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*Research Article*

# Stability of the Wave Equation with a Source

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We prove the generalized Hyers-Ulam stability of the wave equation with a source, $u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t)$, for a class of real-valued functions with continuous second partial derivatives in $x$ and $t$.

## 1. Introduction

The stability problem for functional equations or (partial) differential equations started with the question of Ulam [1]: *Under what conditions does there exist an additive function near an approximately additive function?* In 1941, Hyers [2] answered the question of Ulam in the affirmative for the Banach space cases. Indeed, Hyers' theorem states that the following statement is true for all $\varepsilon \geq 0$: if a function $f$ satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x$, then there exists an exact additive function $F$ such that $\|f(x) - F(x)\| \leq \varepsilon$ for all $x$. In that case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have (satisfy) the Hyers-Ulam stability.

Assume that $V$ is a normed space and $I$ is an open interval of $\mathbb{R}$. The $n$th-order linear differential equation

$$a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) + h(x) = 0 \tag{1}$$

is said to have (satisfy) the Hyers-Ulam stability provided the following statement is true for all $\varepsilon \geq 0$: if a function $u : I \to V$ satisfies the differential inequality

$$\left\| a_n(x)\, u^{(n)}(x) + a_{n-1}(x)\, u^{(n-1)}(x) + \cdots + a_1(x)\, u'(x) + a_0(x)\, u(x) + h(x) \right\| \leq \varepsilon \tag{2}$$

for all $x \in I$, then there exists a solution $u_0 : I \to V$ to the differential equation (1) and a continuous function $K$ such that $\|u(x) - u_0(x)\| \leq K(\varepsilon)$ for any $x \in I$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

When the above statement is true even if we replace $\varepsilon$ and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on $u$ and $u_0$ explicitly, the corresponding differential equation (1) is said to have (satisfy) the generalized Hyers-Ulam stability. (This type of stability is sometimes called the Hyers-Ulam-Rassias stability.)

These terminologies will also be applied for other differential equations and partial differential equations. For more detailed definitions, we refer the reader to [1–9].

To the best of our knowledge, Obłoza was the first author who investigated the Hyers-Ulam stability of differential equations (see [10, 11]): assume that $g, r : (a,b) \to \mathbb{R}$ are continuous functions with $\int_a^b |g(x)|dx < \infty$ and $\varepsilon$ is an arbitrary positive real number. Obłoza's theorem states that there exists a constant $\delta > 0$ such that $|y(x) - y_0(x)| \leq \delta$ for all $x \in (a,b)$ whenever a differentiable function $y : (a,b) \to \mathbb{R}$ satisfies the inequality $|y'(x) + g(x)y(x) - r(x)| \leq \varepsilon$ for all $x \in (a,b)$ and a function $y_0 : (a,b) \to \mathbb{R}$ satisfies $y_0'(x) + g(x)y_0(x) = r(x)$ for all $x \in (a,b)$ and $y(\tau) = y_0(\tau)$ for some $\tau \in (a,b)$. Since then, a number of mathematicians have dealt with this subject (see [3, 12, 13]).

Prástaro and Rassias are the first authors who investigated the Hyers-Ulam stability of partial differential equations (see [14]). Thereafter, the first author [15], together with Lee, proved the Hyers-Ulam stability of the first-order linear

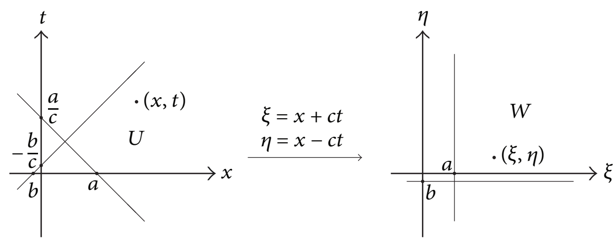


Figure 1

partial differential equation of the form, $au_x(x, y) + bu_y(x, y) + cu(x, y) + d = 0$, where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ are constants with $\Re(c) \neq 0$. As a further step, the first author proved the generalized Hyers-Ulam stability of the wave equation without source (see [16, 17]).

One of typical examples of hyperbolic partial differential equations is the wave equation with a spatial variable $x$ and a time variable $t$,

$$u_{tt}(x, t) - c^2 u_{xx}(x, t) = f(x, t) \tag{3}$$

where $c > 0$ is a constant, whose solution is a scalar function $u = u(x, t)$ describing the propagation of a wave at a speed $c$ in the spatial direction.

In this paper, applying ideas from [16, 18], we investigate the generalized Hyers-Ulam stability of the wave equation (3) with a source, where $x > a$ and $t > b$ with $a, b \in \mathbb{R} \cup \{-\infty\}$. The main advantages of this present paper over the previous papers [16, 17] are that this paper deals with the wave equation with a source and it describes the behavior of approximate solutions of wave equation in the vicinity of origin while the previous one [17] can only deal with domains excluding the vicinity of origin. (Roughly speaking, a solution to a perturbed equation is called an approximate solution.)

