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WHEN A LINE GRAPH ASSOCIATED TO ANNIHILATING-IDEAL GRAPH OF A LATTICE IS PLANAR OR PROJECTIVE

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Abstract. Let $(L,\wedge,\vee)$ be a finite lattice with a least element 0. $\mathbb{A}G(L)$ is an annihilating-ideal graph of L in which the vertex set is the set of all nontrivial ideals of L, and two distinct vertices I and J are adjacent if and only if $I\wedge J=0$. We completely characterize all finite lattices L whose line graph associated to an annihilating-ideal graph, denoted by $\mathfrak{L}(\mathbb{A}G(L))$, is a planar or projective graph.

Keywords: annihilating-ideal graph; lattice; line graph; planar graph; projective graph

MSC 2010: 05C75, 05C10, 06B10

1. Introduction

In the last twenty years, the study of algebraic structures, using the properties of graph theory, tends to an exciting research topic. Associating a graph to an algebraic structure has been the interest of many researchers. For example see [2], [4], and [13]. The notion of an annihilating-ideal graph $\mathbb{A}G(R)$ of a commutative ring R was introduced by Behboodi and Rakeei in [5] and [6]. However, they let all annihilating-ideals of R be vertices of the graph $\mathbb{A}G(R)$, and two distinct vertices I and J be adjacent if and only if $IJ=0$. In [1], Khashyarmanesh et al. introduced and studied the annihilating-ideal graph of a lattice L, denoted by $\mathbb{A}G(L)$. Graf $\mathbb{A}G(L)$ is a graph whose vertex set is the set of all nontrivial ideals of L and two distinct vertices I and J are joined by an edge if and only if $I\wedge J=0$.

First we review some definitions and notation from lattice theory.

Recall that a *lattice* is an algebra $L=(L,\wedge,\vee)$ satisfying the following conditions: for all $a,b,c\in L$:

(1) $a\wedge a=a$, $a\vee a=a$,
(2) $a\wedge b=b\wedge a$, $a\vee b=b\vee a$,

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(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $a \vee (b \vee c) = (a \vee b) \vee c$, and
(4) $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

There is an equivalent definition for a lattice (see for example [15], Theorem 2.1). To do this, for a lattice L, one can define an order on L as follows: For any $a, b \in L$, we set $a \leqslant b$ if and only if $a \wedge b = a$. Then $(L, \leqslant)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, g.l.b.(a, b) and l.u.b.(a, b) belong to P. For each a and b in P, we define $a \wedge b :=$ g.l.b.(a, b) and $a \vee b :=$ l.u.b.(a, b). Then $(P, \wedge, \vee)$ is a lattice. A lattice L is said to be *bounded* if there are elements 0 and 1 in L such that $0 \wedge a = 0$ and $a \vee 1 = 1$ for all $a \in L$. Clearly, every finite lattice is bounded. Let $(L, \wedge, \vee)$ be a lattice with a least element 0 and let I be a nonempty subset of L. I is called an *ideal* of L, denoted by $I \trianglelefteq L$, if and only if the following conditions are satisfied:

(1) For all $a, b \in I$, $a \vee b \in I$.
(2) If $0 \leqslant a \leqslant b$ and $b \in I$, then $a \in I$.

For two distinct ideals I and J of a lattice L, we put $I \wedge J := \{x \wedge y\colon\ x \in I,\ y \in J\}$.

In a lattice $(L, \wedge, \vee)$ with a least element 0, an element a is called an *atom* if $a \neq 0$ and, for an element x in L, the relation $0 \leqslant x \leqslant a$ implies that either $x = 0$ or $x = a$. We denote the set of all atoms of L by $A(L)$. For terminology in lattice theory we refer to [10].

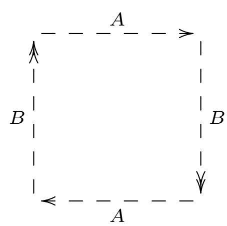
Now, we recall some definitions and notation on graphs. We use the standard terminology of graphs following [7]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. In a graph G, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. Also, the *degree of a vertex* a, denoted by $\deg(a)$, is the number of edges incident to a, and an *isolated vertex* is a vertex with zero degree. A graph with no edges (but at least one vertex) is called an *empty graph*. The graph with no vertices and no edges is the *null graph*. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one of the subsets. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. For notation, we let K_n represent the complete graph on n vertices, and $K_{m,n}$ the complete bipartite graph with part sizes m and n. A complete bipartite graph $K_{1,n}$ is called *star* (see [7] and [12]). A graph G is said to be *contracted to a graph H* if there exists a sequence of elementary contractions which transforms G into H, where an *elementary contraction* consists of deletion of a vertex or an edge or the identification of two adjacent vertices. A *subdivision of a graph* is any graph that can be obtained from the original graph by replacing edges by paths. The *line graph* of a graph G is the graph $\mathfrak{L}(G)$ with the edges of G as its vertices, and two edges of G are adjacent in $\mathfrak{L}(G)$ if and only if they are incident in G.

