Czechoslovak Mathematical Journal, 66 (141) (2016), 169–191

SOME ESTIMATES FOR COMMUTATORS OF RIESZ TRANSFORM ASSOCIATED WITH SCHRÖDINGER TYPE OPERATORS

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(Received February 15, 2015)

Abstract. Let $\mathcal{L}_1 = -\Delta + V$ be a Schrödinger operator and let $\mathcal{L}_2 = (-\Delta)^2 + V^2$ be a Schrödinger type operator on $\mathbb{R}^n$ ($n \geqslant 5$), where $V \neq 0$ is a nonnegative potential belonging to certain reverse Hölder class B_s for $s \geqslant n/2$. The Hardy type space $H^1_{\mathcal{L}_2}$ is defined in terms of the maximal function with respect to the semigroup $\{\mathrm{e}^{-t\mathcal{L}_2}\}$ and it is identical to the Hardy space $H^1_{\mathcal{L}_1}$ established by Dziubański and Zienkiewicz. In this article, we prove the L^p-boundedness of the commutator $\mathcal{R}_b = b\mathcal{R}f - \mathcal{R}(bf)$ generated by the Riesz transform $\mathcal{R} = \nabla^2\mathcal{L}_2^{-1/2}$, where $b \in \mathrm{BMO}_\theta(\varrho)$, which is larger than the space $\mathrm{BMO}(\mathbb{R}^n)$. Moreover, we prove that $\mathcal{R}_b$ is bounded from the Hardy space $H^1_{\mathcal{L}_2}(\mathbb{R}^n)$ into weak $L^1_{\mathrm{weak}}(\mathbb{R}^n)$.

Keywords: commutator; Hardy space; reverse Hölder inequality; Riesz transform; Schrödinger operator; Schrödinger type operator

MSC 2010: 42B35, 35J10, 42B30, 42B20

1. Introduction

The Schrödinger type operator is denoted by

$$\mathcal{L}_k = (-\Delta)^k + V^k \quad \text{on } \mathbb{R}^n,\ n > 2k,$$

where $k \geqslant 1$ is a positive integer and $V(x)$ belongs to the reverse Hölder class B_s for $s \geqslant n/2$. When $k = 1$, $\mathcal{L}_1$ is exactly the Schrödinger operator. In this paper, we focus on the operator $\mathcal{L}_2$, which has been studied in [19], [18], [3], [12], [11] and [14].

The research has been supported by the National Natural Science Foundation of China (No. 10901018, 11471018), the Fundamental Research Funds for the Central Universities (No. FRF-TP-14-005C1), Program for New Century Excellent Talents in University and the Beijing Natural Science Foundation under Grant (No. 1142005).

Let T be a bounded operator on $L^p(\mathbb{R}^n)$ for some p with $p \in (1,\infty)$. In this paper, we consider the commutator operator

$$T_b(f)(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathbb{R}^n,$$

where b belongs to a function space, which needs not be the classical BMO space. See [4], [8] for the investigation of the commutator operator T_b on Euclidean spaces $\mathbb{R}^n$ and [2], [5] on spaces of homogeneous type.

In recent years, many scholars have investigated the singular integral operators related to Schrödinger operators $\mathcal{L}_1$ and their commutators, see, for example, [17], [9], [7], [12], [1], [10], [16], [13], [15] and their references. Especially, when $b \in \mathrm{BMO}$, Guo, Li and Peng [7] investigated the boundedness of the commutators of the Riesz transform $\nabla \mathcal{L}_1^{-1/2}$. Furthermore, Bongioanni, Harboure and Salinas investigated the same problem when b belongs to a larger space than the BMO in [1]. In this paper we denote the Riesz transform associated with the Schrödinger type operator $\mathcal{L}_2$ by $\mathcal{R} = \nabla^2 \mathcal{L}_2^{-1/2}$. A natural problem is to study the boundedness of the commutator $\mathcal{R}_b$, where $b \in \mathrm{BMO}_\theta(\varrho)$. The aim of our paper is to investigate this problem.

A nonnegative locally L^s-integrable function V $(1 < s < \infty)$ is said to belong to B_s if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^s \,\mathrm{d}x\right)^{1/s} \leqslant C\left(\frac{1}{|B|}\int_B V \,\mathrm{d}x\right) \tag{1.1}$$

holds for every ball B in $\mathbb{R}^n$. Obviously, $B_{s_2} \subset B_{s_1}$, if $s_2 > s_1$. But it is important that the B_s class has the property of "self improvement", that is, if $V \in B_s$, then $V \in B_{s+\varepsilon}$ for some $\varepsilon > 0$.

We assume the potential $V \in B_{q_0}$ for $q_0 \geqslant n/2$ throughout the paper. We introduce the auxiliary function $\varrho(x,V) = \varrho(x)$ defined by

$$\varrho(x) = \frac{1}{m(x,V)} = \sup_{r>0}\left\{r\colon \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,\mathrm{d}y \leqslant 1\right\}, \quad x \in \mathbb{R}^n.$$

It is known that $0 < \varrho(x) < \infty$ for any $x \in \mathbb{R}^n$ (from Lemma 2.2 in Section 2).

In order to study the endpoint estimates of the commutator $\mathcal{R}_b$, we recall some basic facts for the Hardy spaces associated with the Schrödinger type operator.

The Schrödinger operator $\mathcal{L}_1 = -\Delta + V$ generates a (C_0) semigroup $\{\mathrm{e}^{-t\mathcal{L}_1}\}_{t>0}$. The maximal function with respect to the semigroup $\{\mathrm{e}^{-t\mathcal{L}_1}\}_{t>0}$ is given by

$$M^{\mathcal{L}_1} f(x) = \sup_{t>0} |\mathrm{e}^{-t\mathcal{L}_1} f(x)|.$$

The Hardy space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ associated with the Schrödinger operator $\mathcal{L}_1$ is defined as follows in terms of the maximal function mentioned above (cf. [6]).

Definition 1.1. A function $f \in L^1(\mathbb{R}^n)$ is said to be in $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ if the semigroup maximal function $M^{\mathcal{L}_1} f$ belongs to $L^1(\mathbb{R}^n)$. The norm of such a function is defined by

$$\|f\|_{H^1_{\mathcal{L}_1}} = \|M^{\mathcal{L}_1} f\|_{L^1}.$$

Correspondingly, the Schrödinger type operator $\mathcal{L}_2 = (-\Delta)^2 + V^2$ also generates a (C_0) semigroup $\{\mathrm{e}^{-t\mathcal{L}_2}\}_{t>0}$ (cf. [12]). The maximal function with respect to the semigroup $\{\mathrm{e}^{-t\mathcal{L}_2}\}_{t>0}$ is given by

$$M^{\mathcal{L}_2} f(x) = \sup_{t>0} |\mathrm{e}^{-t\mathcal{L}_2} f(x)|.$$

The Hardy space $H^1_{\mathcal{L}_2}(\mathbb{R}^n)$ associated with the Schrödinger operator $\mathcal{L}_2$ is defined as follows.

Definition 1.2. A function $f \in L^1(\mathbb{R}^n)$ is said to be in $H^1_{\mathcal{L}_2}(\mathbb{R}^n)$ if the semigroup maximal function $M^{\mathcal{L}_2} f$ belongs to $L^1(\mathbb{R}^n)$. The norm of such a function is defined by

$$\|f\|_{H^1_{\mathcal{L}_2}} = \|M^{\mathcal{L}_2} f\|_{L^1}.$$

Theorem 1.1 in [3] implies that $H^1_{\mathcal{L}_2}(\mathbb{R}^n) = H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ with equivalent norms.

Definition 1.3. Let $1 < q \leqslant \infty$. A measurable function a is called a $(1,q)_\varrho$-atom associated to the ball $B(x,r)$ if $r < \varrho(x)$ and the following conditions hold:

(1) $\operatorname{supp} a \subset B(x,r)$ for some $x \in \mathbb{R}^n$ and $r > 0$,
(2) $\|a\|_{L^q(\mathbb{R}^n)} \leqslant |B(x,r)|^{1/q-1}$,
(3) when $r < \varrho(x)/4$, $\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0$.

The space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ admits the following atomic decompositions.

Proposition 1.1. *Let $f \in L^1(\mathbb{R}^n)$. Then $f \in H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum\limits_j \lambda_j a_j$, where a_j are $(1,q)_\varrho$-atoms and $\sum\limits_j |\lambda_j| < \infty$. Moreover,*

$$\|f\|_{H^1_{\mathcal{L}_1}} \sim \inf\Big\{\sum_j |\lambda_j|\Big\},$$

where the infimum is taken over all atomic decompositions of f into $H^1_{\mathcal{L}_1}$-atoms.

Moreover, the function b in this paper belongs to the new BMO space $\mathrm{BMO}_\theta(\varrho)$, which is defined as follows (cf. Definition 1.4). It follows from [1] that the classical BMO space is a subspace of $\mathrm{BMO}_\theta(\varrho)$.

Definition 1.4. The class $\mathrm{BMO}_\theta(\varrho)$ consists of the locally integrable functions b which satisfy

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y)-b_B|\,\mathrm{d}y \leqslant C\Big(1+\frac{r}{\varrho(x)}\Big)^\theta, \tag{1.2}$$

for all $x\in\mathbb{R}^n$ and $r>0$, where $\theta>0$ and $b_B=(1/|B|)\int_B b(y)\,\mathrm{d}y$.

