fact and Example 2.3 we have the following question.

Question 3.1. Let $I \subseteq S$ be a monomial ideal. Suppose S/I is Cohen-Macaulay. If S/I is sequentially Cohen-Macauly with respect to P, is S/I sequentially Cohen-Macaulay with respect to Q?

We do not know the answer to this question yet, however in this section, we obtain some properties of a Cohen-Macaulay filtration with respect to Q in general provided that the module itself is Cohen-Macaulay.

Fact 3.2. For a Cohen-Macaulay filtration $\mathcal{F}$ with respect to Q we recall the following fact from [6], Fact 2.3,

$$\operatorname{grade}(Q, M_i) = \operatorname{grade}(Q, M) \quad \text{for } i = 1, \ldots, r.$$

Proposition 3.3. *Let M be a finitely generated bigraded Cohen-Macaulay S-module with $|K| = \infty$. Suppose M is sequentially Cohen-Macaulay with respect to Q with the Cohen-Macaulay filtration $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_r = M$ with respect to Q. Then*

(a) $\operatorname{cd}(P, M_i) = \operatorname{cd}(P, M)$ *for* $i = 1, \ldots, r$;
(b) $\operatorname{grade}(Q, M_i) + \operatorname{cd}(P, M_i) = \dim M_i$ *for* $i = 1, \ldots, r$.

P r o o f. In order to prove (a), since M_1 is Cohen-Macaulay with respect to Q, it follows from Fact 1.2 that $\operatorname{cd}(P, M_1) + \operatorname{cd}(Q, M_1) = \dim M_1$. By Fact 3.2 we have $\operatorname{cd}(Q, M_1) = \operatorname{grade}(Q, M_1) = \operatorname{grade}(Q, M)$. Since M is Cohen-Macaulay, it follows from [6], Lemma 1.8, that $\dim M_1 = \dim M$ and $\operatorname{cd}(P, M) = \dim M - \operatorname{grade}(Q, M)$ by Fact 1.1. Thus we conclude that $\operatorname{cd}(P, M_1) = \operatorname{cd}(P, M)$. As by Fact 1.3 we have $\operatorname{cd}(P, M_{i-1}) \leqslant \operatorname{cd}(P, M_i)$ for all i, the first equality follows.

For the proof (b), by [6], Lemma 1.8, we have $\dim M_i = \dim M$ for $i = 1, \ldots, r$. Thus the second equalities follow from Fact 1.1, Fact 3.2 and part (a). $\square$

Proposition 3.4. *Let the assumptions and the notation be as in Proposition* 3.3. *Then the following statements are equivalent:*

(a) $\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M + r - 1$;

(b) $H^s_Q(M) \neq 0$ *for all* $\operatorname{grade}(Q, M) \leqslant s \leqslant \operatorname{cd}(Q, M)$.

Proof. We first assume that $r = 1$. As M is Cohen-Macaulay, by Fact 1.1 and Fact 1.2 we have $\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M$ if and only if M is Cohen-Macaulay with respect to Q. Thus the claim holds in this case. Now let $r \geqslant 2$. By Fact 1.1 we have $\operatorname{cd}(P, M) + \operatorname{cd}(Q, M) = \dim M + r - 1$ if and only if $\operatorname{cd}(Q, M) - \operatorname{grade}(Q, M) = r - 1$. This is equivalent to saying that $\operatorname{cd}(Q, M_{i+1}) = \operatorname{cd}(Q, M_i) + 1$ for $i = 1, \ldots, r-1$ by Fact 3.2. By [6], Lemma 2.2, this is equivalent to saying that $H^s_Q(M) \neq 0$ for all $\operatorname{grade}(Q, M) \leqslant s \leqslant \operatorname{cd}(Q, M)$. □

The following example shows that the condition that "M is Cohen-Macaulay" is required for Proposition 3.4.

Example 3.5. We set $K[x] = K[x_1, \ldots, x_m]$ and $K[y] = K[y_1, \ldots, y_n]$. Let L be a nonzero finitely generated graded $K[x]$-module of depth 0 and dimension 1, and N a nonzero finitely generated graded $K[y]$-module of depth 0 and dimension 1. We set $M = L \otimes_K N$ and consider it as S-module. One has $\operatorname{depth} M = 0$ and $\dim M = 2$. Hence M is not Cohen-Macaulay. On the other hand, $\operatorname{grade}(Q, M) = \operatorname{depth} N = 0$ and $\operatorname{cd}(Q, M) = \dim N = 1 = \dim L = \operatorname{cd}(P, M)$. Hence M is sequentially Cohen-Macaulay with respect to Q which satisfies condition (b) in Proposition 3.4, while the equality (a) does not hold.

The following question is inspired by Proposition 3.4.

Question 3.6. Let M be a finitely generated bigraded Cohen-Macaulay S-module such that $H^k_Q(M) \neq 0$ for all $\operatorname{grade}(Q, M) \leqslant k \leqslant \operatorname{cd}(Q, M)$. Is $H^s_P(M) \neq 0$ for all $\operatorname{grade}(P, M) \leqslant s \leqslant \operatorname{cd}(P, M)$?

Remark 3.7. Of course the question has affirmative answer in the following cases, namely, if M has only one(two) non-vanishing local cohomology with respect to Q. This immediately follows by Fact 1.1. The projective plane $\mathbb{P}^2$ given in Example 2.3 is also the case as module with three non-vanishing local cohomology.

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