Recall that a simple graph is said to be *planar* if it can be drawn in the plane or on the surface of a sphere so that its edges intersect only at their ends. A remarkable characterization of the planar graphs was given by Kuratowski in 1930 (cf. [7], page 153). In 1962, Sedláček characterized the planarity of a line graph $\mathfrak{L}(G)$ by using the planarity of G and its vertex degrees. In the sequel, we give the following theorem from [18] which will be used later.

Theorem 1.1 ([18], Lemma 2.6). *A nonempty graph G has a planar line graph $\mathfrak{L}(G)$ if and only if*

(i) *G is planar,*
(ii) *$\triangle(G) \leqslant 4$, and*
(iii) *if $\deg(v) = 4$, then v is a cut vertex in the graph G.*

By a *surface*, we mean a connected compact 2-dimensional real manifold without boundary, that is a connected topological space such that each point has a neighborhood homeomorphic to an open disc. It is well-known that every compact surface is homeomorphic to a sphere, or to a connected sum of g tori (S_g), or to a connected sum of k projective planes (N_k) (see [14], Theorem 5.1). This number k is called the *crosscap number* of the surface. The projective plane can be thought of as a sphere with one crosscap. This means that the crosscap number of the projective plane is 1.



The canonical representation of a projective plane.

A graph G is *embeddable* in a surface S if the vertices of G are assigned to distinct points in S so that every edge of G is a simple arc in S connecting the two vertices which are joined in G. A *projective graph* is a graph that can be embedded in a projective plane. The least number k that G can be embedded in N_k is called the *crosscap number of* G. We denote the crosscap number of a graph G by $\overline{\gamma}(G)$. One easy observation is that $\overline{\gamma}(H) \leqslant \overline{\gamma}(G)$ for any subgraph H of G. If G cannot be embedded in S, then G has at least two edges intersecting at a point which is not a vertex of G. We say a graph G is *irreducible* for a surface S if G does not embed in S, but any proper subgraph of G embeds in S. The set of 103 irreducible graphs for the projective plane has been found by Glover, Huneke and Wang in [11], and Archdeacon in [3] proved that this list is complete. This list also has been checked by Myrvold and Roth in [17]. Hence a graph embeds in the projective plane if and only if it contains no subdivision of 103 graphs in [11]. Also, a complete graph K_n is projective if $n = 5$ or 6, and the only projective complete bipartite graphs are $K_{3,3}$ and $K_{3,4}$ (see [8] or [16]). Note that a planar graph is not considered as a projective graph. For more detailes on the notions concerning embedding of graphs following [19].

In this paper, we assume that L is a finite lattice and $A(L) = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of L. We denote the line graph associated with $\mathbb{A}G(L)$ by $\mathfrak{L}(\mathbb{A}G(L))$ and we denote $w_{I,J}$ for the vertices $I, J \in \mathbb{A}G(L)$, where I and J are adjacent vertices in $\mathbb{A}G(L)$. In the second section of this work, we completely characterize all finite lattices L such that the line graphs associated with their annihilating-ideal graphs $\mathbb{A}G(L)$, are planar or projective.

2. On the planarity and projectivity of $\mathfrak{L}(\mathbb{A}G(L))$

In this section, we explore the planarity and projectivity of the line graph associated with the graph $\mathbb{A}G(L)$, which is denoted by $\mathfrak{L}(\mathbb{A}G(L))$. If $|A(L)| = 1$, then $\mathbb{A}G(L)$ is an empty graph, and hence $\mathfrak{L}(\mathbb{A}G(L))$ is a null graph. We begin this section with the following notation, which is needed in the rest of the paper.