A norm for $b\in\mathrm{BMO}_\theta(\varrho)$, denoted by $[b]_\theta$, is given by the infimum of the constants satisfying (1.2), after identifying functions that differ upon a constant. If we let $\theta=0$ in (1.2), then $\mathrm{BMO}_\theta(\varrho)$ is exactly the John-Nirenberg space BMO. Denote $\mathrm{BMO}_\infty(\varrho)=\bigcup\limits_{\theta>0}\mathrm{BMO}_\theta(\varrho)$.

Theorem 1.1. *Let $V\in B_{q_0}$ with $q_0\geqslant n/2$, $b\in\mathrm{BMO}_\infty(\varrho)$ and p_0 such that $1/p_0=2/q_0-2/n$.*

(i) *If $1<p<p_0$, then*

$$\|\mathcal{R}_b f\|_p\leqslant C_b\|f\|_p,$$

for all $f\in L^p$.

(ii) *If $p_0'<p<\infty$ with $p_0'=p_0/(p_0-1)$, then*

$$\|\mathcal{R}_b^* f\|_p\leqslant C_b\|f\|_p,$$

for all $f\in L^p$, where

$$\mathcal{R}_b^* f(x)=b(x)\mathcal{R}^*f(x)-\mathcal{R}^*(bf)(x),\ x\in\mathbb{R}^n.$$

Moreover, $C_b\leqslant C[b]_\theta$ whenever $b\in\mathrm{BMO}_\theta(\varrho)$.

Theorem 1.2. *Suppose $V\in B_{q_0}$ for some $q_0\geqslant n/2$. Let $b\in\mathrm{BMO}_\infty(\varrho)$. Then, for any $\lambda>0$,*

$$|\{x\in\mathbb{R}^n\colon\ |\mathcal{R}_b(f)(x)|>\lambda\}|\leqslant C\frac{[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}_1}(\mathbb{R}^n)},\quad f\in H^1_{\mathcal{L}_1}(\mathbb{R}^n),$$

where the constant $C>0$ is independent of f. Namely, the commutator $\mathcal{R}_b$ is bounded from $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ into $L^1_{\mathrm{weak}}(\mathbb{R}^n)$.

This paper is organized as follows. In this section, we set some notations and state our main results. In Section 2, we collect some lemmas which we need. In Section 3, we give the estimates of the kernels of $\mathcal{R}$ and $\mathcal{R}^*$. In Section 4, we prove the main results of this paper.

Throughout this paper, unless otherwise indicated, we will use C to denote a positive constant, which is not necessarily the same at each occurrence and even can be different in the same line, and which depends at most on the dimension n and the constant in (1.1). By $A \sim B$ we mean that there exists some constant $C > 0$ such that $1/C \leqslant A/B \leqslant C$.

2. Some lemmas

In this section, we collect some known results about the auxiliary function $\varrho(x)$, which have been proved in [17].

Lemma 2.1. *$V \in B_{q_0}$, $q_0 \geqslant n/2$, is a doubling measure, i.e., there exists a constant $C > 0$ such that*

$$\int_{B(x,2r)} V(y)\,\mathrm{d}y \leqslant C \int_{B(x,r)} V(y)\,\mathrm{d}y.$$

Especially, there exist constants $\mu \geqslant 1$ and C such that

$$\int_{B(x,tr)} V(y)\,\mathrm{d}y \leqslant Ct^{n\mu} \int_{B(x,r)} V(y)\,\mathrm{d}y,$$

holds for every ball $B(x,r)$ and $t > 1$.

Lemma 2.2. *There exist constants C, $k_0 > 0$ such that*

$$\frac{1}{C}\Big(1 + \frac{|x-y|}{\varrho(x)}\Big)^{-k_0} \leqslant \frac{\varrho(y)}{\varrho(x)} \leqslant C\Big(1 + \frac{|x-y|}{\varrho(x)}\Big)^{k_0/(k_0+1)}.$$

In particular, $\varrho(y) \sim \varrho(x)$ if $|x-y| < C\varrho(x)$.

Lemma 2.3.

$$\int_{B(x,R)} \frac{V(y)\,\mathrm{d}y}{|x-y|^{n-2}} \leqslant \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)\,\mathrm{d}y.$$

Moreover, if $V \in B_n$, then there exists $C > 0$ such that

$$\int_{B(x,R)} \frac{V(y)\,\mathrm{d}y}{|x-y|^{n-1}} \leqslant \frac{C}{R^{n-1}} \int_{B(x,R)} V(y)\,\mathrm{d}y.$$

Lemma 2.4. (1) *For* $0 < r < R < \infty$,

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,\mathrm{d}y \leqslant C\Big(\frac{r}{R}\Big)^{2-n/q_0}\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,\mathrm{d}y$$

and

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,\mathrm{d}y \sim 1 \quad \text{if and only if} \quad r\sim\varrho(x);$$

(2) *There exist* $C>0$ *and* $k_0'>0$ *such that*

$$\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,\mathrm{d}y \leqslant C\Big(1+\frac{R}{\varrho(x)}\Big)^{k_0'}.$$

We also need the following propositions (cf. [6]).

Proposition 2.1. *There exists a sequence of points* $\{x_k\}_{k=1}^{\infty}$ *in* $\mathbb{R}^n$ *such that the family of critical balls* $Q_k = B(x_k,\varrho(x_k))$, $k\geqslant 1$, *satisfies*

(i) $\bigcup\limits_k Q_k = \mathbb{R}^n$,

(ii) *there exists* N *such that for every* $k\in\mathbb{N}$, $|\{j\colon\ 4Q_j\cap 4Q_k\neq\emptyset\}|\leqslant N$.

Proposition 2.2 ([1], Proposition 3)**.** *Let* $\theta>0$ *and* $1\leqslant s<\infty$. *If* $b\in \mathrm{BMO}_\theta(\varrho)$, *then there exists a constant* $C>0$ *such that*

$$\Big(\frac{1}{|B|}\int_B |b-b_B|^s\Big)^{1/s} \leqslant C[b]_\theta\Big(1+\frac{r}{\varrho(x)}\Big)^{\theta'}$$

for all $B=B(x,r)$, *with* $x\in\mathbb{R}^n$ *and* $r>0$, *where* $\theta'=(k_0+1)\theta$ *and* k_0 *is the constant appearing in Lemma* 2.2.

Lemma 2.5 ([1], Lemma 1)**.** *Let* $b\in\mathrm{BMO}_\theta(\varrho)$, $B=B(x_0,r)$ *and* $s\geqslant 1$. *Then there exists a constant* $C>0$ *such that*

$$\Big(\frac{1}{|2^kB|}\int_{2^kB} |b-b_B|^s\Big)^{1/s} \leqslant C[b]_\theta k\Big(1+\frac{2^kr}{\varrho(x_0)}\Big)^{\theta'}$$

for all $k\in\mathbb{N}$, *with* θ' *as in Proposition* 2.2.

Given $\alpha > 0$, we define the following maximal functions for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$M_{\varrho,\alpha} g(x) = \sup_{x\in B\in\mathcal{B}_{\varrho,\alpha}} \frac{1}{|B|}\int_B |g|,$$
$$M^{\sharp}_{\varrho,\alpha} g(x) = \sup_{x\in B\in\mathcal{B}_{\varrho,\alpha}} \frac{1}{|B|}\int_B |g-g_B|,$$

where $\mathcal{B}_{\varrho,\alpha} = \{B(y,r)\colon\ y\in\mathbb{R}^n, \text{ and } r\leqslant \alpha\varrho(y)\}$.

Lemma 2.6 ([1], Lemma 2)**.** *For $1<p<\infty$, there exist β and γ such that if $\{Q_k\}_{k=1}^{\infty}$ is a sequence of balls as in Proposition* 2.1, *then there exists a constant $C>0$ such that*

$$\int_{\mathbb{R}^n} |M_{\varrho,\beta} g(x)|^p \leqslant C\biggl(\int_{\mathbb{R}^n} |M^{\sharp}_{\varrho,\gamma} g(x)|^p + \sum_k |Q_k|\Bigl(\frac{1}{|Q_k|}\int_{2Q_k}|g|\Bigr)^p\biggr)$$

for all $g\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

3. Estimates for the kernels of $\mathcal{R}$ and $\mathcal{R}^*$

In this section, we give some estimates for the kernels of $\mathcal{R}$ and $\mathcal{R}^*$. These two operators have been investigated in [19], [18], [12], [3]. We denote by $\mathcal{K}$ and $\mathcal{K}^*$ the kernels of $\mathcal{R}$ and $\mathcal{R}^*$, respectively.

Lemma 3.1. *Suppose $V\in B_{q_0}$, $q_0>n/2$. Assume that $(-\Delta)^2u+(V(x)^2+\lambda)u=0$ in $B(x_0,2R)$ for some $x_0\in\mathbb{R}^n$. If $n/2<q_0<2n/(4-j)$, there exists a k_0' such that*

$$\biggl(\int_{B(x_0,R)} |\nabla^j u|^t \,\mathrm{d}x\biggr)^{1/t} \leqslant CR^{(n/q_0)-2}\{1+Rm(x_0,V)\}^{k_0'} \sup_{B(x_0,2R)}|u|,$$

where $j=1,2$ and $1/t = 2/q_0-(4-j)/n$.

