

BOUNDEDNESS OF PARA-PRODUCT OPERATORS
ON SPACES OF HOMOGENEOUS TYPE

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Abstract. We obtain the boundedness of Calderón-Zygmund singular integral operators T of non-convolution type on Hardy spaces $H^p(\mathcal{X})$ for $1/(1+\varepsilon) < p \leq 1$, where \mathcal{X} is a space of homogeneous type in the sense of Coifman and Weiss (1971), and ε is the regularity exponent of the kernel of the singular integral operator T . Our approach relies on the discrete Littlewood-Paley-Stein theory and discrete Calderón's identity. The crucial feature of our proof is to avoid atomic decomposition and molecular theory in contrast to what was used in the literature.

Keywords: boundedness; Calderón-Zygmund singular integral operator; para-product; spaces of homogeneous type

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1. INTRODUCTION AND STATEMENTS OF RESULTS

In the 1970's, in order to extend the theory of Calderón-Zygmund singular integrals on \mathbb{R}^n to a more general setting, R. Coifman and G. Weiss introduced spaces of homogeneous type which are equipped with a quasi-metric defined as follows.

For a set \mathcal{X} , we say that a function $\varrho: \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is a quasi-metric on \mathcal{X} if it satisfies that

- (i) $\varrho(x, y) = 0$ if and only if $x = y$;
- (ii) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in \mathcal{X}$;
- (iii) there exists a constant $A \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$,

$$\varrho(x, y) \leq A[\varrho(x, z) + \varrho(z, y)].$$

Any quasi-metric ϱ defines a topology, for which the balls $B(x, r) = \{y \in \mathcal{X}: \varrho(x, y) < r\}$ for all $x \in \mathcal{X}$ and all $r > 0$ form a basis.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [2].

Definition 1. Let $\theta \in (0, 1]$. A space of homogeneous type, $(\mathcal{X}, \varrho, \mu)_\theta$, is a set \mathcal{X} together with a quasi-metric ϱ and a nonnegative measure μ on \mathcal{X} , and there exists a constant $C_0 > 0$ such that for all $0 < r < \text{diam } \mathcal{X}$ and all $x, y, z \in \mathcal{X}$,

$$\mu(B(x, r)) \sim r \quad \text{and} \quad |\varrho(x, y) - \varrho(z, y)| \leq C_0 \varrho(x, z)^\theta [\varrho(x, y) + \varrho(z, y)]^{1-\theta}.$$

In the following, let $(\mathcal{X}, \varrho, \mu)_\theta$ be a space of homogeneous type as in Definition 1. The Hölder spaces on \mathcal{X} are defined as follows.

Definition 2. Let $C_0^\eta(\mathcal{X})$, $\eta > 0$, be the space of all continuous functions on \mathcal{X} with compact support and

$$\|f\|_{C^\eta} = \sup_{x, y \in \mathcal{X}; x \neq y} \frac{|f(x) - f(y)|}{\varrho(x, y)^\eta} < \infty.$$

Remark 1. For $\eta \in (0, \theta]$, $C_0^\eta(\mathcal{X})$ is not empty. To see this, we can consider the function $g(x) = f(\varrho(x, x_0))$ with any fixed $x_0 \in \mathcal{X}$, where f is a C^1 function defined on \mathbb{R} with a compact support. It is easy to check that $g \in C_0^\eta(\mathcal{X})$ with $0 < \eta \leq \theta \leq 1$.

Remark 2. The dual space of $C^\beta(\mathbb{R})$ is not a functional space for $0 < \beta \leq 1$. However, it suffices to replace $C^\beta(\mathbb{R})$ by the closure $\dot{C}^\beta(\mathbb{R})$ for the $C^\beta(\mathbb{R})$ norm of functions in $C^\gamma(\mathbb{R})$ where $\gamma > \beta$, and this closure does not depend on γ . Following this argument we define the function space $\dot{C}_0^\eta(\mathcal{X})$ as the closure for the $C_0^\eta(\mathcal{X})$ norm of functions in $C_0^s(\mathcal{X})$ where $s > \eta$, and let $(\dot{C}_0^\eta(\mathcal{X}))'$ be the dual space of $\dot{C}_0^\eta(\mathcal{X})$. Here these two spaces do not depend on s . For more detail, see [11].

We now introduce the Calderón-Zygmund operator on \mathcal{X} . For convenience, in the following, we use C to denote all constants only dependent on \mathcal{X} , which may vary from line to line.

Definition 3 ([2]). A continuous function $K: \mathcal{X} \times \mathcal{X} \setminus \{(x, y): x = y\} \rightarrow \mathbb{C}$ is said to be a Calderón-Zygmund singular integral kernel on \mathcal{X} if there exist $\varepsilon \in (0, \theta]$ and constants $C > 0$ such that

$$\begin{aligned} |K(x, y)| &\leq C \varrho(x, y)^{-1} \text{ for all } x \neq y; \\ |K(x, y) - K(x', y)| &\leq C \varrho(x, x')^\varepsilon \varrho(x, y)^{-(1+\varepsilon)} \text{ for } \varrho(x, x') \leq \frac{1}{2A} \varrho(x, y); \\ |K(x, y) - K(x, y')| &\leq C \varrho(y, y')^\varepsilon \varrho(x, y)^{-(1+\varepsilon)} \text{ for } \varrho(y, y') \leq \frac{1}{2A} \varrho(x, y). \end{aligned}$$

The smallest such constant C is denoted by $\|K\|_{CZ}$. And ε is said to be the regularity exponent of the kernel K .

Definition 4 ([2]). A continuous linear operator $T: \dot{C}_0^\eta(\mathcal{X}) \rightarrow (\dot{C}_0^\eta(\mathcal{X}))'$ for all $\eta \in (0, \theta]$ is said to be a Calderón-Zygmund singular integral operator on \mathcal{X} , if T is associated with a Calderón-Zygmund kernel K so that

$$\langle Tf, g \rangle = \iint K(x, y) f(y) g(x) \, d\mu(y) \, d\mu(x)$$

for all f and $g \in \dot{C}_0^\eta(\mathcal{X})$ with disjoint supports.