Notation. Let $i_1, i_2, \ldots, i_n$ be integers with $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. The notation $U_{i_1 i_2 \ldots i_k}$ stands for the set

$$\{I \trianglelefteq L\colon \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \subseteq I \text{ and } a_j \notin I \text{ for } j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}\}.$$

Note that no two distinct elements in $U_{i_1 i_2 \ldots i_k}$ are adjacent in $\mathbb{A}G(L)$. Also, if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \ldots i_k}$ and $U_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \ldots i_k} \cap U_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Moreover, $V(\mathbb{A}G(L)) = \bigcup U_{i_1 i_2 \ldots i_k}$ for all $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. Suppose that L has n atoms. We denote the ideal $\{0, a_i\} \in U_i$, where a_i is an atom and U_i is an ideal, with $1 \leqslant i \leqslant n$, by u_i. Note that $U_{12\ldots n}$ consist of isolated vertices. Clearly, the isolated points do not affect planarity and projectivity. Hence, we ignore the set of isolated vertices from the vertex-set of $\mathfrak{L}(\mathbb{A}G(L))$, and so we do not show these points in our figures.

Now, we state the following lemma.

Lemma 2.1. *If $\mathfrak{L}(\mathbb{A}G(L))$ is planar or projective, then the size of $A(L)$ is at most four.*

Proof. Assume on the contrary that $|A(L)| \geqslant 5$. Then the graph $\mathbb{A}G(L)$ contains a copy of K_5 with vertices $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$ and $u_5 \in U_5$. So the contraction of the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subdivision of $K_{3,3}$ (see Figure 1). Therefore it is not a planar graph, which is a contradiction.

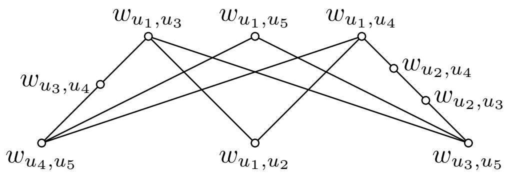


Figure 1.

Also, the contraction of the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of E_{20}, one of the graphs listed in [11] (see Figure 2). Therefore $\mathfrak{L}(\mathbb{A}G(L))$ is not a projective graph, which is again a contradiction. □

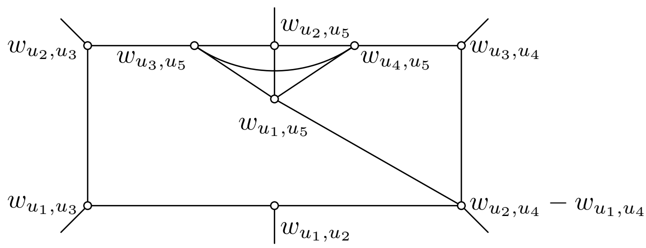


Figure 2.

By Lemma 2.1, it is sufficient for us to investigate the planarity and projectivity of the graph $\mathfrak{L}(\mathbb{A}G(L))$ in the cases in which the size of $A(L)$ is 2, 3, or 4.

First we state necessary and sufficient conditions for the planarity and projectivity of the graph $\mathfrak{L}(\mathbb{A}G(L))$, when $|A(L)| = 2$.

Theorem 2.1. *Suppose that $|A(L)| = 2$. Then $\mathfrak{L}(\mathbb{A}G(L))$ is a planar graph if and only if* $\left|\bigcup\limits_{j=1}^{2} U_j\right| \leqslant 5$.

Proof. First, assume that $\mathfrak{L}(\mathbb{A}G(L))$ is planar and assume on the contrary that $\left|\bigcup\limits_{j=1}^{2} U_j\right| \geqslant 6$. By [1], Theorem 2.6, we know that as $|A(L)| = 2$, the graph $\mathbb{A}G(L)$ is a complete bipartite graph. If $\mathbb{A}G(L)$ is a star graph, then the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subgraph isomorphic to K_5, which is not planar. Otherwise, $\mathbb{A}G(L)$ is not a star graph. Then it contains a subgraph isomorphic to $K_{2,4}$ or $K_{3,3}$. In these two cases, $\mathfrak{L}(\mathbb{A}G(L))$ contains a subdivision of $K_{3,3}$. Hence $\mathfrak{L}(\mathbb{A}G(L))$ is not planar, which is a contradiction.