P r o o f. We prove this lemma by a method similar to the one used in the proof of Lemma 7 in [18]. Let $\varphi\in C_0^{\infty}(B(x_0,R))$ such that $\varphi\equiv 1$ on $B(x_0,3R/4)$, $0<\varphi\leqslant 1$. Also, φ satisfies $|\nabla^j\varphi|\leqslant CR^{-j}$. Denote by $\Gamma_0(x,y,\lambda)$ the fundamental solution of $(-\Delta)^2+\lambda$. Following Theorem 5 in [12], we have

$$|\nabla^l\Gamma_0(x,y,\lambda)|\leqslant \frac{C_l}{|x-y|^{n-4+l}}, \tag{3.1}$$

where $l=1,2,3$.

Note that

$$
\begin{aligned}
u(x)\varphi(x) &= \int_{\mathbb{R}^n} \Gamma_0(x,y,\lambda)(-\Delta)^2(u\varphi)(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \Gamma_0(x,y,\lambda)\big(- V^2(y)u(y)\varphi(y) + 4\Delta\nabla u(y)\cdot\nabla\varphi(y) \\
&\qquad + 2\Delta(u(y)\Delta\varphi(y)) - 4\nabla^2 u(y)\cdot\nabla^2\varphi(y) - 4\nabla u(y)\cdot\nabla(\Delta\varphi(y)) \\
&\qquad - u(y)\Delta^2\varphi(y)\big)\,\mathrm{d}y.
\end{aligned}
$$

Then by integration by parts and (3.1) we get, for $x \in B(x_0, R/2)$,

$$
\begin{aligned}
|\nabla^j u(x)| &\leqslant C\int_{B(x_0,R)} \frac{V^2(y)|u(y)||\varphi(y)|}{|x-y|^{n-4+j}}\,\mathrm{d}y + \frac{C}{R^{n+j}}\int_{B(x_0,R)} |u(y)|\,\mathrm{d}y \\
&\leqslant C\sup_{B(x_0,R)}|u(y)|\bigg(\int_{B(x_0,R)} \frac{V^2(y)|\varphi(y)|}{|x-y|^{n-4+j}}\,\mathrm{d}y + \frac{1}{R^j}\bigg).
\end{aligned}
$$

Then by the theorem on fractional integration, (1.1) and Lemma 2.4, we have

$$
\begin{aligned}
\bigg(\int_{B(x_0,R/2)} |\nabla^j u(x)|^t\,\mathrm{d}x\bigg)^{1/t} &\leqslant C\sup_{B(x_0,R)}|u(x)|\bigg(\bigg(\int_{B(x_0,R)} V(x)^q\,\mathrm{d}x\bigg)^{2/q} + R^{2n/q-4}\bigg) \\
&\leqslant CR^{2n/q-4}\sup_{B(x_0,R)}|u(x)|\bigg(\frac{1}{R^{n-2}}\int_{B(x_0,R)} V(x)\,\mathrm{d}x + 1\bigg) \\
&\leqslant CR^{2n/q-4}\big(1+Rm(x_0,V)\big)^{k_0'}\sup_{B(x_0,R)}|u(x)|,
\end{aligned}
$$

where $1/t = 2/q_0 - (4-j)/n$. □

Lemma 3.2. *If $V \in B_{q_0}$ for $q_0 > n/2$, then we have:*

(i) *For every N, there exists a positive constant C_N such that*

(3.2)
$$
|\mathcal{K}^*(x,z)| \leqslant \frac{C_N\big(1+|x-z|/\varrho(x)\big)^{-N}}{|x-z|^{n-2}}\bigg(\int_{B(z,|x-z|/4)} \frac{V^2(u)}{|u-z|^{n-2}}\,\mathrm{d}u + \frac{1}{|x-z|^2}\bigg).
$$

Moreover, the last inequality also holds with $\varrho(x)$ replaced by $\varrho(z)$:

(ii) *For every N and $0 < \delta < \min\{1, 2-n/q_0\}$, there exists a positive constant C_N such that*

$$
\begin{aligned}
|\mathcal{K}^*(x,z) - \mathcal{K}^*(y,z)| &\leqslant \frac{C_N|x-y|^\delta\big(1+|x-z|/\varrho(x)\big)^{-N}}{|x-z|^{n-2+\delta}} \\
&\quad\times\bigg(\int_{B(z,|x-z|)} \frac{V^2(u)}{|u-z|^{n-2}}\,\mathrm{d}u + \frac{1}{|x-z|^2}\bigg),
\end{aligned}
\tag{3.3}
$$

whenever $|x-y| < |x-z|/16$.

(iii) *When* $q_0 > n$, *the term involving* V *can be dropped from inequalities* (3.2) *and* (3.3).

Proof. (i) The proof can be found in the proof of Lemma 9 in [12].

(ii) Fix $x, z \in \mathbb{R}^n$ such that $|x-z| < \varrho(x)$ and let $R = |x-z|/8$, $1/t = 1/q_0 - 3/n$ and $\delta = 4 - 2n/q_0$.

By the functional calculus, we have

$$|\mathcal{K}^*(x,z) - \mathcal{K}^*(y,z)| \leqslant \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}|\nabla_z^2\Gamma(x,z,\lambda) - \nabla_z^2\Gamma(y,z,\lambda)|\,\mathrm{d}\lambda.$$

We need to estimate $|\nabla_z^2\Gamma(x,z,\lambda) - \nabla_z^2\Gamma(y,z,\lambda)|$ in advance. It follows from the Morrey embedding theorem and Lemma 3.1 that

$$\begin{aligned}
&|\nabla_z^2\Gamma(x,z,\lambda) - \nabla_z^2\Gamma(y,z,\lambda)| \\
&\qquad \leqslant C|x-y|^{1-n/t}\Big(\int_{B(x,R)} |\nabla_x\nabla_z^2\Gamma(u,z,\lambda)|^t\,\mathrm{d}u\Big)^{1/t} \\
&\qquad \leqslant C|x-y|^{1-n/t}R^{n/q-2}\{1+Rm(x,V)\}^{k_0'} \sup_{u\in B(x,2R)} |\nabla_z^2\Gamma(u,z,\lambda)|.
\end{aligned}$$

Since $\Gamma(u,z,\lambda) = \Gamma(z,u,\lambda)$, we have $\nabla_z^2\Gamma(u,z,\lambda) = \nabla_x^2\Gamma(z,u,\lambda)$. Hence, by the proof of Lemma 3.1 and Theorem 4 in [12],

$$\begin{aligned}
&\sup_{u\in B(x,2R)} |\nabla_z^2\Gamma(u,z,\lambda)| \\
&\quad \leqslant \sup_{u\in B(x,2R)} |\nabla_x^2\Gamma(z,u,\lambda)| \\
&\quad \leqslant \sup_{u\in B(x,2R)} \bigg\{\frac{C_N}{(1+\lambda^{1/2}|z-u|^2)^N\{1+|z-u|m(z,V)\}^N} \\
&\qquad \times \bigg\{\frac{1}{|z-u|^{n-4}}\int_{B(z,|z-u|/2)} \frac{V^2(\xi)}{|\xi-z|^{n-2}}\,\mathrm{d}\xi + \frac{1}{|z-u|^{n-2}}\bigg\}\bigg\} \\
&\quad \leqslant \frac{C_N}{(1+\lambda^{1/2}R^2)^N\{1+Rm(z,V)\}^N}\bigg\{\frac{1}{R^{n-4}}\int_{B(z,5R)} \frac{V^2(\xi)}{|\xi-z|^{n-2}}\,\mathrm{d}\xi + \frac{1}{R^{n-2}}\bigg\},
\end{aligned}$$

where we use the fact that $6R \leqslant |z-u| \leqslant 10R$ and $|\xi - z| \sim |x-\xi|$.