Remark 3 ([2]). Any Calderón-Zygmund singular integral operator which is bounded on $L^2(\mathcal{X})$ is also bounded on $L^p(\mathcal{X})$ for $1 < p < \infty$; and is of weak type $(1, 1)$.

We call an operator T a Calderón-Zygmund operator if T is a Calderón-Zygmund singular integral operator and is bounded on L^2 .

From Remark 3 a question arises: Under what conditions a Calderón-Zygmund singular integral operator is bounded on L^2 ? This question was answered by the well-known T1 theorems of G. David and J.L. Journé, and G. David, J.L. Journé and S. Semmes in the standard case of \mathbb{R}^n and in spaces of homogeneous type, respectively.

To introduce the generalization of the T1 theorem to spaces of homogeneous type, we first need to define $T(1)$: The difficulty is that 1 is not a function in $\dot{C}_0^\eta(\mathcal{X})$, hence $T(1)$ is not a distribution in $(\dot{C}_0^\eta(\mathcal{X}))'$, but is a distribution modulo constant function. The definition is based on the following lemma (see [12]).

Lemma 1. *Let S be a distribution in $(\dot{C}_0^\eta(\mathcal{X}))'$. Suppose that there exists $R > 0$ such that the restriction of S to the open set $\{x \in \mathcal{X}: \varrho(x, x_0) > R\}$, where x_0 is a fixed point in \mathcal{X} , is a continuous function such that $S(x) = O(\varrho(x, x_0))^{-1-\gamma}$ as $\varrho(x, x_0) \rightarrow \infty$. If $\gamma > 0$, then the integral*

$$\int_{\mathcal{X}} S(x) \, d\mu(x) = \langle S, 1 \rangle$$

converges.

We first write $1 = \varphi_1(x) + \varphi_2(x)$, where $\varphi_1 \in \dot{C}_0^\eta(\mathcal{X})$ for some $\eta > 0$ and $\varphi_1(x) = 1$ for $\varrho(x, x_0) \leq R$. Then $\langle S, 1 \rangle$ is defined by

$$\langle S, \varphi_1 \rangle + \langle S, \varphi_2 \rangle = \langle S, \varphi_1 \rangle + \int_{\mathcal{X}} S(x) \varphi_2(x) \, d\mu(x)$$

since the integral converges absolutely. It is easy to check that $\langle S, 1 \rangle$ is independent of the decomposition.

Before defining $T1$, we define

$$\dot{C}_{0,0}^\eta(\mathcal{X}) = \left\{ f \in \dot{C}_0^\eta(\mathcal{X}) : \int_{\mathcal{X}} f(x) d\mu(x) = 0 \right\}.$$

If $f \in \dot{C}_{0,0}^\eta(\mathcal{X})$, we define $\langle T1, f \rangle = \langle 1, T^*f \rangle$. Indeed, if the support of f is contained in $\{x \in \mathcal{X} : \varrho(x, x_0) \leq R\}$, then

$$T^*(f)(x) = \int_{\mathcal{X}} [K(y, x) - K(x_0, x)] f(y) d\mu(y) = O(\varrho(x, x_0)^{-1-\varepsilon})$$

for $\varrho(x, x_0) > R$ and $\varepsilon > 0$.

Now $T1$ is a continuous linear form on $\dot{C}_{0,0}^\eta(\mathcal{X}) \subset \dot{C}_0^\eta(\mathcal{X})$. We extend $T1$ to a distribution $S \in (\dot{C}_0^\eta(\mathcal{X}))'$ as follows: let $\varphi \in \dot{C}_0^\eta(\mathcal{X})$ be a function with $\int_{\mathcal{X}} \varphi(x) d\mu(x) = 1$, then for all $f \in \dot{C}_0^\eta(\mathcal{X})$, f can be written uniquely as $f = \lambda\varphi + g$, where $\lambda = \int f(x) d\mu(x)$ and $g \in \dot{C}_{0,0}^\eta(\mathcal{X})$. Now we choose S such that $\langle S, f \rangle = \lambda \langle S, \varphi \rangle + \langle T1, g \rangle$, then $T1 = S$ on $\dot{C}_{0,0}^\eta(\mathcal{X})$, and is a distribution modulo the constant. T^*1 can be defined in a similar way.

For $\delta \in (0, \theta]$, $x_0 \in \mathcal{X}$ and $r > 0$, we define $A(\delta, x_0, r)$ to be the set of all $\varphi \in \dot{C}_0^\delta(\mathcal{X})$ supported in $B(x_0, r)$ satisfying $\|\varphi\|_\infty < 1$ and $\|\varphi\|_{C^\delta} < r^{-\delta}$. To introduce $T1$ theorem on \mathcal{X} , we also need the following definition of weak boundedness.

Definition 5. An operator T is weakly bounded if there exist $\delta \in (0, \theta]$ and $C < \infty$ such that for all $x_0 \in \mathcal{X}$, $r > 0$ and $\varphi, \psi \in A(\delta, x_0, r)$,

$$|\langle T\varphi, \psi \rangle| \leq C\mu(B(x_0, r)).$$

Remark 4. It is easy to see that weak boundedness is obviously implied by L^2 boundedness. And Calderón-Zygmund singular integral operator whose is antisymmetrical kernel, i.e., $K(x, y) = -K(y, x)$, has the weak boundedness property.

In 1985, using Coifman's idea on decomposition of the identity operator, G. David, J.L. Journé and S. Semmes developed the Littlewood-Paley analysis on spaces of homogeneous type and used it to give a proof of the following $T1$ theorem in this general setting.

Theorem A ([4]). Let T be a Calderón-Zygmund singular integral operator on \mathcal{X} . Then a necessary and sufficient condition for the extension of T as a continuous linear operator on $L^2(\mathcal{X})$ is that the following conditions are all satisfied: (a) $T1 \in \text{BMO}$; (b) $T^*1 \in \text{BMO}$; (c) T is weakly bounded. Here

$$\text{BMO}(\mathcal{X}) = \left\{ f \in L_{\text{loc}}^1(\mathcal{X}) : \sup_{r>0, x \in \mathcal{X}} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f_B| d\mu(y) < \infty \right\},$$

where $f_B = \mu(B(x, r))^{-1} \int_{B(x, r)} f(y) d\mu(y)$.