Conversely, suppose that $\left|\bigcup_{j=1}^{2} U_j\right| \leqslant 5$. If $\left|\bigcup_{j=1}^{2} U_j\right| = 2$, then $\mathfrak{L}(\mathbb{A}G(L))$ is isomorphic to $\mathfrak{L}(K_2)$, which is an empty graph with one vertex. Also, if $\left|\bigcup_{j=1}^{2} U_j\right| = 3$, then $\mathfrak{L}(\mathbb{A}G(L)) \cong \mathfrak{L}(K_{1,2}) \cong K_2$. In addition, if $\left|\bigcup_{j=1}^{2} U_j\right| = 4$, then $\mathbb{A}G(L)$ is isomorphic to $K_{1,3}$ or $K_{2,2}$. Hence $\mathfrak{L}(\mathbb{A}G(L))$ is isomorphic to K_3 or $K_{2,2}$, respectively. Finally, assume that $\left|\bigcup_{j=1}^{2} U_j\right| = 5$. If $\mathbb{A}G(L)$ is a star graph, then $\mathfrak{L}(\mathbb{A}G(L)) \cong K_4$. Otherwise, the graph $\mathbb{A}G(L)$ is isomorphic to $K_{2,3}$ with vertices $u_1, I_1, I_1' \in U_1$ and $u_2, I_2 \in U_2$. In this case, the graph $\mathfrak{L}(\mathbb{A}G(L))$ is pictured in Figure 3.

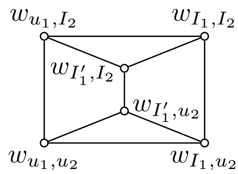


Figure 3.

In all of the above situations, $\mathfrak{L}(\mathbb{A}G(L))$ is a planar graph. $\square$

Theorem 2.2. *Suppose that $|A(L)| = 2$. Then $\mathfrak{L}(\mathbb{A}G(L))$ is a projective graph if and only if one of the following conditions holds:*

(i) $\left|\bigcup_{j=1}^{2} U_j\right| = 6$ *and* $|U_i| = 1$ *for some unique* $i \in \{1,2\}$ *or* $|U_i| = |U_j| = 3$ *for* $i, j \in \{1,2\}$.

(ii) $\left|\bigcup_{j=1}^{2} U_j\right| = 7$ *and* $|U_i| = 1$ *for some unique* $i \in \{1,2\}$.

P r o o f. First, assume that the graph $\mathfrak{L}(\mathbb{A}G(L))$ is projective and on the contrary, $\left|\bigcup_{j=1}^{2} U_j\right| \leqslant 5$. Then, by Theorem 2.1, the graph $\mathfrak{L}(\mathbb{A}G(L))$ is planar, which is not projective. Now, assume that $\left|\bigcup_{j=1}^{2} U_j\right| = 6$ and $\mathbb{A}G(L) \cong K_{2,4}$. By [9], Example 2.14, $\overline{\gamma}(\mathfrak{L}(K_{2,4})) = 2$, and so the graph $\mathfrak{L}(\mathbb{A}G(L))$ is not projective. Hence, if $\left|\bigcup_{j=1}^{2} U_j\right| = 6$, then the statement (i) holds. Now, suppose that $\left|\bigcup_{j=1}^{2} U_j\right| = 7$. If $\mathbb{A}G(L)$ is not a star graph, then it is isomorphic to $K_{2,5}$ or $K_{3,4}$. By [9], Corollary 2.11, $\overline{\gamma}(\mathfrak{L}(K_{2,5})) = 2$ and, by [9], Example 2.14, $\overline{\gamma}(\mathfrak{L}(K_{3,4})) = 2$. So if $\left|\bigcup_{j=1}^{2} U_j\right| = 7$, then the statement (ii) holds. Finally, we may assume that $\left|\bigcup\limits_{j=1}^{2} U_j\right| \geqslant 8$. If $\mathbb{A}G(L)$ is a star graph, then the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subgraph isomorphic to K_7, which is not projective. Otherwise, $\mathbb{A}G(L)$ is not a star graph. Then it contains a subgraph isomorphic to $K_{2,6}$, $K_{3,5}$ or $K_{4,4}$. In these cases, $\mathbb{A}G(L)$ contains a copy of $K_{2,4}$. Clearly, $\overline{\gamma}(\mathfrak{L}(\mathbb{A}G(L))) \geqslant \overline{\gamma}(\mathfrak{L}(K_{2,4}))$, and we have $\overline{\gamma}(\mathfrak{L}(K_{2,4})) = 2$. It means that the graph $\mathfrak{L}(\mathbb{A}G(L))$ is not projective. Therefore, if $\mathfrak{L}(\mathbb{A}G(L))$ is projective, then one of the conditions (i) or (ii) holds.