## 2. Main Results

We know that if we introduce the characteristic coordinates

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct, \end{aligned} \tag{4}$$

then the wave equation, $u_{tt}(x, t) = c^2 u_{xx}(x, t)$, is transformed into $u_{\xi\eta}(\xi, \eta) = 0$, which seems to be handled easily.

Given real constants $a$ and $b$ with $a, b \in \mathbb{R} \cup \{-\infty\}$, we define

$$\begin{aligned} U &:= \{(x, t) \in \mathbb{R} \times \mathbb{R} : x + ct > a,\ x - ct > b\}, \\ W &:= \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi > a,\ \eta > b\}. \end{aligned} \tag{5}$$

We note that the map $(x, t) \mapsto (\xi, \eta)$, where $\xi = x + ct$ and $\eta = x - ct$, is a one-to-one correspondence from $U$ onto $W$ (see Figure 1).

**Theorem 1.** *Assume that $f : U \to \mathbb{R}$ and $\varphi : U \to [0, \infty)$ are continuous functions with the properties*

$$\begin{aligned} &\int\int_U |f(x, t)|\, dx\, dt < \infty, \\ &\int\int_U \varphi(x, t)\, dx\, dt < \infty. \end{aligned} \tag{6}$$

*If a function $u : U \to \mathbb{R}$ has continuous second partial derivatives and satisfies the inequality*

$$\left| u_{tt}(x, t) - c^2 u_{xx}(x, t) - f(x, t) \right| \le \varphi(x, t) \tag{7}$$

*for all $(x, t) \in U$, then there exists a function $v : U \to \mathbb{R}$ with continuous second partial derivatives such that $v$ is a solution to the wave equation (3) and*

$$|u(x, t) - v(x, t)| \le \frac{1}{2c} \int\int_{U_{a,b,x,t}} \varphi(p, q)\, dp\, dq \tag{8}$$

*for all $(x, t) \in U$, where $U_{a,b,x,t}$ is the interior of the parallelogram having the points*

$$\begin{gathered} \left( \frac{a + b}{2}, \frac{a - b}{2c} \right), \\ \left( \frac{x + a - ct}{2}, \frac{-x + a + ct}{2c} \right), \\ (x, t), \\ \left( \frac{x + b + ct}{2}, \frac{x - b + ct}{2c} \right) \end{gathered} \tag{9}$$

*as its vertices.*

*Proof.* We introduce the characteristic coordinates (4) and we set

$$\begin{aligned} w(\xi, \eta) &:= u\left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = u(x, t), \\ g(\xi, \eta) &:= f\left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = f(x, t), \\ \psi(\xi, \eta) &:= \varphi\left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = \varphi(x, t) \end{aligned} \tag{10}$$

for all $(x, t) \in U$ and corresponding $(\xi, \eta) \in W$ with the relations in (4).

By the chain rule, we get

$$\begin{aligned} u_x(x, t) &= w_\xi(\xi, \eta) \frac{\partial \xi}{\partial x} + w_\eta(\xi, \eta) \frac{\partial \eta}{\partial x} \\ &= w_\xi(\xi, \eta) + w_\eta(\xi, \eta), \\ u_t(x, t) &= w_\xi(\xi, \eta) \frac{\partial \xi}{\partial t} + w_\eta(\xi, \eta) \frac{\partial \eta}{\partial t} \\ &= c w_\xi(\xi, \eta) - c w_\eta(\xi, \eta) \end{aligned} \tag{11}$$

and hence,

$$\begin{aligned} &u_{tt}(x,t) - c^2 u_{xx}(x,t) \\ &\quad = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u(x,t) \\ &\quad = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(2cw_\xi(\xi,\eta)\right) = -4c^2 w_{\xi\eta}(\xi,\eta) \end{aligned} \tag{12}$$

for all $(x,t) \in U$ and corresponding $(\xi,\eta) \in W$ with the relations in (4).

It then follows from (7) and (10) that

$$\begin{aligned} &\left|u_{tt}(x,t) - c^2 u_{xx}(x,t) - f(x,t)\right| \\ &\quad = \left|-4c^2 w_{\xi\eta}(\xi,\eta) - g(\xi,\eta)\right| \le \psi(\xi,\eta) \end{aligned} \tag{13}$$

or

$$\left|w_{\xi\eta}(\xi,\eta) + \frac{1}{4c^2} g(\xi,\eta)\right| \le \frac{1}{4c^2}\psi(\xi,\eta) \tag{14}$$

or

$$\begin{aligned} &-\frac{1}{4c^2}\psi(\xi,\eta) - \frac{1}{4c^2} g(\xi,\eta) \le w_{\xi\eta}(\xi,\eta) \\ &\quad \le \frac{1}{4c^2}\psi(\xi,\eta) - \frac{1}{4c^2} g(\xi,\eta) \end{aligned} \tag{15}$$

for any $(\xi,\eta) \in W$.