According to Lemma 2.2,

$$\begin{aligned}
\sup_{u\in B(x,2R)} |\nabla_z^2\Gamma(u,z,\lambda)| &\leqslant \frac{C_{N'}}{(1+\lambda^{1/2}R^2)^N\{1+Rm(x,V)\}^{N'}} \\
&\qquad \times \bigg\{\frac{1}{R^{n-4}}\int_{B(z,5R)} \frac{V^2(\xi)}{|\xi-z|^{n-2}}\,\mathrm{d}\xi + \frac{1}{R^{n-2}}\bigg\}.
\end{aligned}$$

Then we have

$$
\begin{aligned}
&|\nabla_z^2\Gamma(x,z,\lambda)-\nabla_z^2\Gamma(y,z,\lambda)| \\
&\quad\leqslant C|x-y|^{1-n/t}\biggl(\int_{B(x,R)}|\nabla_y\nabla_z^2\Gamma(u,z,\lambda)|^t\,\mathrm{d}u\biggr)^{1/t} \\
&\quad\leqslant C|x-y|^{1-d/t}R^{d/q-2}\{1+Rm(x,V)\}^{k_0} \\
&\qquad\times\frac{C_{N'}}{(1+\lambda^{1/2}R^2)^{N'}\{1+Rm(x,V)\}^{N'}}\biggl\{\frac{1}{R^{n-4}}\int_{B(z,5R)}\frac{V^2(\xi)}{|\xi-z|^{n-2}}\,\mathrm{d}\xi+\frac{1}{R^{n-2}}\biggr\} \\
&\quad\leqslant\frac{C_{N''}R^{-\delta}|x-y|^{\delta}}{(1+\lambda^{1/2}R^2)^{N'}\{1+Rm(x,V)\}^{N''}}\biggl\{\frac{1}{R^{n-4}}\int_{B(z,5R)}\frac{V^2(\xi)}{|\xi-z|^{n-2}}\,\mathrm{d}\xi+\frac{1}{R^{n-2}}\biggr\}.
\end{aligned}
$$

For $k>2$,

$$\int_0^\infty\lambda^{-1/2}\{1+\lambda^{1/2}R^2\}^{-k}\,\mathrm{d}\lambda\leqslant\frac{C_k}{R^2}.$$

Then we obtain

$$
\begin{aligned}
|\mathcal{K}^*(x,z)-\mathcal{K}^*(y,z)|&\leqslant\biggl[\int_0^\infty\lambda^{-1/2}\{1+\lambda^{1/2}R^2\}^{-k}\,\mathrm{d}\lambda\biggr] \\
&\qquad\times\frac{C_{N''}R^{-\delta}|x-y|^{\delta}}{\{1+Rm(x,V)\}^{N''}}\biggl\{\frac{1}{R^{n-4}}\int_{B(z,5R)}\frac{V^2(\xi)}{|\xi-z|^{n-2}}\,\mathrm{d}\xi+\frac{1}{R^{n-2}}\biggr\} \\
&\leqslant\frac{C|x-y|^{\delta}}{R^{n-2+\delta}\{1+Rm(x,V)\}^{N''}}\biggl\{\int_{B(z,5R)}\frac{V^2(\xi)}{|\xi-z|}\,\mathrm{d}\xi+\frac{1}{R^2}\biggr\},
\end{aligned}
$$

which concludes the proof of the lemma. □

Furthermore, we immediately have:

Lemma 3.3. *If $V\in B_{q_0}$ for $q_0>n/2$, then we have:*

(i) *For every N, there exists a positive constant C_N such that*

(3.4)
$$|\mathcal{K}(x,z)|\leqslant\frac{C_N\bigl(1+|x-z|/\varrho(x)\bigr)^{-N}}{|x-z|^{n-2}}\biggl(\int_{B(z,|x-z|/4)}\frac{V^2(u)}{|u-z|^{n-2}}\,\mathrm{d}u+\frac{1}{|x-z|^2}\biggr).$$

Moreover, the last inequality also holds with $\varrho(x)$ replaced by $\varrho(z)$.

(ii) *For every N and $0<\delta<\min\{1,2-n/q_0\}$, there exists a positive constant C_N such that*

$$
\begin{aligned}
|\mathcal{K}(x,y)-\mathcal{K}(x,z)|&\leqslant\frac{C_N|z-y|^{\delta}\bigl(1+|x-z|/\varrho(x)\bigr)^{-N}}{|x-z|^{n-2+\delta}} \\
&\qquad\times\biggl(\int_{B(x,|x-z|)}\frac{V^2(u)}{|u-z|^{n-2}}\,\mathrm{d}u+\frac{1}{|x-z|^2}\biggr),
\end{aligned}\tag{3.5}
$$

whenever $|z-y|<|x-z|/16$.

(iii) *When $q_0 > n$, the term involving V can be dropped from inequalities* (3.4) *and* (3.5).

4. The proof of our main results

Before we prove our main results, we need to give some necessary lemmas. The method we adopted is similar to the one in [1].

Lemma 4.1. *Let $V(x) \in B_{q_0}$ for $q_0 > n/2$, $1/p_0 = 2/q_0 - 2/n$, and $b \in \mathrm{BMO}_\theta(\varrho)$. Then, for any $p_0' \leqslant s < \infty$, there exists a constant $C > 0$ such that*

$$\frac{1}{|Q|}\int_Q |\mathcal{R}_b^* f| \leqslant C[b]_\theta \inf_{y\in Q} M_s f(y)$$

for all $f \in L^s_{\mathrm{loc}}(\mathbb{R}^n)$ and every ball $Q = B(x_0, \varrho(x_0))$. Additionally, if $q_0 > n$, the above estimate also holds for $\mathcal{R}$ instead of $\mathcal{R}^*$.

Proof. Let $f \in L^p(\mathbb{R}^n)$ and $Q = B(x_0, \varrho(x_0))$. First, we consider

$$\mathcal{R}_b^* f = (b - b_Q)\mathcal{R}^* f - \mathcal{R}^*(f(b - b_Q)) \tag{4.1}$$

and therefore, we need to deal with the average on Q for each term. By Hölder's inequality with $s > p_0'$ and Lemma 2.5,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(b-b_Q)\mathcal{R}^* f| &\leqslant \Big(\frac{1}{|Q|}\int_Q |(b-b_Q)|^{s'}\Big)^{1/s'}\Big(\frac{1}{|Q|}\int_Q |\mathcal{R}^* f|^s\Big)^{1/s}\\ &\leqslant C[b]_\theta\Big(\frac{1}{|Q|}\int_Q |\mathcal{R}^* f|^s\Big)^{1/s}.\end{aligned}$$

If we write $f = f_1 + f_2$ with $f_1 = f_{\chi_{2Q}}$, then by using the fact that $\mathcal{R}^*$ is bounded on $L^s(\mathbb{R}^n)$ with $p_0' \leqslant s$, we have

$$\Big(\frac{1}{|Q|}\int_Q |\mathcal{R}^* f_1|^s\Big)^{1/s} \leqslant C\Big(\frac{1}{|Q|}\int_{2Q} |f|^s\Big)^{1/s} \leqslant C \inf_{y\in Q} M_s f(y).$$

Now, for $x \in Q$ and using (3.2) in Lemma 3.2, we have

$$|\mathcal{R}^* f_2| = \Big|\int_{|x_0 - z| > 2\varrho(x_0)} \mathcal{K}^*(x,z) f(z)\,\mathrm{d}z\Big| \leqslant C(I_1(x) + I_2(x)),$$

where

$$I_1(x) = \int_{|x_0 - z| > 2\varrho(x_0)} \frac{|f(z)|}{|x-z|^n \big(1 + |x-z|/\varrho(x)\big)^N} \, \mathrm{d}z$$

and

$$I_2(x) = \int_{|x_0 - z| > 2\varrho(x_0)} \frac{|f(z)|}{|x-z|^{n-2} \big(1 + |x-z|/\varrho(x)\big)^N} \int_{B(z,|x-z|/4)} \frac{V^2(u)}{|u-z|^{n-2}} \, \mathrm{d}u \, \mathrm{d}z.$$

To deal with $I_1(x)$, noting that $\varrho(x) \sim \varrho(x_0)$ and $|x-z| \sim |x_0 - z|$, we split into annuli to obtain

$$\begin{aligned} I_1(x) &= \int_{|x_0 - z| > 2\varrho(x_0)} \frac{|f(z)|}{|x-z|^n \big(1 + |x-z|/\varrho(x)\big)^N} \, \mathrm{d}z \\ &\leqslant \sum_{k \geqslant 1} \frac{2^{-Nk}}{(2^k \varrho(x_0))^n} \int_{|x_0 - z| < 2^k \varrho(x_0)} |f(z)| \, \mathrm{d}z \\ &\leqslant C \inf_{y \in Q} Mf(y). \end{aligned}$$

For $I_2(x)$, we assume $n/2 < q_0 < n$ because of (iii) in Lemma 3.2. Then, since $x \in Q$,

$$\begin{aligned} I_2(x) &= \int_{|x_0 - z| > 2\varrho(x_0)} \frac{|f(z)|}{|x-z|^{n-2} \big(1 + |x-z|/\varrho(x)\big)^N} \int_{B(z,|x-z|/4)} \frac{V^2(u)}{|u-z|^{n-2}} \, \mathrm{d}u \, \mathrm{d}z \\ &\leqslant C \int_{|x_0 - z| > 2\varrho(x_0)} \frac{|f(z)|}{|x_0-z|^{n-2} \big(1 + |x_0-z|/\varrho(x)\big)^N} \int_{B(z,4|x_0-z|)} \frac{V^2(u)}{|u-z|^{n-2}} \, \mathrm{d}u \, \mathrm{d}z \\ &\leqslant C \sum_{k \geqslant 1} \frac{2^{-Nk}}{(2^k \varrho(x_0))^{n-2}} \int_{|x_0 - z| < 2^{k+1} \varrho(x_0)} |f(z)| \int_{B(z,2^{k+3}|x_0-z|)} \frac{V^2(u)}{|u-z|^{n-2}} \, \mathrm{d}u \, \mathrm{d}z \\ &\leqslant C \sum_{k \geqslant 1} \frac{2^{-Nk}}{(2^k \varrho(x_0))^{n-2}} \int_{|x_0 - z| < 2^{k+1} \varrho(x_0)} |f| \mathcal{I}_2(V^2 \chi_{B(x_0, 2^k \varrho(x_0))}). \end{aligned}$$