Conversely, suppose that $\left|\bigcup\limits_{j=1}^{2} U_j\right| = 6$, and the graph $\mathbb{A}G(L)$ is a star graph. Then $\mathfrak{L}(\mathbb{A}G(L)) \cong K_5$, and so it is a projective graph. Now, suppose that $\mathbb{A}G(L) \cong K_{3,3}$. By [9], Example 2.12, $\overline{\gamma}(\mathfrak{L}(K_{3,3})) = 1$, and so the graph $\mathfrak{L}(\mathbb{A}G(L))$ is projective. Finally, suppose that $\left|\bigcup\limits_{j=1}^{2} U_j\right| = 7$, and the graph $\mathbb{A}G(L)$ is a star graph. Then $\mathfrak{L}(\mathbb{A}G(L)) \cong K_6$, and so it is a projective graph. $\square$

Now, we investigate the planarity of $\mathfrak{L}(\mathbb{A}G(L))$, when $|A(L)| = 3$. Let $\left|\bigcup\limits_{j=1}^{3} U_j\right| \geqslant 5$. It is easy to see that $\mathbb{A}G(L)$ contains a subgraph isomorphic to a complete 3-partite graph $K_{3,1,1}$ or $K_{2,2,1}$. Therefore the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subdivision of $K_{3,3}$ or a subdivision of K_5, respectively. Hence it is not planar, and so we have the following lemma.

Lemma 2.2. *If $\mathfrak{L}(\mathbb{A}G(L))$ is planar, then $\left|\bigcup\limits_{j=1}^{3} U_j\right| \leqslant 4$.*

Theorem 2.3. *Suppose that $|A(L)| = 3$. Then $\mathfrak{L}(\mathbb{A}G(L))$ is a planar graph if and only if one of the following conditions holds:*

(i) $\left|\bigcup\limits_{j=1}^{3} U_j\right| = 3$ *and* $|U_{ij}| \leqslant 2$ *for* $1 \leqslant i, j \leqslant 3$.

(ii) $\left|\bigcup\limits_{j=1}^{3} U_j\right| = 4$ *and* $|U_{ij}| \leqslant 1$ *for* $1 \leqslant i, j \leqslant 3$.

Proof. First, assume that one of the conditions (i) or (ii) holds. Suppose that $\left|\bigcup\limits_{j=1}^{3} U_j\right| = 3$ and $|U_{12}| = |U_{13}| = |U_{23}| = 2$. The graph $\mathbb{A}G(L)$ with vertices $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $I_{12}, I'_{12} \in U_{12}$, $I_{13}, I'_{13} \in U_{13}$ and $I_{23}, I'_{23} \in U_{23}$ is pictured in Figure 4.

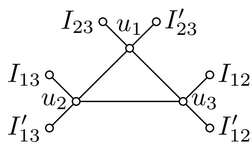


Figure 4.

Hence the graph $\mathfrak{L}(\mathbb{A}G(L))$ pictured in Figure 5 is planar.

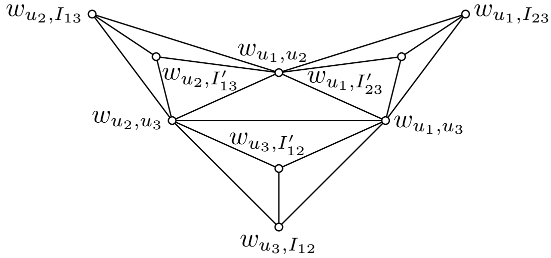


Figure 5.

Now, suppose that $\left|\bigcup\limits_{j=1}^{3} U_j\right| = 4$, $|U_1| = 2$ and $|U_{12}| = |U_{13}| = |U_{23}| = 1$. The graph $\mathbb{A}G(L)$ with vertices $u_1, I_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$ and $I_{23} \in U_{23}$ is pictured in Figure 6 and $\mathfrak{L}(\mathbb{A}G(L))$, which is a planar graph is pictured in Figure 7.

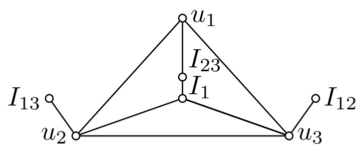


Figure 6.