Considering the conditions in (6) and Figure 2, we can integrate each term of the last inequality from $a$ to $\xi$ with respect to the first variable and then we integrate each term of the resulting inequality from $b$ to $\eta$ with respect to the second variable to obtain

$$\begin{aligned} &-\frac{1}{4c^2}\int_b^\eta\int_a^\xi \psi(\sigma,\tau)\,d\sigma\,d\tau - \frac{1}{4c^2}\int_b^\eta\int_a^\xi g(\sigma,\tau)\,d\sigma\,d\tau \\ &\quad \le \int_b^\eta\int_a^\xi w_{\sigma\tau}(\sigma,\tau)\,d\sigma\,d\tau \\ &\quad \le \frac{1}{4c^2}\int_b^\eta\int_a^\xi \psi(\sigma,\tau)\,d\sigma\,d\tau \\ &\qquad - \frac{1}{4c^2}\int_b^\eta\int_a^\xi g(\sigma,\tau)\,d\sigma\,d\tau \end{aligned} \tag{16}$$

for any $(\xi,\eta) \in W$.

If we define the function $z : W \to \mathbb{R}$ by

$$\begin{aligned} z(\xi,\eta) &:= -\frac{1}{4c^2}\int_b^\eta\int_a^\xi g(\sigma,\tau)\,d\sigma\,d\tau + w(\xi,b) \\ &\quad + w(a,\eta) - w(a,b), \end{aligned} \tag{17}$$

then we have

$$\left|w(\xi,\eta) - z(\xi,\eta)\right| \le \frac{1}{4c^2}\int_b^\eta\int_a^\xi \psi(\sigma,\tau)\,d\sigma\,d\tau \tag{18}$$

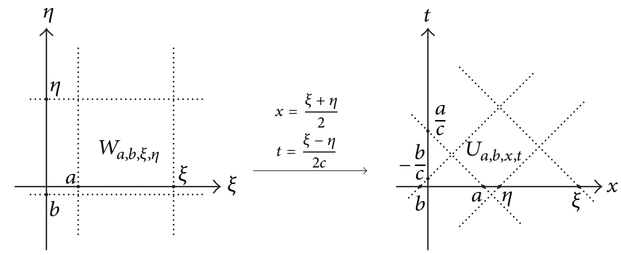


Figure 2

for all $(\xi,\eta) \in W$. Moreover, we get

$$z_{\eta\xi}(\xi,\eta) = -\frac{1}{4c^2} g(\xi,\eta). \tag{19}$$

We now set $v(x,t) := z(\xi,\eta) = z(x+ct, x-ct)$ and, analogously to (11), we compute the partial derivatives:

$$\begin{aligned} v_x(x,t) &= z_\xi(\xi,\eta) + z_\eta(\xi,\eta), \\ v_{xx}(x,t) &= z_{\xi\xi}(\xi,\eta) + 2z_{\xi\eta}(\xi,\eta) + z_{\eta\eta}(\xi,\eta), \\ v_t(x,t) &= cz_\xi(\xi,\eta) - cz_\eta(\xi,\eta), \\ v_{tt}(x,t) &= c^2 z_{\xi\xi}(\xi,\eta) - 2c^2 z_{\xi\eta}(\xi,\eta) + c^2 z_{\eta\eta}(\xi,\eta). \end{aligned} \tag{20}$$

In view of (10), (12), (19), and (20), we get

$$\begin{aligned} v_{tt}(x,t) - c^2 v_{xx}(x,t) &= -4c^2 z_{\xi\eta}(\xi,\eta) = g(\xi,\eta) \\ &= f(x,t) \end{aligned} \tag{21}$$

for all $(x,t) \in U$, that is, $v$ is a solution to wave equation (3).

We compute the Jacobian determinant

$$J(x,t) = \det\begin{pmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\xi}{\partial t} \\ \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\eta}{\partial t} \end{pmatrix} = \det\begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} = -2c. \tag{22}$$

By (10) and (18), we obtain

$$\begin{aligned} |u(x,t) - v(x,t)| &\le \frac{1}{4c^2}\int_b^\eta\int_a^\xi \psi(\sigma,\tau)\,d\sigma\,d\tau \\ &= \frac{1}{2c}\iint_{U_{a,b,x,t}} \varphi(p,q)\,dp\,dq \end{aligned} \tag{23}$$

for all $(x,t) \in U$ (see Figure 2). □

*Remark 2.* In general, it is somewhat tedious to estimate the upper bound of inequality (8). However, in view of (10) and (18), we can compute the upper bound less tediously:

$$\begin{aligned} |u(x,t) - v(x,t)| &\le \int_b^\eta\int_a^\xi \frac{1}{4c^2}\psi(\sigma,\tau)\,d\sigma\,d\tau \\ &= \int_b^{x-ct}\int_a^{x+ct} \frac{1}{4c^2}\varphi\left(\frac{\sigma+\tau}{2}, \frac{\sigma-\tau}{2c}\right)d\sigma\,d\tau \end{aligned} \tag{24}$$

for all $(x,t) \in U$.