Let $p_0' < s < d$. First, using Hölder's inequality and the boundedness of the fractional integral $\mathcal{I}_2$: $L^{s'} \to L^{q_0/2}$ with $1/s' = 2/q_0 - 2/n$, we have

$$\begin{aligned} &\int_{|x_0 - z| > 2^{k+1} \varrho(x_0)} |f| \mathcal{I}_2(V^2 \chi_{B(x_0, 2^k \varrho(x_0))}) \\ &\qquad \leqslant \|f \chi_{B(x_0, 2^k \varrho(x_0))}\|_s \|\mathcal{I}_2(V^2 \chi_{B(x_0, 2^k \varrho(x_0))})\|_{s'} \\ &\qquad \leqslant C \|f \chi_{B(x_0, 2^k \varrho(x_0))}\|_s \|V^2 \chi_{B(x_0, 2^k \varrho(x_0))}\|_{q_0/2}. \end{aligned}$$

For $V \in B_{q_0}$, we get

$$\begin{aligned} \|V^2 \chi_{B(x_0, 2^k \varrho(x_0))}\|_{q_0/2} &= \Big[\Big(\int_{B(x_0, 2^k \varrho(x_0))} V^{q_0} \Big)^{1/q_0} \Big]^2 \\ &\leqslant C \Big(\frac{1}{|B(x_0, 2^k \varrho(x_0))|} \Big)^2 |B(x_0, 2^k \varrho(x_0))|^{2/q_0} \Big(\int_{B(x_0, 2^k \varrho(x_0))} V \Big)^2 \end{aligned}$$

$$
\begin{aligned}
&= C\frac{1}{(2^k\varrho(x_0))^{2n}}(2^k\varrho(x_0))^{2n/q_0}\left(\int_{B(x_0,2^k\varrho(x_0))} V\right)^2\\
&\leqslant C(2^k\varrho(x_0))^{-2n+2n/q_0}\left(2^{kn\mu}\int_{B(x_0,\varrho(x_0))} V\right)^2\\
&\leqslant C2^{-2kn+2n/q_0+2kn\mu}\varrho(x_0)^{-2n+2n/q_0}\varrho(x_0)^{2(n-2)}\\
&= C2^{kn(2\mu-2+2/q_0)}\varrho(x_0)^{2n/q_0-4},
\end{aligned}
$$

where in the second last inequality we have used Lemma 2.1 and the definition of ϱ. Therefore,

$$
\begin{aligned}
(4.2)\quad I_2(x) &\leqslant C\sum_{k\geqslant 1}\frac{2^{-Nk}}{(2^k\varrho(x_0))^{n-2}}2^{kn(2\mu-2+2/q_0)}\varrho(x_0)^{2n/q_0-4}\|f\chi_{B(x_0,2^k\varrho(x_0))}\|_s\\
&= C\varrho(x_0)^{2-n+2n/q_0-4}\sum_{k\geqslant 1}2^{-Nk+kn(2\mu-2+2/q_0)}\|f\chi_{B(x_0,2^k\varrho(x_0))}\|_s\\
&= C\varrho(x_0)^{-n+2n/q_0-2}\sum_{k\geqslant 1}2^{-Nk+kn(2\mu-2+2/q_0)}\|f\chi_{B(x_0,2^k\varrho(x_0))}\|_s.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\|f\chi_{B(x_0,2^k\varrho(x_0))}\|_s &\leqslant C(2^k\varrho(x_0))^{n/s}\inf_{y\in Q}M_sf(y)\\
&= C(2^k)^{n/s}(\varrho(x_0))^{n/s}\inf_{y\in Q}M_sf(y)
\end{aligned}
$$

and using the fact that $n/s' = 2n/q_0 + 2$, we have

$$
\begin{aligned}
(4.3)\quad I_2(x) &\leqslant C\varrho(x_0)^{n/s-n+2n/q_0-2}\sum_{k\geqslant 1}2^{-Nk+kn(2\mu-2+2/q_0+1/s)}\inf_{y\in Q}M_sf(y)\\
&\leqslant C\inf_{y\in Q}M_sf(y),
\end{aligned}
$$

where we choose N large enough such that the above series converges.

To deal with the second term of (4.1), we also split $f = f_1 + f_2$. Choose $p_0' < \tilde{s} < s$ and denote $v = \tilde{s}s/(s-\tilde{s})$. Using the boundedness of $\mathcal{R}^*$ on $L^{\tilde{s}}(\mathbb{R}^n)$ (cf. [12]) and using Hölder's inequality, we get

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q|\mathcal{R}^*f_1(b-b_Q)| &\leqslant \left(\frac{1}{|Q|}\int_Q|\mathcal{R}^*f_1(b-b_Q)|^{\tilde{s}}\right)^{1/\tilde{s}} \leqslant C\left(\frac{1}{|Q|}\int_{2Q}|f(b-b_Q)|^{\tilde{s}}\right)^{1/\tilde{s}}\\
&\leqslant C\left(\frac{1}{|Q|}\int_{2Q}|f|^s\right)^{1/s}\left(\frac{1}{|Q|}\int_{2Q}|(b-b_Q)|^v\right)^{1/v}\\
&\leqslant C[b]_\theta\inf_{y\in Q}M_sf(y),
\end{aligned}
$$

where in the last inequality we have used Proposition 2.2.

For the remaining part we have to deal with

$$\tilde{I}_1(x) = \int_{|x_0-z|>2\varrho(x_0)} \frac{|f(z)(b-b_Q)|}{|x-z|^n\big(1+|x-z|/\varrho(x)\big)^N}\,\mathrm{d}z$$

and

$$\tilde{I}_2(x) = \int_{|x_0-z|>2\varrho(x_0)} \frac{|f(z)(b-b_Q)|}{|x-z|^{n-2}\big(1+|x-z|/\varrho(x)\big)^N}\int_{B(z,|x-z|/4)} \frac{V^2(u)}{|u-z|^{n-2}}\,\mathrm{d}u\,\mathrm{d}z.$$

We start by observing that for $1 \leqslant \tilde{s} < s$, $v = \tilde{s}s/(s-\tilde{s})$, and using Lemma 2.5, we obtain

$$\begin{aligned} \|f(b-b_Q)\chi_{B(x_0,2^k\varrho(x_0))}\|_{\tilde{s}} &\leqslant \|f\chi_{B(x_0,2^k\varrho(x_0))}\|_s\|(b-b_Q)\chi_{B(x_0,2^k\varrho(x_0))}\|_v \\ &\leqslant C(2^k\varrho(x_0))^{n/\tilde{s}}\inf_{y\in Q} M_s f(y)k2^{k\theta'}[b]_\theta. \end{aligned}\tag{4.4}$$

For $\tilde{I}_1(x)$, using (4.4) with $\tilde{s}=1$, we have

$$\begin{aligned} \tilde{I}_1(x) &\leqslant C\sum_{k\geqslant 1}\frac{2^{-Nk}}{(2^k\varrho(x_0))^n}\int_{|x_0-z|>2^k\varrho(x_0)} |b(z)-b_Q||f(z)|\,\mathrm{d}z \\ &\leqslant C[b]_\theta \inf_{y\in Q} M_s f(y)\sum_{k\geqslant 1} k2^{k(-N+\theta')} \\ &\leqslant C[b]_\theta \inf_{y\in Q} M_s f(y). \end{aligned}$$

To deal with $\tilde{I}_2(x)$, we proceed as in the estimate for $I_2(x)$ with $f(b-b_Q)$ instead of f and $\tilde{s}$ and $\tilde{q}$ instead of s and q_0, where $1/\tilde{s}' = 2/\tilde{q}-2/n$. Similar to (4.2) and using also (4.4), we have

$$\begin{aligned} \tilde{I}_2(x) &\leqslant C\varrho(x_0)^{-n+2n/\tilde{q}-2}\sum_{k\geqslant 1} 2^{-Nk+kn(2\mu-2+2/\tilde{q})}\|(b-b_Q)f\chi_{B(x_0,2^k\varrho(x_0))}\|_{\tilde{s}} \\ &\leqslant C[b]_\theta \inf_{y\in Q} M_s f(y)\sum_{k\geqslant 1} k2^{k(-N+\theta')+kn(2\mu-2+2/\tilde{q}+1/\tilde{s})} \leqslant C[b]_\theta \inf_{y\in Q} M_s f(y), \end{aligned}$$

where we choose N large enough to ensure that the above series converges. □

Remark 4.1. Similarly, we can conclude that the above lemma also holds if the critical ball Q is replaced by $2Q$.

Lemma 4.2. *Let $V(x) \in B_{q_0}$ for $q_0 > n/2$, and $b \in \mathrm{BMO}_\infty(\varrho)$. Then, for any $s > p_0'$ and $\gamma > 1$, there exists a positive constant C such that*

$$\int_{(2B)^c} |\mathcal{K}^*(x,z)-\mathcal{K}^*(y,z)||b(z)-b_B||f(z)|\,\mathrm{d}z \leqslant C[b]_\theta \inf_{y\in B} M_s f(y)\tag{4.5}$$

for all f and $x,y\in B = B(x_0,r)$. Additionally, if $q_0 > n$, the above estimate also holds for $\mathcal{K}$ instead of $\mathcal{K}^*$.