Deng and Han gave a new $T1$ theorem for the general spaces of homogeneous type as follows.

Theorem B ([5]). *Let T be a Calderón-Zygmund singular integral operator on \mathcal{X} with $T1 = T^*1 = 0$, and T is weakly bounded. Then T is bounded on L^p for $1 < p < \infty$ and H^p for $1/(1 + \varepsilon) < p \leq 1$, where ε is the regularity exponent of the kernel of the singular integral operator T .*

In the above theorem, the conditions $T1 = 0$ and $T^*1 = 0$ are sufficient conditions. A natural problem is when these conditions are also necessary. The following theorem answers this problem.

Theorem 1 ([5]). *Let T be a Calderón-Zygmund operator on \mathcal{X} , then T is bounded on $H^p(\mathcal{X})$ for all $1/(1 + \varepsilon) < p \leq 1$ if and only if $T^*1 = 0$.*

We remark here that the main tool used in the literature to prove Theorem 1 is the molecular theory of the Hardy space $H^p(\mathcal{X})$, see [3], [5].

In this paper, we will use a different approach to prove Theorem 1 without using atomic decomposition or molecular theory of $H^p(\mathcal{X})$. Moreover, we can get

Theorem 2. *If T is a Calderón-Zygmund operator on \mathcal{X} , then T is bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $1/(1 + \varepsilon) < p \leq 1$.*

The main ideas are using almost estimates, the discrete Littlewood-Paley-Stein theory and discrete Calderón's identity together with the maximal and Littlewood-Paley characterizations of the Hardy spaces $H^p(\mathcal{X})$ to get the boundedness of the para-product which will be defined later (see Definition 8).

Our new approach includes the following steps.

Step 1. The discrete Calderón's identity, almost orthogonality estimates and the H^p boundedness.

To recall the classical continuous Calderón's identity, we begin with introducing the approximation to identity on the space of homogeneous type.

Definition 6 ([10]). A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\varepsilon \in (0, \theta]$ on \mathcal{X} if there exists $C > 0$ such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x, y)$, the kernel of S_k , is a function from $\mathcal{X} \times \mathcal{X}$ into \mathbb{C} satisfying

- (1) $|S_k(x, y)| \leq C \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{1+\varepsilon}};$
- (2) $|S_k(x, y) - S_k(x', y)| \leq C \left(\frac{\varrho(x, x')}{2^{-k} + \varrho(x, y)} \right)^\varepsilon \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{1+\varepsilon}}$
for $\varrho(x, x') \leq (2A)^{-1}(2^{-k} + \varrho(x, y));$

- (3) $|S_k(x, y) - S_k(x, y')| \leq C \left(\frac{\varrho(y, y')}{2^{-k} + \varrho(x, y)} \right)^\varepsilon \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{1+\varepsilon}}$
for $\varrho(y, y') \leq (2A)^{-1}(2^{-k} + \varrho(x, y))$;
(4) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]|$
 $\leq C \left(\frac{\varrho(x, x')}{2^{-k} + \varrho(x, y)} \right)^\varepsilon \left(\frac{\varrho(y, y')}{2^{-k} + \varrho(x, y)} \right)^\varepsilon \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{1+\varepsilon}}$
for $\varrho(x, x') \leq (2A)^{-1}(2^{-k} + \varrho(x, y))$ and $\varrho(y, y') \leq (2A)^{-1}(2^{-k} + \varrho(x, y))$;
(5) $\int_{\mathcal{X}} S_k(x, y) d\mu(y) = 1$;
(6) $\int_{\mathcal{X}} S_k(x, y) d\mu(x) = 1$.

Next let us recall the definition of the space of test functions on spaces of homogeneous type.

Definition 7 ([8]). Fix $0 < \gamma, \beta < \theta$. A function f defined on \mathcal{X} is said to be a test function of type (x_0, r, β, γ) with $x_0 \in \mathcal{X}$ and $r > 0$, if f satisfies the following conditions:

- (i) $|f(x)| \leq C \frac{r^\gamma}{(r + \varrho(x, x_0))^{1+\gamma}}$;
(ii) $|f(x) - f(y)| \leq C \left(\frac{\varrho(x, y)}{r + \varrho(x, x_0)} \right)^\beta \frac{r^\gamma}{(r + \varrho(x, x_0))^{1+\gamma}}$
for $\varrho(x, y) \leq (2A)^{-1}[r + \varrho(x, x_0)]$;
(iii) $\int_{\mathcal{X}} f(x) d\mu(x) = 0$.

If f is a test function of type (x_0, r, β, γ) , we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} = \inf\{C: \text{(i) and (ii) hold}\}.$$

Now fix $x_0 \in \mathcal{X}$ and let $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with an equivalent norm for all $x_1 \in \mathcal{X}$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Also, let the dual space $(\mathcal{G}(\beta, \gamma))'$ consist of all linear functionals \mathcal{L} from $\mathcal{G}(\beta, \gamma)$ to \mathbb{C} with the property that there exists $C \geq 0$ such that for all $f \in \mathcal{G}(\beta, \gamma)$,

$$|\mathcal{L}(f)| \leq C \|f\|_{\mathcal{G}(\beta, \gamma)}.$$

We denote by $\langle h, f \rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta, \gamma))'$ and $f \in \mathcal{G}(\beta, \gamma)$. Clearly, for all $h \in (\mathcal{G}(\beta, \gamma))'$, $\langle h, f \rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in \mathcal{X}$ and $r > 0$.

It is well-known that even when $\mathcal{X} = \mathbb{R}^n$, $\mathcal{G}(\beta_1, \gamma)$ is not dense in $\mathcal{G}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will cause us some inconvenience. To overcome this defect, in what follows, for a given $\varepsilon \in (0, \theta]$, we let $\mathring{\mathcal{G}}(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\varepsilon, \varepsilon)$ in $\mathcal{G}(\beta, \gamma)$ when $0 < \beta, \gamma < \varepsilon$.