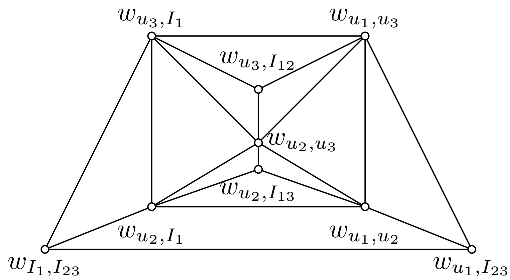


Figure 7.

Conversely, suppose that $\mathfrak{L}(\mathbb{A}G(L))$ is a planar graph. By Lemma 2.2, $\left|\bigcup\limits_{j=1}^{3} U_j\right| \leqslant 4$.

Hence we have the following cases.

Case 1. $\left|\bigcup\limits_{j=1}^{3} U_j\right| = 3$. If U_{12}, U_{13} or U_{23} has at least three elements, then there exists at least a vertex of degree 5 in the graph $\mathbb{A}G(L)$. Hence the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subgraph isomorphic to K_5, and so it is not planar, which is a contradiction.

Case 2. $\left|\bigcup_{j=1}^{3} U_j\right| = 4$. Without loss of generality, we may assume that $|U_1| = 2$. If U_{12} or U_{13} has at least two elements, then there exists at least a vertex of degree 5 in the graph $\mathbb{A}G(L)$. Hence the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of K_5, and so it is not planar, which is a contradiction. In addition, if U_{23} has at least two elements, then the contraction of $\mathbb{A}G(L)$ contains a subgraph isomorphic to $K_{2,4}$. Therefore $\mathbb{A}G(L)$ has a vertex of degree 4 which is not a cut vertex. By Theorem 1.1, $\mathfrak{L}(\mathbb{A}G(L))$ is not a planar graph, which is a contradiction. □

Now, we investigate the projectivity of $\mathfrak{L}(\mathbb{A}G(L))$, when $|A(L)| = 3$.

Suppose that $\left|\bigcup_{j=1}^{3} U_j\right| \geqslant 6$. Then the graph $\mathbb{A}G(L)$ contains a subgraph isomorphic to $K_{4,1,1}$, $K_{3,2,1}$ or $K_{2,2,2}$. If $\mathbb{A}G(L)$ contains a subgraph isomorphic to $K_{4,1,1}$, then one can easily find a copy of A_1, one of the listed graphs in [11], in the graph $\mathfrak{L}(\mathbb{A}G(L))$, which is not projective. Also, if $\mathbb{A}G(L)$ contains a subgraph isomorphic to $K_{3,2,1}$, then one can easily find a copy of E_{20}, one of the graphs listed in [11], in the contraction of $\mathfrak{L}(\mathbb{A}G(L))$, which is not projective. Now, if $\mathbb{A}G(L)$ contains a subgraph isomorphic to $K_{2,2,2}$, then the contraction of $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of E_3, one of the listed graphs in [11], which is not projective. Therefore $\mathfrak{L}(\mathbb{A}G(L))$ is not a projective graph.

As a consequence of the above discussion, we state the following lemma.

Lemma 2.3. *If $\mathfrak{L}(\mathbb{A}G(L))$ is projective, then* $\left|\bigcup_{j=1}^{3} U_j\right| \leqslant 5$.

Theorem 2.4. *Suppose that $|A(L)| = 3$. Then $\mathfrak{L}(\mathbb{A}G(L))$ is a projective graph if and only if one of the following conditions holds:*

(i) $\left|\bigcup_{j=1}^{3} U_j\right| = 3$, *there exist unique i and j, with $1 \leqslant i, j \leqslant 3$, such that $3 \leqslant |U_{ij}| \leqslant 4$ and $|U_{kk'}| \leqslant 2$ for $k \in \{i, j\}$ and $\{k'\} = \{1, 2, 3\} \setminus \{i, j\}$.*

(ii) $\left|\bigcup_{j=1}^{3} U_j\right| = 4$, *there exists a unique i, with $1 \leqslant i \leqslant 3$, such that $|U_i| = 2$, and for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, if $2 \leqslant |U_{ij}| \leqslant 3$, then $|U_{ik}| \leqslant 1$ and $|U_{jk}| \leqslant 1$.*

(iii) $\left|\bigcup_{j=1}^{3} U_j\right| = 5$,

  (a) *there exists a unique i, with $1 \leqslant i \leqslant 3$, such that $|U_i| = 3$, and for all $1 \leqslant j, k \leqslant 3$, $U_{jk} = \emptyset$;*

  (b) *there exists a unique i, with $1 \leqslant i \leqslant 3$, such that $|U_i| = 1$, and for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, $|U_{jk}| \leqslant 1$ and $U_{ij} = U_{ik} = \emptyset$.*

Proof. First we assume that $\mathfrak{L}(\mathbb{A}G(L))$ is a projective graph. By Lemma 2.3, $\left|\bigcup_{j=1}^{3} U_j\right| \leqslant 5$. Hence we have the following cases.