P r o o f. Denoting $Q = B(x_0, \gamma\varrho(x_0))$, noting that $\varrho(x) \sim \varrho(x_0)$ and $|x - z| \sim |x_0 - z|$, by (3.3), we need to deal with four terms

$$\begin{aligned}
I_1 &= r^{\delta} \int_{Q\setminus 2B} \frac{|f(z)||b(z) - b_B|}{|x_0 - z|^{n+\delta}} \,\mathrm{d}z,\\
I_2 &= r^{\delta} \varrho(x_0)^N \int_{Q^c} \frac{|f(z)||b(z) - b_B|}{|x_0 - z|^{n+\delta+N}} \,\mathrm{d}z,\\
I_3 &= r^{\delta} \int_{Q\setminus 2B} \frac{|f(z)||b(z) - b_B|}{|x_0 - z|^{n+\delta-2}} \,\mathrm{d}z \int_{B(x_0,4|x_0-z|)} \frac{V^2(u)}{|u - z|^{n-2}} \,\mathrm{d}u \,\mathrm{d}z,\\
I_4 &= r^{\delta} \varrho(x_0)^N \int_{Q^c} \frac{|f(z)||b(z) - b_B|}{|x_0 - z|^{n+\delta+N-2}} \int_{B(x_0,4|x_0-z|)} \frac{V^2(u)}{|u - z|^{n-2}} \,\mathrm{d}u \,\mathrm{d}z.
\end{aligned}$$

For I_1, by splitting into annuli, we have

$$I_1 \leqslant \frac{1}{r^n} \sum_{j=2}^{j_0} 2^{-j(n+\delta)} \int_{2^j B} |f||b - b_B|,$$

where j_0 is the least integer such that $2^{j_0} \geqslant \gamma\varrho(x_0)/r$.

By Hölder's inequality and Lemma 2.5 we obtain for $j \leqslant j_0$

$$\int_{2^j B} |f||b - b_B| \leqslant j[b]_{\theta} |2^j B| \inf_{y\in B} M_s f(y).$$

Then

$$I_1 \leqslant C[b]_{\theta} \inf_{y\in B} M_s f(y) \sum_{j=2}^{\infty} j 2^{-j\delta} \leqslant C[b]_{\theta} \inf_{y\in B} M_s f(y).$$

To deal with I_2, splitting into annuli, using Lemma 2.5 and choosing $N > \theta'$, we have

$$\begin{aligned}
I_2 &\leqslant C \frac{\varrho(x_0)^N}{r^{N+n}} \sum_{j=j_0-1}^{\infty} 2^{-j(n+\delta+N)} \int_{2^j B} |f||b - b_B|\\
&\leqslant C[b]_{\theta} \inf_{y\in B} M_s f(y) \Big(\frac{\varrho(x_0)}{r}\Big)^{N-\theta'} \sum_{j=j_0-1}^{\infty} j 2^{-j(\delta+N-\theta')}\\
&\leqslant C[b]_{\theta} \inf_{y\in B} M_s f(y) \sum_{j=j_0-1}^{\infty} j 2^{-j\delta} \leqslant C[b]_{\theta} \inf_{y\in B} M_s f(y).
\end{aligned}$$

Now we are in a position to consider the terms I_3 and I_4. Following (iii) in Lemma 3.2, we may assume $n/2 < q_0 < n$. Therefore,

$$
\begin{aligned}
I_3 &\leqslant Cr^{\delta} \sum_{j=2}^{j_0} \int_{2^j B} (2^j r)^{(-n-\delta+2)} |f||b-b_B| \mathcal{I}_2 (V^2_{\chi_{2^{j+2}B}}) \\
&= Cr^{2-n} \sum_{j=2}^{j_0} \int_{2^j B} (2^{-j(-n-\delta+2)}) |f||b-b_B| \mathcal{I}_2 (V^2_{\chi_{2^{j+2}B}}).
\end{aligned}
$$

For $p_0' < \tilde{s} < s$, $v = \tilde{s}s/(s-\tilde{s})$ and $2/q = 1/\tilde{s}' + 2/n$, by using Lemma 2.5 and $j < j_0$, we have

$$
\begin{aligned}
&\int_{2^j B} (2^j r)^{(-n-\delta+2)} |f||b-b_B| \mathcal{I}_2 (V^2_{\chi_{2^{j+2}B}}) \\
&\qquad \leqslant \|f\|_s \|(b-b_B)\chi_{2^j B}\|_v \|\mathcal{I}_2 (V^2_{\chi_{2^{j+2}B}})\|_{\tilde{s}'} \\
&\qquad \leqslant Cj|2^j B|^{1/\tilde{s}} [b]_\theta \inf_{y\in B} M_s f(y) \|(V^2_{\chi_{2^{j+2}B}})\|_{q/2}.
\end{aligned}
\tag{4.6}
$$

Note that $V \in B_q$ for $q > q_0$. From our assumption on $\tilde{s}$, we get

$$
\begin{aligned}
\|(V^2_{\chi_{2^{j+2}B}})\|_{q/2} &\leqslant C \frac{1}{|Q|^2} |Q|^{2/q} \Big(\int_Q V \Big)^2 \\
&\leqslant C(\varrho(x_0))^{-2n+2n/q} (\varrho(x_0))^{2n-4} \\
&= C(\varrho(x_0))^{-4+2n/q}
\end{aligned}
$$

for all $j \leqslant j_0$. Therefore, due to the fact that $2n/q = n/\tilde{s}' + 2 \Longleftrightarrow n/\tilde{s} = 2 + n - 2n/q$ and $4 - 2n/q > 0$, we have

$$
\begin{aligned}
I_3 &\leqslant C[b]_\theta \inf_{y\in B} M_s f(y) r^{2-n} \sum_{j=2}^{j_0} j|2^j B|^{1/\tilde{s}} 2^{-j(n-2+\delta)} \big(\varrho(x_0)\big)^{-4+2n/q} \\
&= C[b]_\theta \inf_{y\in B} M_s f(y) \frac{r^{2-n+n/\tilde{s}}}{\varrho(x_0)^{4-2n/q}} \sum_{j=2}^{j_0} j 2^{(jn/\tilde{s})} 2^{(-jn+2j-j\delta)} \\
&= C[b]_\theta \inf_{y\in B} M_s f(y) \Big(\frac{r}{\varrho(x_0)}\Big)^{4-2n/q} \sum_{j=2}^{j_0} j 2^{j(n/\tilde{s}-n+2)} 2^{-j\delta} \\
&= C[b]_\theta \inf_{y\in B} M_s f(y) \Big(\frac{r}{\varrho(x_0)}\Big)^{4-2n/q} 2^{j_0(4-2n/q)} \sum_{j=2}^{j_0} j 2^{-j\delta} \\
&\leqslant C[b]_\theta \inf_{y\in B} M_s f(y).
\end{aligned}
$$

Finally, for I_4 we have

$$\begin{aligned} I_4 &\leqslant C\frac{\varrho(x_0)^N}{r^{n-2+N}}\sum_{j=j_0-1}^{\infty}2^{-j(n-2+\delta+N)}\int_{2^jB}|f||b-b_B|\mathcal{I}_2(V^2_{\chi_{2^{j+2}B}}) \\ &\leqslant C[b]_\theta \inf_{y\in B} M_s f(y)\sum_{j=j_0-1}^{\infty} j2^{-j(n-2+\delta+N)}\frac{(2^jr)^{\theta'+n/\tilde{s}}}{\varrho(x_0)^{\theta'}}\|(V^2_{\chi_{2^{j+2}B}})\|_{q/2}, \end{aligned}$$

where in the last inequality we use (4.6) and $j > j_0$.

Moreover,

$$\begin{aligned} \|(V^2_{\chi_{2^{j+2}B}})\|_{q/2} &\leqslant C(2^jr)^{-2n+2n/q}\Big(\int_{2^jB}V\Big)^2 \\ &= C(2^jr)^{-2n+2n/q}\Big(\Big(\frac{2^jr}{\varrho(x_0)}\Big)^{n\mu}\int_Q V\Big)^2 \\ &= C(2^jr)^{-2n+2n/q}\Big(\frac{2^jr}{\varrho(x_0)}\Big)^{2n\mu}\Big(\int_Q V\Big)^2 \\ &\leqslant C(2^j)^{-2n+2n/q}r^{-2n+2n/q}r^{2n\mu}\frac{2^{2jn\mu}}{\varrho(x_0)^{2n\mu}}\varrho(x_0)^{2n-4} \\ &\leqslant C\frac{(2^{-2jn+2jn/q+2jn\mu})(r^{2n\mu-2n+2n/q})}{\varrho(x_0)^{2n\mu+4-2n}}. \end{aligned}$$

Then we have

$$\begin{aligned} I_4 &\leqslant C[b]_\theta \inf_{y\in B} M_s f(y)\frac{\varrho(x_0)^N}{\varrho(x_0)^{2n\mu+4+\theta'-2n}}\frac{r^{2n\mu-2n+2n/q+\theta'+n/\tilde{s}}}{r^{n-2+N}} \\ &\quad\times\sum_{j=j_0-1}^{\infty}\frac{j2^{j\theta'+nj/\tilde{s}-2jn+2jn/q+2jn\mu}}{2^{j(n-2+\delta+N)}} \\ &\leqslant C[b]_\theta \inf_{y\in B} M_s f(y)\Big(\frac{r}{\varrho(x_0)}\Big)^{4+2n\mu-2n+\theta'-N}\sum_{j=j_0-1}^{\infty} j2^{j(\theta'+n/\tilde{s}+2n/q-3n+2-\delta-N)} \\ &\leqslant C[b]_\theta \inf_{y\in B} M_s f(y), \end{aligned}$$

where in the last inequality we choose N large enough so that $4+2n\mu-2n+\theta'-N<0$.