We also need the following construction given by Christ in [1], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. A similar construction was independently given by Sawyer and Wheeden in [14].

Lemma 2. *For every integer $k \in \mathbb{Z}_+$, there exists a collection of open subsets $\{Q_\tau^k \subset \mathcal{X} : \tau \in I_k\}$, where I_k denotes some index set depending on k , and $c_1, c_2 > 0$, are such that*

- (i) $\mu(\{X \setminus \bigcup Q_\tau^k\}) = 0$;
- (ii) if $l \geq k$, then for all $\tau' \in I_l$ and $\tau \in I_k$ either $Q_{\tau'}^l \subset Q_\tau^k$ or $Q_{\tau'}^l \cap Q_\tau^k = \emptyset$;
- (iii) if $l < k$, for each $\tau \in I_k$, there is a unique $\tau' \in I_l$ such that $Q_\tau^k \subset Q_{\tau'}^l$, $\text{diam}(Q_\tau^k) \leq c_1 2^{-k}$, and each $Q_{\tau'}^l$ contains some ball $B(z_{\tau'}^l, c_2 2^{-l})$.

In the following, we say that a cube $Q \subset \mathcal{X}$ is a dyadic cube in \mathcal{X} if $Q = Q_\tau^k$ for some $k \in \mathbb{Z}_+$ and $\tau \in I_k$, and denote it by $\text{diam } Q \sim 2^{-k}$. Denote by $Q_\tau^{k,\nu}$, $\nu = 1, 2, \dots, N(k, \tau)$, the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$ where j is a fixed large positive integer, and denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$.

We now recall the discrete Calderón reproducing formulae on spaces of homogeneous type in [9].

Lemma 3. *Let $\varepsilon \in (0, \theta]$ for $k \in \mathbb{Z}$, let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity of order ε , $D_k = S_k - S_{k-1}$, let $\{Q_\tau^{k,\nu} : \tau \in I_k, \nu = 1, \dots, N(k, \tau)\}$ be the dyadic cubes of \mathcal{X} defined in Lemma 2 with $j \in \mathbb{N}$ large enough. Then there are two families of linear operators $\{\tilde{D}_k\}_{k \in \mathbb{Z}}$, $\{\overline{D}_k\}_{k \in \mathbb{Z}}$ on \mathcal{X} such that for all $f \in \mathcal{G}(\beta, \gamma)$ with $\beta, \gamma \in (0, \varepsilon)$ and any point any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,*

$$\begin{aligned}
 (1) \quad f(x) &= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \tilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \\
 &= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(x, y_\tau^{k,\nu}) \overline{D}_k(f)(y_\tau^{k,\nu}),
 \end{aligned}$$

where the series converge in the norm of both the space $\mathcal{G}(\beta', \gamma')$ with $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$ and the space $L^p(X)$ with $p \in (1, \infty)$.

By an argument of duality, Han in [9] also established the following discrete Calderón reproducing formulae on spaces of distributions, $(\mathring{\mathcal{G}}(\beta, \gamma))'$ with $\beta, \gamma \in (0, \varepsilon)$.

Lemma 4. *With all the notation as in Lemma 3, for all $f \in (\mathring{\mathcal{G}}(\beta, \gamma))'$ with $\beta, \gamma \in (0, \varepsilon)$, (1) holds in $(\mathring{\mathcal{G}}(\beta', \gamma'))'$ with $\beta < \beta' < \varepsilon$ and $\gamma < \gamma' < \varepsilon$.*

Applying the above lemma, it was proved in [5] that $H^p(\mathcal{X})$ can be characterized by discrete Littlewood-Paley square functions

Proposition 1. *Let $\theta' \in (0, \theta)$, let D_k and $Q_\tau^{k, \nu}$ be the same as in Lemma 3. Then for $1/(1 + \theta') < p \leq 1$, $f \in H^p(\mathcal{X})$ if and only if $f \in (\mathring{\mathcal{G}}(\beta, \gamma))'$ with $\beta, \gamma \in (0, \theta')$ and*

$$\|f\|_{H^p} \sim \left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} |D_k(f)|^2 \chi_{Q_\tau^{k, \nu}}(\cdot) \right\}^{1/2} \right\|_p < \infty.$$

Remark 5. $H^p(\mathcal{X})$ also can be characterized by classical continuous Littlewood-Paley square functions, i.e.,

$$\|f\|_{H^p(\mathcal{X})} \sim \left\| \left\{ \sum_{k=-\infty}^{\infty} |D_k(f)(\cdot)|^2 \right\}^{1/2} \right\|_p.$$

These two kinds of definition of $H^p(\mathcal{X})$ are both independent of the choice of the approximation to identity, see [5] for the proof.

Proposition 1 and the almost orthogonality estimates provide a direct proof of the following $H^p(\mathcal{X})$ boundedness.

Theorem 3. *If T is a Calderón-Zygmund operator with regularity exponent $\varepsilon > 0$ and $T1 = T^*1 = 0$, then T is bounded on $H^p(\mathcal{X})$ for $1/(1 + \varepsilon) < p \leq 1$.*

The proof of this theorem is elementary. The basic idea is to apply the orthogonality estimates stated as follows.

Lemma 5. *Let D_k be the same as in Lemma 3. If T satisfies the conditions in Theorem 3, then*

$$|D_k T(D_l)(x, y)| \leq C 2^{-|k-l|\varepsilon'} \frac{2^{-(k \wedge l)\varepsilon'}}{(2^{-(k \wedge l)} + \varrho(x, y))^{1+\varepsilon'}},$$

where $\varepsilon' \in (0, \varepsilon)$, and the constant depends only on ε' and D_k .

Remark 6. We remark that the conditions $T1 = T^*1 = 0$ are crucial in deriving Lemma 5. The classical orthogonality estimates are

$$|D_k T(D_l)(x, y)| \leq C 2^{-|k-l|L} \frac{2^{-(k \wedge l)M}}{(2^{-(k \wedge l)} + \varrho(x, y))^{1+M}},$$

for any L, M and the constant C depends only on L, M and D_k . See [4], [10], [5] for details of its proof.