When $a = b = -\infty$ in Theorem 1, $U = W = \mathbb{R} \times \mathbb{R}$. In that case, by Theorem 1 and Remark 2, we have the following corollary.

**Corollary 3.** *Assume that* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *are continuous functions satisfying the conditions*

$$\begin{gathered} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f(x,t) \right| dx\, dt < \infty, \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x,t)\, dx\, dt < \infty. \end{gathered} \tag{25}$$

*If a function* $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *has continuous second partial derivatives and satisfies the inequality*

$$\left| u_{tt}(x,t) - c^2 u_{xx}(x,t) - f(x,t) \right| \le \varphi(x,t) \tag{26}$$

*for all* $x, t \in \mathbb{R}$, *then there exists a function* $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *with continuous second partial derivatives such that* $v$ *is a solution to the wave equation (3) and*

$$\begin{aligned} &\left| u(x,t) - v(x,t) \right| \\ &\quad \le \frac{1}{4c^2} \int_{-\infty}^{x-ct} \int_{-\infty}^{x+ct} \varphi\left( \frac{\sigma+\tau}{2}, \frac{\sigma-\tau}{2c} \right) d\sigma\, d\tau \end{aligned} \tag{27}$$

*for all* $x, t \in \mathbb{R}$.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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## References

[1] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, Chapter 6, Wiley, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Acadamy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] C. Alsina and R. Ger, "On some inequalities and stability results related to the exponential function," *Journal of Inequalities and Applications*, vol. 2, no. 4, pp. 373–380, 1998.

[4] J. Brzdek, D. Popa, and I. Rasa, "Hyers-Ulam stability with respect to gauges," *Journal of Mathematical Analysis and Applications*, vol. 453, no. 1, pp. 620–628, 2017.

[5] J. Brzdek, D. Popa, I. Rasa, and B. Xu, "Ulam Stability of Operators," in *Mathematical Analysis and Its Applications*, vol. 1, Academic Press, Elsevier, 2018.

[6] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Mass, USA, 1998.

[7] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, vol. 48 of *Springer Optimization and Its Applications*, Springer, New York, NY, USA, 2011.

[8] Z. Moszner, "Stability has many names," *Aequationes Mathematicae*, vol. 90, no. 5, pp. 983–999, 2016.

[9] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[10] M. Obłoza, "Hyers stability of the linear differential equation," *Rocznik Naukowo-Dydaktyczny - Wyższa Szkoła Pedagogiczna im. Komisji Edukacji Narodowej. Prace Matematyczne*, no. 13, pp. 259–270, 1993.

[11] M. Obłoza, "Connections between Hyers and Lyapunov stability of the ordinary differential equations," *Rocznik Naukowo-Dydaktyczny - Wyższa Szkoła Pedagogiczna im. Komisji Edukacji Narodowej. Prace Matematyczne*, no. 14, pp. 141–146, 1997.

[12] D. Popa and I. Rasa, "On the Hyers-Ulam stability of the linear differential equation," *Journal of Mathematical Analysis and Applications*, vol. 381, no. 2, pp. 530–537, 2011.

[13] S. E. Takahasi, T. Miura, and S. Miyajima, "On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$," *Bulletin of the Korean Mathematical Society*, vol. 39, no. 2, pp. 309–315, 2002.

[14] A. Prástaro and T. M. Rassias, "Ulam stability in geometry of PDE's," *Nonlinear Functional Analysis and Applications. An International Mathematical Journal for Theory and Applications*, vol. 8, no. 2, pp. 259–278, 2003.

[15] S.-M. Jung and K.-S. Lee, "Hyers-Ulam stability of first order linear partial differential equations with constant coefficients," *Mathematical Inequalities & Applications*, vol. 10, no. 2, pp. 261–266, 2007.

[16] S.-M. Jung, "On the stability of one-dimensional wave equation," *The Scientific World Journal*, vol. 2013, Article ID 978754, 3 pages, 2013.

[17] S.-M. Jung, "On the stability of wave equation," *Abstract and Applied Analysis*, vol. 2013, 6 pages, 2013.

[18] G. Wang, M. Zhou, and L. Sun, "Hyers-Ulam stability of linear differential equations of first order," *Applied Mathematics Letters*, vol. 21, no. 10, pp. 1024–1028, 2008.