For the case of $q_0 > n$, it is easier than for the case of $n/2 < q_0 < n$ to complete the proof. We omit the details. $\square$

Now we are in a position to give the proof of the main results.

Proof of Theorem 1.1. We will prove part (ii), and (i) follows by duality. We start with a function $f \in L^p(\mathbb{R}^n)$ with $p_0' < p < \infty$. Due to Lemma 4.1 we have

$\mathcal{R}_b^* f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. By using Lemma 2.6, Lemma 4.1 with $p_0' < s < p$ and Remark 4.1, we have

$$\begin{aligned}
\|\mathcal{R}_b^* f\|_p^p &\leqslant \int_{\mathbb{R}^n} |M_{\varrho,\beta}(\mathcal{R}_b^* f)|^p \\
&\leqslant C\int_{\mathbb{R}^n} |M^\sharp_{\varrho,\gamma}(\mathcal{R}_b^* f)|^p + C\sum_k |Q_k|\bigg(\frac{1}{|Q_k|}\int_{2Q_k} |\mathcal{R}_b^* f|\bigg)^p \\
&\leqslant C\int_{\mathbb{R}^n} |M^\sharp_{\varrho,\gamma}(\mathcal{R}_b^* f)|^p + C[b]_\theta^p \sum_k \int_{2Q_k} |M_s f|^p.
\end{aligned}$$

By the finite overlapping property given by Proposition 2.1 and the boundedness of M_s in $L^p(\mathbb{R}^n)$, the second term is controlled by $[b]_\theta^p \|f\|_p^p$. Thus, we need to consider the first term.

Our goal is to find a pointwise estimate of $M^\sharp_{\varrho,\gamma}(\mathcal{R}_b^* f)$. Let $x \in \mathbb{R}^n$ and $B = B(x_0, r)$, with $r < \gamma\varrho(x_0)$ such that $x \in B$. If $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$, then we write

$$\mathcal{R}_b^* f = (b - b_B)\mathcal{R}^* f - \mathcal{R}^*(f_1(b - b_B)) - \mathcal{R}^*(f_2(b - b_B)).$$

Therefore, we need to control the mean oscillation on B of each term, let us denote them $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

Let $s > p_0'$, applying Hölder's inequality and Proposition 2.2 we get

$$\begin{aligned}
\mathcal{O}_1 &\leqslant \frac{2}{|B|}\int_B |(b - b_B)\mathcal{R}^* f| \\
&\leqslant C\bigg(\frac{1}{|B|}\int_B |b - b_B|^{s'}\bigg)^{1/s'} \bigg(\frac{1}{|B|}\int_B |\mathcal{R}^* f|^s\bigg)^{1/s} \\
&\leqslant C[b]_\theta M_s \mathcal{R}^* f(x),
\end{aligned}$$

where $r/\varrho(x_0) < \gamma$.

To estimate $\mathcal{O}_2$, let $p_0' < \tilde{s} < s$ and $v = \tilde{s}s/(s - \tilde{s})$. Then,

$$\begin{aligned}
\mathcal{O}_2 &\leqslant \frac{2}{|B|}\int_B |\mathcal{R}^*\big((b - b_B)f_1\big)| \\
&\leqslant C\bigg(\frac{1}{|B|}\int_B |\mathcal{R}^*\big((b - b_B)f_1\big)|^{\tilde{s}}\bigg)^{1/\tilde{s}} \\
&\leqslant C\bigg(\frac{1}{|B|}\int_{2B} |(b - b_B)f|^{\tilde{s}}\bigg)^{1/\tilde{s}} \\
&\leqslant C\bigg(\frac{1}{|B|}\int_{2B} |b - b_B|^{v}\bigg)^{1/v} \bigg(\frac{1}{|B|}\int_{2B} |f|^s\bigg)^{1/s} \\
&\leqslant C[b]_\theta M_s f(x).
\end{aligned}$$

For $\mathcal{O}_3$ we see that

$$\mathcal{O}_3 \leqslant C\frac{1}{|B|^2}\int_B\int_B |\mathcal{R}^*\big(f_2(b-b_B)\big)(u) - \mathcal{R}^*\big(f_2(b-b_B)\big)(z)|\,\mathrm{d}u\,\mathrm{d}z$$

and the integral is clearly bounded by the left-hand side of (4.5). Therefore, Lemma 4.2 asserts

$$\mathcal{O}_3 \leqslant C[b]_\theta M_s f(x).$$

Therefore, we have proved that

$$|M^\sharp_{\varrho,\gamma}(\mathcal{R}^*_b f)| \leqslant C[b]_\theta(M_s\mathcal{R}^* f + M_s f).$$

Since $s < p$, we conclude the desired result. □

Proof of Theorem 1.2. For $f \in H^1_{\mathcal{L}_1}(\mathbb{R}^n)$, we can write $f = \sum\limits_{j=-\infty}^{\infty} \lambda_j a_j$, where each a_j is a $(1,q)_\varrho$-atom and $\sum\limits_{j=-\infty}^{\infty} |\lambda_j| \leqslant 2\|f\|_{H^1_{\mathcal{L}_1}}$. Suppose that $\operatorname{supp} a_j \subseteq B_j = B(x_j, r_j)$ with $r_j < \varrho(x_j)$. Write

$$\begin{aligned}
\mathcal{R}_b f(x) = &\sum_{j=-\infty}^{\infty} \lambda_j(b(x) - b_{B_j})\mathcal{R}a_j(x)\chi_{8B_j}(x)\\
&+ \sum_{\{j\colon r_j \geqslant \varrho(x_j)/4\}} \lambda_j(b(x) - b_{B_j})\mathcal{R}a_j(x)\chi_{(8B_j)^c}(x)\\
&+ \sum_{\{j\colon r_j < \varrho(x_j)/4\}} \lambda_j(b(x) - b_{B_j})\mathcal{R}a_j(x)\chi_{(8B_j)^c}(x)\\
&- \mathcal{R}\bigg(\sum_{j=-\infty}^{\infty} \lambda_j(b(x) - b_{B_j})a_j\bigg)(x)\\
= &\, A_1(x) + A_2(x) + A_3(x) + A_4(x).
\end{aligned}$$

Using Hölder's inequality, the (L^q, L^q)-boundedness of $\mathcal{R}$ with $1 < q < p_0$ and Lemma 2.5,

$$\begin{aligned}
\|(b(x) - b_{B_j})\mathcal{R}a_j(x)\chi_{8B_j}(x)\|_{L^1(\mathbb{R}^n)} &\leqslant \bigg(\int_{8B_j} |b(x) - b_{B_j}|^{q'}\,\mathrm{d}x\bigg)^{1/q'} \|\mathcal{R}a_j\|_{L^q}\\
&\leqslant C\bigg(\int_{8B_j} |b(x) - b_{B_j}|^{q'}\,\mathrm{d}x\bigg)^{1/q'} \|a_j\|_{L^q}\\
&\leqslant C\bigg(\frac{1}{|B_j|}\int_{8B_j} |b(x) - b_{B_j}|^{q'}\,\mathrm{d}x\bigg)^{1/q'} \leqslant C[b]_\theta,
\end{aligned}$$

since $r_j < \varrho(x_j)$. When we consider the term $A_2(x)$, we note that $\varrho(x_j) > r_j \geqslant \varrho(x_j)/4$. Similar to the proof of Lemma 9 in [12], by Lemma 3.3 we obtain

$$
\begin{aligned}
&\|(b(x)-b_{B_j})\mathcal{R}a_j(x)\chi_{(8B_j)^c}(x)\|_{L^1(\mathbb{R}^n)}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\int_{|x-x_j|\geqslant 8r_j}|K(x,y)||b(x)-b_{B_j}|\,\mathrm{d}x\Big\}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\sum_{k=4}^{\infty}\Big(\int_{2^{k-1}r_j\leqslant|x-x_j|<2^k r_j}|K(x,y)|^{p_1'}\,\mathrm{d}x\Big)^{1/p_1'}\\
&\qquad\times\Big(\int_{|x-x_j|<2^k r_j}|b(x)-b_{B_j}|^{p_1}\,\mathrm{d}x\Big)^{1/p_1}\Big\}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\sum_{k=4}^{\infty}\Big(\frac{C_N}{(2^k)^N}(2^k r_j)^{-n/p_1}\Big)\\
&\qquad\times\Big(\frac{1}{|2^k B_j|}\int_{2^k B_j}|b(x)-b_{B_j}|^{p_1}\,\mathrm{d}x\Big)^{1/p_1}|2^k B_j|^{1/p_1}\Big\}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\sum_{k=4}^{\infty}\Big(\frac{C_N}{(2^k)^N}(2^k r_j)^{-n/p_1}\Big)[b]_\theta k\Big(1+\frac{2^k r_j}{\varrho(x_0)}\Big)^{\theta'}|2^k r_j|^{n/p_1}\Big\}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\sum_{k=4}^{\infty}\frac{1}{(2^k)^N}[b]_\theta\Big\}\leqslant C[b]_\theta,
\end{aligned}
$$

where we choose N sufficiently large such that the above series converges.