We also need the following lemma, which can be found in [7], pages 147–148, for \mathbb{R}^n and [5], page 93, for spaces of homogeneous type.

Lemma 6. *Let $k, \eta \in \mathbb{Z}_+$ with $\eta \leq k$. If for any dyadic cube $Q_\tau^{k,\nu} \subset \mathcal{X}$,*

$$|f_{Q_\tau^{k,\nu}}(x)| \leq (1 + 2^\eta \varrho(x, y_\tau^{k,\nu}))^{-1-\varepsilon},$$

where $x \in \mathcal{X}$, $y_\tau^{k,\nu}$ is any point in $Q_\tau^{k,\nu}$ and $\varepsilon > 0$, then

$$\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}| |f_{Q_\tau^{k,\nu}}(x)| \leq C 2^{(k-\eta)} \left[M \left(\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}| \chi_{Q_\tau^{k,\nu}}(x) \right)^r \right]^{1/r},$$

where $r > 1/(1+\varepsilon)$, C is independent of x , k and η , $\lambda_{Q_\tau^{k,\nu}}$ is any constant only depending on $Q_\tau^{k,\nu}$. Here and in the sequel, M is the Hardy-Littlewood maximal operator on \mathcal{X} , which is defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y).$$

We now return to the proof of Theorem 3. By Proposition 1, we only need to show that for $1/(1+\varepsilon) < p \leq 1$, $f \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$, we have

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |D_k(Tf)|^2 \chi_{Q_\tau^{k,\nu}}(\cdot) \right\}^{1/2} \right\|_p < \|f\|_{H^p(\mathcal{X})}.$$

Note that T is bounded on $L^2(\mathcal{X})$. Therefore, by Lemma 4, we can rewrite $D_k(Tf)$ as

$$\begin{aligned} D_k(Tf) &= D_k \left(T \sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \mu(Q_\tau^{k',\nu}) D_{k'}(\cdot, y_\tau^{k',\nu}) \overline{D_{k'}(f)}(y_\tau^{k',\nu}) \right) \\ &= \sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \mu(Q_\tau^{k',\nu}) D_k T D_{k'}(\cdot, y_\tau^{k',\nu}) \overline{D_{k'}(f)}(y_\tau^{k',\nu}). \end{aligned}$$

Using the orthogonality estimates yields

$$|D_k(Tf)| \leq C \sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} 2^{-|k-k'|\varepsilon'} \frac{2^{-(k \wedge k')\varepsilon'} \mu(Q_\tau^{k',\nu})}{(2^{-(k \wedge k')} + \varrho(\cdot, y))^{1+\varepsilon'}} \overline{D_{k'}(f)}(y_\tau^{k',\nu}),$$

where $\varepsilon' \in (0, \varepsilon)$.

Then by applying Lemma 6, we have

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} |D_k(Tf)|^2 \\ & \leq C \sum_{k=-\infty}^{\infty} \left[\sum_{k'=-\infty}^{\infty} 2^{-|k-k'|\varepsilon'} \left\{ M \left(\sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \overline{D}_{k'}(f)(y_{\tau}^{k',\nu}) \chi_{Q_{\tau}^{k',\nu}} \right)^r \right\}^{1/r} \right]^2. \end{aligned}$$

Finally, by the Fefferman-Stein vector valued maximal function inequality in [6] on $L^2(\mathcal{X})$, we obtain

$$\begin{aligned} & \left\| \left\{ \sum_{k=-\infty}^{\infty} |D_k(Tf)|^2 \right\}^{1/2} \right\|_p \\ & \leq C \left\| \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{k'=-\infty}^{\infty} 2^{-|k-k'|\varepsilon'} \left\{ M \left(\sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \overline{D}_{k'}(f)(y_{\tau}^{k',\nu}) \chi_{Q_{\tau}^{k',\nu}}(\cdot) \right)^r \right\}^{1/r} \right]^2 \right\}^{1/2} \right\|_p \\ & \leq C \left\| \left(\sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} |\overline{D}_{k'}(f)(y_{\tau}^{k',\nu})|^2 \chi_{Q_{\tau}^{k',\nu}}(\cdot) \right)^{1/2} \right\|_p \leq C \|f\|_{H^p}. \end{aligned}$$

Since $L^2(\mathcal{X}) \cap H^p(\mathcal{X})$ is dense in $H^p(\mathcal{X})$, the above estimates give the proof of Theorem 3.

Step 2. A new discrete Calderón's identity for $\text{BMO}(\mathcal{X})$.

Proposition 2. *Let $\theta' \in (0, \theta)$, $1/(1 + \theta') < p \leq 1$. Then for any $f \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$, there exists some $\tilde{f} \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$ with $\|f\|_2 \sim \|\tilde{f}\|_2$ and $\|f\|_{H^p} \sim \|\tilde{f}\|_{H^p}$ and*

$$(2) \quad f = \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_{\tau}^{k,\nu}) D_k(\cdot, y_{\tau}^{k,\nu}) \overline{D}_k(\tilde{f})(y_{\tau}^{k,\nu})$$

where $Q_{\tau}^{k,\nu}, y_{\tau}^{k,\nu}, D_k$ are the same as in Lemma 3, and the series converges in $L^2(\mathcal{X}) \cap H^p(\mathcal{X})$.