Case 1. $\left|\bigcup_{j=1}^{3} U_j\right| = 3$. In this case, if $|U_{ij}| \leqslant 2$ for all $i, j \in \{1, 2, 3\}$, then by Theorem 2.3, the graph $\mathfrak{L}(\mathbb{A}G(L))$ is planar, which is not projective. Also, without loss of generality we may assume that $|U_{12}|, |U_{13}| \in \{3, 4\}$. Then one can easily check that the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of A_1, one of the graphs listed in [11], which is not projective. In addition, if we assume that U_{12}, U_{13} or U_{23} has at least five elements, then the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subgraph isomorphic to K_7, which is not projective. Therefore, for the projectivity of $\mathfrak{L}(\mathbb{A}G(L))$, it is necessary that there exist unique i and j, with $1 \leqslant i, j \leqslant 3$, such that $3 \leqslant |U_{ij}| \leqslant 4$ and $|U_{kk'}| \leqslant 2$ for $k \in \{i, j\}$ and $\{k'\} = \{1, 2, 3\} \setminus \{i, j\}$.

Case 2. $\left|\bigcup_{j=1}^{3} U_j\right| = 4$. In this case, if $|U_{ij}| \leqslant 1$ for all $i, j \in \{1, 2, 3\}$, then, by Theorem 2.3, the graph $\mathfrak{L}(\mathbb{A}G(L))$ is planar, which is not projective. Now, suppose that there exists a unique U_i, with $1 \leqslant i \leqslant 3$, say U_1, such that $|U_1| = 2$. If $|U_{23}| \geqslant 2$, then $\mathbb{A}G(L)$ contains a copy of $K_{2,4}$. Clearly, $\overline{\gamma}(\mathfrak{L}(\mathbb{A}G(L))) \geqslant \overline{\gamma}(\mathfrak{L}(K_{2,4}))$, and we have $\overline{\gamma}(\mathfrak{L}(K_{2,4})) = 2$. This implies that the graph $\mathfrak{L}(\mathbb{A}G(L))$ is not projective. Now, we may assume that $U_{23} = \emptyset$. If U_{12} or U_{13} has at least four elements, then the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subgraph isomorphic to K_7, which is not projective. Also, if $|U_{12}| = |U_{13}| = 2$, then the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of A_1, one of the graphs listed in [11], which is not projective. Therefore, for the projectivity of $\mathfrak{L}(\mathbb{A}G(L))$, it is necessary that $2 \leqslant |U_{ij}| \leqslant 3$, $|U_{ik}| \leqslant 1$ and $|U_{jk}| \leqslant 1$, for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, when $|U_i| = 2$.

Case 3. $\left|\bigcup_{j=1}^{3} U_j\right| = 5$. Suppose that $|U_1| = 3$. If U_{12} or U_{13} has at least one element, then the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of D_{17}, one of the graphs listed in [11], which is not projective. Also, if U_{23} has at least one element, then the contraction of $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of E_{20}, one of the graphs listed in [11], which is not a projective graph. Therefore, for the projectivity of $\mathfrak{L}(\mathbb{A}G(L))$, it is necessary that $U_{12} = U_{13} = U_{23} = \emptyset$, when $|U_1| = 3$. On the other hand, suppose that there exists a unique U_i, with $1 \leqslant i \leqslant 3$, say U_1, such that $|U_1| = 1$. If U_{12} or U_{13} has at least one element, then the contraction of $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of E_{20}, one of the listed graphs in [11], which is not a projective graph. Also, if $|U_{23}| \geqslant 2$, then the contraction of $\mathfrak{L}(AG(L))$ contains a copy of D_{17}, one of the graphs listed in [11], which is not a projective graph. Therefore, for the projectivity of $\mathfrak{L}(\mathbb{A}G(L))$, it is necessary that $U_{12} = U_{13} = \emptyset$ and $|U_{23}| \leqslant 1$, when $|U_1| = 1$.