For A_3, by using the vanishing condition of a_j and Lemma 3.3, we get

$$
\begin{aligned}
&\|(b(x)-b_{B_j})\mathcal{R}a_j(x)\chi_{(8B_j)^c}(x)\|_{L^1(\mathbb{R}^n)}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\int_{|x-x_j|\geqslant 8r_j}|K(x,y)-K(x,x_j)||b(x)-b_{B_j}|\,\mathrm{d}x\Big\}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\Big\{\int_{|x-x_j|\geqslant 8r_j}\frac{C_N}{\big(1+|x-y|/\varrho(x)\big)^N}\frac{|y-x_j|^\delta}{|x-y|^{n-2+\delta}}\\
&\qquad\times\Big(\int_{B(x,|x-y|)}\frac{V^2(z)}{|x-z|^{n-2}}\,\mathrm{d}z+\frac{1}{|x-y|^2}\Big)|b(x)-b_{B_j}|\,\mathrm{d}x\Big\}\\
&\quad=\int_{B_j}|a_j(y)|\,\mathrm{d}y\{\tilde I_1+\tilde I_2\}.
\end{aligned}
$$

First of all, we need to obtain the following new estimate.

$$
\begin{aligned}
\|V^2\chi_{B(x_j,2^{k+3}r_j)}\|_{q_0/2}&=\Big(\int_{B(x_j,2^{k+3}r_j)}V^{q_0}\,\mathrm{d}z\Big)^{2/q_0}\\
&\leqslant C(2^{k+3}r_j)^{2n/q_0-2n}\Big(\int_{B(x_j,2^{k+3}r_j)}V\,\mathrm{d}z\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C(2^{k+3}r_j)^{2n/q_0-4}\bigg(\frac{1}{(2^{k+3}r_j)^{n-2}}\int_{B(x_j,2^{k+3}r_j)} V\,\mathrm{d}z\bigg)^2\\
&\leqslant C(2^{k+3}r_j)^{2n/q_0-4}(1+2^{k+3}r_jm(x_j,V))^{2k_0'},
\end{aligned}
$$

where we use assumption (2) from Lemma 2.4.

Note that $|x-x_j|\sim|x-y|$.

$$
\begin{aligned}
\tilde{I}_1 &\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\\
&\quad\times\sum_{k=1}^{\infty}\int_{2^{k+3}r_j\leqslant|x-x_j|<2^{k+4}r_j}\frac{\mathcal{I}_2(V^2\chi_{2^{k+3}B_j})(x)}{(1+|x-x_j|/\varrho(x_j))^{N/(k_0+1)}}\frac{|x_j-y|^{\delta}|b-b_{B_j}|}{|x-x_j|^{n-2}|x-x_j|^{\delta}}\,\mathrm{d}x\\
&\leqslant C\sum_{k=1}^{\infty}\frac{2^{-(k+3)\delta}(2^{k+3}r_j)^{2-n/q'}}{(1+2^{k+3}r_j/\varrho(x_j))^{N/(k_0+1)}}\\
&\quad\times\bigg(\frac{1}{|B(x,2^{k+3}r_j)|}\int_{|x-x_j|<2^{k+4}r_j}|b-b_{B_j}|^q\,\mathrm{d}x\bigg)^{1/q}\|\mathcal{I}_2(V^2\chi_{2^{k+3}B_j})\|_{q'}\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}\frac{(2^{k+3}r_j)^{2-n/q'}}{(1+2^{k+3}r_j/\varrho(x_j))^{N/(k_0+1)}}\\
&\quad\times\bigg(\frac{1}{|B(x,2^{k+3}r_j)|}\int_{|x-x_j|<2^{k+4}r_j}|b-b_{B_j}|^q\,\mathrm{d}x\bigg)^{1/q}\|V^2\chi_{2^{k+3}B_j}\|_{q_0/2}\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}\frac{(2^{k+3}r_j)^{2-n/q'+2n/q_0-4}}{(1+2^{k+3}r_j/\varrho(x_j))^{N/(k_0+1)}}[b]_\theta\\
&\qquad\times k\Big(1+\frac{2^{k+4}r_j}{\varrho(x_j)}\Big)^{(k_0+1)\theta}(1+2^{k+3}r_jm(x_j,V))^{2k_0'}\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}[b]_\theta k(2^{k+3}r_j)^{2-n/q'+2n/q_0-4}(1+2^{k+3}r_jm(x_j,V))^{2k_0'-N/(k_0+1)+(k_0+1)\theta}\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}[b]_\theta k\leqslant C[b]_\theta,
\end{aligned}
$$

where we use the fact that $1/q'=2/q_0-2/n$ and we choose $N>(2k_0'+(k_0+1)\theta)\times(k_0+1)$. Second,

$$
\begin{aligned}
\tilde{I}_2\leqslant C\int_{B_j}|a_j(y)|\,\mathrm{d}y\sum_{k=1}^{\infty}\int_{2^{k+3}r_j\leqslant|x-x_j|<2^{k+4}r_j}&\frac{C_N}{(1+|x-x_j|/\varrho(x_j))^{N/(k_0+1)}}\\
&\times\frac{|x_j-y|^{\delta}|b(x)-b_{B_j}|}{|x-x_j|^{n}|x-x_j|^{\delta}}\,\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}\Big(1+\frac{2^{k+3}r_j}{\varrho(x_j)}\Big)^{-N/(k_0+1)}\frac{1}{|B(x,2^{k+3}r_j)|}\int_{|x-x_j|<2^{k+4}r_j}|b-b_{B_j}|\,\mathrm{d}x\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}\Big(1+\frac{2^{k+3}r_j}{\varrho(x_j)}\Big)^{-N/(k_0+1)}\frac{1}{|B(x,2^{k+4}r_j)|}\int_{|x-x_j|<2^{k+4}r_j}|b-b_{B_j}|\,\mathrm{d}x\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}[b]_\theta k\Big(1+\frac{2^{k+4}r_j}{\varrho(x_j)}\Big)^{(k_0+1)\theta-N/(k_0+1)}\\
&\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta+k(k_0+1)\theta}[b]_\theta k\leqslant C[b]_\theta,
\end{aligned}$$

where we have also chosen N large enough such that the above series converges. Therefore, if $r_j\leqslant\varrho(x_j)/4$, then

$$\|(b(x)-b_{B_j})\mathcal{R}a_j(x)\chi_{(8B_j)^c}(x)\|_{L^1(\mathbb{R}^n)}\leqslant C[b]_\theta.$$

Thus, we have

$$\Big|\Big\{x\in\mathbb{R}^n\colon\ |A_i(x)|>\frac{\lambda}{4}\Big\}\Big|\leqslant\frac{C}{\lambda}\|A_i(x)\|_{L^1}\leqslant\frac{C[b]_\theta}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j|,\quad i=1,2,3.$$

Note that

$$\begin{aligned}
\|(b-b_{B_j})a_j\|_{L^1}&\leqslant\Big(\int_{B_j}|b(x)-b_{B_j}|^{q'}\,\mathrm{d}x\Big)^{1/q'}\|a_j\|_{L^q}\\
&\leqslant\Big(\frac{1}{|B_j|}\int_{B_j}|b(x)-b_{B_j}|^{q'}\,\mathrm{d}x\Big)^{1/q'}\leqslant C[b]_\theta\Big(1+\frac{r_j}{\varrho(x_j)}\Big)^{\theta'}\leqslant C[b]_\theta,
\end{aligned}$$

where $r_j<\varrho(x_j)$. By the weak $(1,1)$-boundedness of $\mathcal{R}$ (see Theorem 2 in [12]), we get

$$\Big|\Big\{x\in\mathbb{R}^n\colon\ |A_4(x)|>\frac{\lambda}{4}\Big\}\Big|\leqslant\frac{C}{\lambda}\Big\|\sum_{j=-\infty}^{\infty}\lambda_j(b-b(x_j))a_j\Big\|_{L^1}\leqslant\frac{C[b]_\theta}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j|.$$

Therefore,

$$\begin{aligned}
|\{x\in\mathbb{R}^n\colon\ |[b,\mathcal{R}]f(x)|>\lambda\}|&\leqslant C\sum_{i=1}^{4}\Big|\Big\{x\in\mathbb{R}^n\colon\ |A_i(x)|>\frac{\lambda}{4}\Big\}\Big|\\
&\leqslant\frac{C[b]_\theta}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j|\leqslant\frac{C[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}_1}}.
\end{aligned}$$

This completes the proof of Theorem 1.2. $\square$

Acknowledgment. The authors would like to thank the referee for useful comments and suggestions which improved the presentation of this paper.

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