Proof. We begin with the classical Calderón's identity on $L^2(\mathcal{X})$:

$$f = \sum_{k=-\infty}^{\infty} D_k \overline{D}_k(f).$$

Using Coifman's idea of decomposition of identity yields

$$\begin{aligned}
f(x) &= \sum_{k=-\infty}^{\infty} \int_{\mathcal{X}} D_k \overline{D}_k(x, y)(f)(y) \, d\mu(y) \\
&= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \int_{Q_{\tau}^{k, \nu}} D_k \overline{D}_k(x, y)(f)(y) \, d\mu(y) \\
&= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_k(x, y_{\tau}^{k, \nu}) \overline{D}_k(f)(y_{\tau}^{k, \nu}) + R(f)(x).
\end{aligned}$$

It was proved by Deng and Han in [5] that R is a Calderón-Zygmund operator on \mathcal{X} . Note that $R(1) = R^*(1) = 0$, hence by Theorem 3, R is bounded on $H^p(\mathcal{X})$. Moreover, there exists $\delta > 0$ such that $\|R(f)\|_2 \leq C2^{-N\delta}\|f\|_2$ and $\|R(f)\|_{H^p} \leq C2^{-N\delta}\|f\|_{H^p}$.

See [4], [5], [10] for details of the proofs. Now for any $f \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$, we set $\tilde{f} = \sum_{n=0}^{\infty} R^n(f)$. This implies

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_k(x, y_{\tau}^{k, \nu}) \overline{D}_k(\tilde{f})(y_{\tau}^{k, \nu}).$$

We remark that R is also bounded on $\text{BMO}(\mathcal{X})$ with the inequality $\|R(f)\|_{\text{BMO}} \leq C2^{-N\delta}\|f\|_{\text{BMO}}$. For any $f \in L^2(\mathcal{X}) \cap H^1(\mathcal{X})$, the same proof implies

$$\tilde{f}(x) = \sum_{n=0}^{\infty} R^n \left(\sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_k(\cdot, y_{\tau}^{k, \nu}) \overline{D}_k(\tilde{f})(y_{\tau}^{k, \nu}) \right)(x),$$

where the series converges in $H^1(\mathcal{X})$. Therefore, for any $h \in \text{BMO}(\mathcal{X})$,

$$\begin{aligned}
\langle \tilde{f}, h \rangle &= \left\langle \sum_{n=0}^{\infty} R^n \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_k(\cdot, y_{\tau}^{k, \nu}) \overline{D}_k(\tilde{f})(y_{\tau}^{k, \nu}), h \right\rangle \\
&= \left\langle \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_k(\cdot, y_{\tau}^{k, \nu}) \overline{D}_k(\tilde{f})(y_{\tau}^{k, \nu}), \tilde{h} \right\rangle \\
&= \left\langle \tilde{f}, \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_k(\cdot, y_{\tau}^{k, \nu}) \overline{D}_k(\tilde{h})(y_{\tau}^{k, \nu}) \right\rangle
\end{aligned}$$

where $\tilde{h} = \sum_{n=0}^{\infty} R^n(h) \in \text{BMO}(\mathcal{X})$ with $\|h\|_{\text{BMO}} \sim \|\tilde{h}\|_{\text{BMO}}$.

We now obtain the discrete Calderón's identity for BMO functions: for any $h \in \text{BMO}(\mathcal{X})$, there exists $\tilde{h} \in \text{BMO}(\mathcal{X})$ such that

$$h = \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\tilde{h})(y_\tau^{k,\nu})$$

where the series converges in (H^1, BMO) sense. \square

Step 3. The discrete para-product operators.

We now introduce the discrete para-product operators.

Definition 8. Let $\varepsilon \in (0, \theta]$ for $k \in \mathbb{Z}$, let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity of order ε , $D_k = S_k - S_{k-1}$, \overline{D}_k , $Q_\tau^{k,\nu}$ and let $y_\tau^{k,\nu}$ be the same as in Lemma 3. For the convenience, let

$$\Lambda = \{\lambda = (k, \tau, \nu) : k \in \mathbb{Z}, \tau \in I_k, \nu = 1, \dots, N(k, \tau)\},$$

and Q_λ, y_λ are used to denote the associated $Q_\tau^{k,\nu}$ and $y_\tau^{k,\nu}$.

Then the discrete para-product π_b for $b \in \text{BMO}(\mathcal{X})$ is defined by

$$\pi_b(f)(x) = \sum_{\lambda \in \Lambda} \mu(Q_\lambda) D_k(x, y_\lambda) \overline{D}_k(\tilde{b})(y_\lambda) S_k(f)(y_\lambda)$$

and

$$\pi_b^*(f)(x) = \sum_{\lambda \in \Lambda} \mu(Q_\lambda) S_k(x, y_\lambda) \overline{D}_k(\tilde{b})(y_\lambda) D_k(f)(y_\lambda),$$

where \tilde{h} is the same as in Proposition 2.

Note that for $b \in \text{BMO}(\mathcal{X})$, π_b is a Calderón-Zygmund operator on \mathcal{X} . Then for $b \in \text{BMO}(\mathcal{X})$ and $g \in \mathring{C}_{0,0}^\eta(\mathcal{X})$, by Proposition 2 we have

$$\begin{aligned} \langle T_b(1), g \rangle &= \left\langle \sum_{\lambda \in \Lambda} \mu(Q_\lambda) D_k(x, y_\lambda) \overline{D}_k(\tilde{b})(y_\lambda) S_k(1)(y_\lambda), g \right\rangle \\ &= \left\langle \sum_{\lambda \in \Lambda} \mu(Q_\lambda) D_k(x, y_\lambda) \overline{D}_k(\tilde{b})(y_\lambda)(y_\lambda), g \right\rangle = \langle b, g \rangle. \end{aligned}$$

Therefore, we have $\pi_b(1) = b$. Similarly, we can get $\pi_b^*(1) = 0$. Then using an idea of the proof of the $T1$ theorem given by David and Journé, one can decompose a Calderón-Zygmund singular integral operator T into

$$T = \tilde{T} + \pi_{T1} + \pi_{T^*1}^*,$$

where \tilde{T} is a Calderón-Zygmund singular integral operator. Moreover, note that

$$\langle T1, g \rangle = \langle \tilde{T}1, g \rangle + \langle \pi_{T1}(1), g \rangle + \langle \pi_{T^*1}^*(1), g \rangle = \langle \tilde{T}1, g \rangle + \langle T1, g \rangle,$$

so we have $\tilde{T}1 = 0$. And similarly we get $\tilde{T}^*1 = 0$. Moreover, if T is bounded on $L^2(\mathcal{X})$, then $T1$ and T^*1 are bounded on $\text{BMO}(\mathcal{X})$ and \tilde{T} , π_{T1} and $\pi_{T^*1}^*$ are all bounded on $L^2(\mathcal{X})$.

Note that Theorem B implies that \tilde{T} is bounded on $H^p(\mathcal{X})$, since L^2 boundedness of \tilde{T} implies its weak boundedness. Therefore, to prove Theorem 1 we only need to show that π_b is bounded on $H^p(\mathcal{X})$, to prove Theorem 2 we only need to show that π_b^* and π_b are bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leq 1$ and $b \in \text{BMO}(\mathcal{X})$.

Lemma 7. *Let $b \in \text{BMO}(\mathcal{X})$. Then π_b is bounded on $H^p(\mathcal{X})$, π_b^* and π_b are bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leq 1$.*

Proof. We first show that π_b is bounded on $H^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leq 1$ and $b \in \text{BMO}(\mathcal{X})$.

By the Littlewood-Paley characterization of $H^p(\mathcal{X})$ in Proposition 1, we only need to prove that

$$\left\| \left\{ \sum_{\lambda=(k,\tau,\nu) \in \Lambda} |D_k(\pi_b(f))(y_\lambda)|^2 \chi_{Q_\lambda}(\cdot) \right\}^{1/2} \right\|_p^p \leq C_p \|f\|_{H^p}^p.$$

Using the almost orthogonality, we get

$$\begin{aligned} & \left\| \left\{ \sum_{\lambda \in \Lambda} |D_k(\pi_b(f))(y_\lambda)|^2 \chi_{Q_\lambda}(\cdot) \right\}^{1/2} \right\|_p^p \\ &= \left\| \left\{ \sum_{\lambda \in \Lambda} \left| D_k \left(\sum_{\lambda' \in \Lambda'} \mu(Q_{\lambda'}) D_{k'}(\cdot, y_{\lambda'}) \overline{D}_{k'}(\tilde{b})(y_{\lambda'}) S_{k'}(f)(y_{\lambda'}) \right) (y_\lambda) \right|^2 \chi_{Q_\lambda}(\cdot) \right\}^{1/2} \right\|_p^p \\ &\leq C \left\| \left\{ \sum_{\lambda' \in \Lambda'} |\overline{D}_{k'}(\tilde{b})(y_{\lambda'}) S_{k'}(f)(y_{\lambda'})|^2 \chi_{Q_{\lambda'}}(\cdot) \right\}^{1/2} \right\|_p^p. \end{aligned}$$

Set

$$\Omega_l = \left\{ x \in \mathcal{X} : \sup_k |S_k(f)(x)|^2 > 2^l \right\}$$

and

$$B_l = \left\{ Q' \text{ is a dyadic cube in } \mathcal{X} : \mu(Q' \cap \Omega_l) > \frac{1}{2} \mu(Q') \text{ and } \mu(Q' \cap \Omega_{l+1}) \leq \frac{1}{2} \mu(Q') \right\}.$$

Then by Remark 5, i.e., the maximal characterization of the Hardy space given in [5], we get

$$\sum_l 2^{lp} \mu(\Omega_l) \leq \|f\|_{H^p}^p.$$

Then

$$\begin{aligned} & \sum_{\lambda' \in \Lambda'} |\overline{D}_{k'}(\tilde{b})(y_{\lambda'}) S_{k'}(f)(y_{\lambda'})|^2 \chi_{Q_{\lambda}}(\cdot) \\ &= \sum_{k'} \sum_l \sum_{\tilde{Q} \in B_l} \sum_{Q' \subset \tilde{Q}, Q' \in B_l} |\overline{D}_{k'}(\tilde{b})(y_{Q'}) S_{k'}(f)(y_{Q'})|^2 \chi_{Q'}(\cdot), \end{aligned}$$

where \tilde{Q} are maximal dyadic cubes in B_l and $y_{Q'}$ is any point in Q' . This leads to the estimate

$$\begin{aligned} & \left\| \sum_{\lambda' \in \Lambda'} \{ |\overline{D}_{k'}(\tilde{b})(y_{\lambda'}) S_{k'}(f)(y_{\lambda'})|^2 \chi_{Q_{\lambda}}(\cdot) \}^{1/2} \right\|_p^p \\ & \leq \sum_l \sum_{\tilde{Q} \in B_l} \left\| \sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \{ |\overline{D}_{k'}(\tilde{b})(y_{Q'}) S_{k'}(f)(y_{Q'})|^2 \chi_{Q'}(\cdot) \}^{1/2} \right\|_p^p, \end{aligned}$$

where the inequality $(a + b)^p \leq a^p + b^p$ for $0 < p \leq 1$ is used. Using the Hölder inequality to control the L^p norm by the L^2 norm for functions with compact support, we get

$$\begin{aligned} & \left\| \sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \{ |\overline{D}_{k'}(\tilde{b})(y_{Q'}) S_{k'}(f)(y_{Q'})|^2 \chi_{Q'}(\cdot) \}^{1/2} \right\|_p^p \\ & \leq C \mu(\tilde{Q})^{1-p/2} \left(\sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q') |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 |S_{k'}(f)(y_{Q'})|^2 \right)^{p/2}. \end{aligned}$$

This yields

$$\begin{aligned} & \sum_l \sum_{\tilde{Q} \in B_l} \left\| \sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \{ |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 |S_{k'}(f)(y_{Q'})|^2 \chi_{Q'}(\cdot) \}^{1/2} \right\|_p^p \\ & \leq \sum_l \sum_{\tilde{Q} \in B_l} C \mu(\tilde{Q})^{1-p/2} \left(\sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q') |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 |S_{k'}(f)(y_{Q'})|^2 \right)^{p/2} \\ & \leq \sum_l \left(\sum_{\tilde{Q} \in B_l} C \mu(\tilde{Q}) \right)^{1-p/2} \left(\sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q') |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 |S_{k'}(f)(y_{Q'})|^2 \right)^{p/2}. \end{aligned}$$

Note that if $Q' \in B_l$, then

$$Q' \subset \tilde{\Omega}_l = \left\{ x \in \mathcal{X} : M_{\chi_{\Omega_l}}(x) > \frac{1}{2} \right\}$$

and since $y_{Q'}$ is any fixed point in $Q' \in B_l$, where $\mu(Q' \cap \Omega_{l+1}) \leq \mu(Q')/2$ so we can take $y_{Q'} \in \Omega_{l+1}$, then $|S_{k'}(f)(y_{Q'})| \leq 2^{l+1}$. Therefore,

$$\begin{aligned} & \left(\sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q') |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 |S_{k'}(f)(y_{Q'})|^2 \right)^{p/2} \\ & \leq C 2^{lp} \left(\sum_{Q' \subset \tilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q') |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 \right)^{p/2} \\ & \leq C 2^{lp} \left(\sum_{\tilde{Q} \in B_l} \mu(\tilde{Q}) \right)^{p/2} \leq C 2^{lp} \mu(\tilde{\Omega}_l)^{p/2} \leq C 2^{lp} \mu(\Omega_l)^{p/2} \end{aligned}$$

where we have used the fact that $b \in \text{BMO}$ and a result concerning the Carleson measure ([5], page 118, Theorem 4.13)

$$\sum_{Q' \subset \tilde{Q}} \sum_{k'} \mu(Q') |\overline{D}_{k'}(\tilde{b})(y_{Q'})|^2 \leq C \mu(\tilde{Q}).$$

Substituting all these estimates into the above inequality we get

$$\begin{aligned} & \left\| \sum_{\lambda' \in \Lambda'} \{ |\overline{D}_{k'}(\tilde{b})(y_{\lambda'}) S_{k'}(f)(y_{\lambda'})|^2 \chi_{Q_{\lambda}}(\cdot) \}^{1/2} \right\|_p^p \\ & \leq \sum_l \left(\sum_{\tilde{Q} \in B_l} C \mu(\tilde{Q}) \right)^{1-p/2} 2^{lp} \mu(\Omega_l)^{p/2} \\ & \leq C \sum_l 2^{lp} \mu(\tilde{\Omega})^{1-p/2} \mu(\Omega_l)^{p/2} \leq C \sum_l 2^{lp} \mu(\Omega_l) \leq C \|f\|_{H^p}^p. \end{aligned}$$

This shows that π_b is bounded on $H^p(\mathcal{X})$.

We now prove that π_b^* is bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$. A similar result for π_b can be obtained by the same method. We first note that π_b^* is bounded on L^2 , thus

$$\|\pi_b^* f\|_p^p \leq \sum_l \sum_{\tilde{Q} \in B_l} \left\| \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_p^p.$$

Set

$$\Omega_l = \left\{ x \in \mathcal{X} : \left\{ \sum_k \sum_Q |D_k(f)(y_Q)|^2 \chi_Q(x) \right\}^{1/2} > 2^l \right\}$$

and

$$B_l = \left\{ Q \text{ is a dyadic cube in } \mathcal{X}: \mu(Q \cap \Omega_l) > \frac{1}{2}\mu(Q) \text{ and } \mu(Q \cap \Omega_{l+1}) \leq \frac{1}{2}\mu(Q) \right\},$$

using the Hölder inequality yields

$$\begin{aligned} & \sum_l \sum_{\tilde{Q} \in B_l} \left\| \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_p^p \\ & \leq C \mu(\tilde{Q})^{1-p/2} \left(\left\| \sum_l \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_2^2 \right)^{p/2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \sum_l \sum_{\tilde{Q} \in B_l} \left\| \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_p^p \\ & \leq C \sum_l \sum_{\tilde{Q} \in B_l} \mu(\tilde{Q})^{1-p/2} \left(\left\| \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_2^2 \right)^{p/2} \\ & \leq C \sum_l \left(\sum_{\tilde{Q} \in B_l} \mu(\tilde{Q}) \right)^{1-p/2} \left(\left\| \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_2^2 \right)^{p/2}. \end{aligned}$$

We claim that

$$\left\| \sum_{Q \subset \tilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_2^2 \leq C 2^{2l} \mu(\tilde{\Omega}_l),$$

which implies

$$\| \pi_b^* f \|_p^p \leq C 2^{2l} \mu(\Omega_l) \leq C \left\| \sum_k \sum_Q |D_k(f)(y_Q)|^2 \chi_Q(\cdot) \right\|_p^{1/2} \Bigg|_p^p \leq C \|f\|_{H^p}^p.$$

To show the claim, we use the duality argument to get

$$\begin{aligned} & \left\| \sum_{Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \right\|_2 \\ & \quad \times \sup_{\|h\|_2 \leq 1} \left| \left\langle \sum_{Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q), h \right\rangle \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\|h\|_2 \leq 1} \sum_{Q \in B_l} \mu(Q) S_k(h)(y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q) \\
&\leq \sup_{\|h\|_2 \leq 1} \left(\sum_{Q \in B_l} |\mu(Q) S_k(h)(y_Q)|^2 |\overline{D}_k(b)(y_Q)|^2 \right)^{1/2} \left(\sum_{Q \in B_l} \mu(Q) |D_k(f)(y_Q)|^2 \right)^{1/2} \\
&\leq C \left(\sum_{Q \in B_l} \mu(Q) |D_k(f)(y_Q)|^2 \right)^{1/2},
\end{aligned}$$

where the last inequality follows from the fact that $b \in \text{BMO}(\mathcal{X})$ and from the Carleson measure estimate. To complete the proof of the claim, we have

$$\begin{aligned}
C2^{2l} \mu(\tilde{\Omega}) &\geq \int_{\tilde{\Omega}_l \setminus \Omega_{l+1}} \left\{ \sum_k \sum_Q |D_k(f)(y_Q)|^2 \chi_Q(x) \right\} d\mu(x) \\
&\geq \frac{1}{2} \sum_{Q \in B_l} \mu(Q) |D_k(f)(y_Q)|^2.
\end{aligned}$$

This finishes the proof of Lemma 7. □

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