Conversely, if one of the statements (i), (ii) or (iii) holds, then we will show that $\mathfrak{L}(\mathbb{A}G(L))$ is a projective graph.

First suppose that $\left|\bigcup_{j=1}^{3} U_j\right| = 3$. If $|U_{12}| = |U_{13}| = 2$ and $|U_{23}| = 4$, then the graph $\mathbb{A}G(L)$ is pictured in Figure 8, which is planar and the graph $\mathfrak{L}(\mathbb{A}G(L))$ is pictured in Figure 9, which is projective. We have $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $I_{12}, I'_{12} \in U_{12}$, $I_{13}, I'_{13} \in U_{13}$ and $I_{23}, I'_{23}, I''_{23}, I'''_{23} \in U_{23}$.

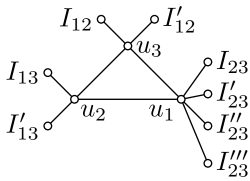


Figure 8.

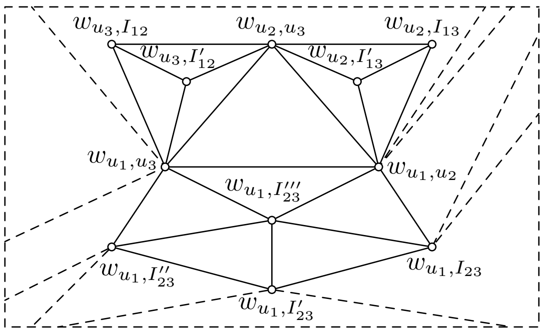


Figure 9.

Now, suppose that $\left|\bigcup_{j=1}^{3} U_j\right| = 4$ and $|U_1| = 2$. If $|U_{12}| = 3$ and $|U_{13}| = |U_{23}| = 1$, then the graph $\mathbb{A}G(L)$ with vertices $u_1, I_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $I_{12}, I'_{12}, I''_{12} \in U_{12}$, $I_{13} \in U_{13}$ and $I_{23} \in U_{23}$ is planar and the graph $\mathfrak{L}(\mathbb{A}G(L))$ is projective (see Figure 10).

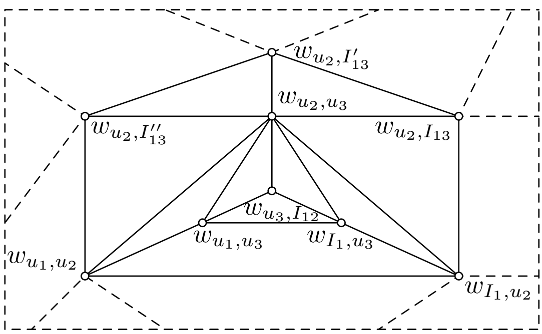


Figure 10.

Finally, suppose that $\left|\bigcup_{j=1}^{3} U_j\right| = 5$ and consider the following cases.

Case 1. There exists a unique U_i, with $1 \leqslant i \leqslant 3$, say U_1, such that $|U_1| = 3$, and also $U_{12} = U_{13} = U_{23} = \emptyset$. Then the graph $\mathbb{A}G(L)$ with vertices $u_1, I_1, I_1' \in U_1$, $u_2 \in U_2$ and $u_3 \in U_3$ is planar. As observed, in Figure 11, the graph $\mathfrak{L}(\mathbb{A}G(L))$ is projective.

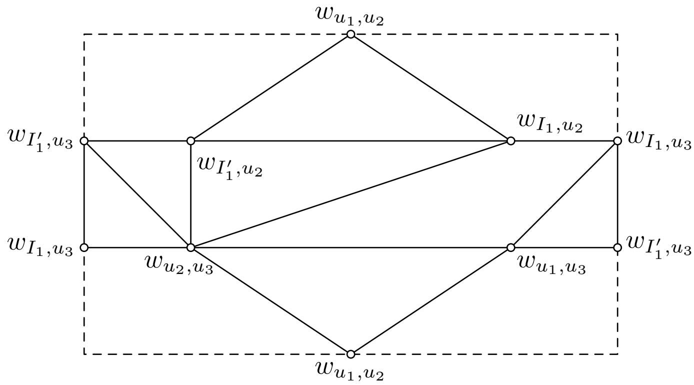


Figure 11.

Case 2. There exists a unique U_i, with $1 \leqslant i \leqslant 3$, say U_1, such that $|U_1| = 1$, also $U_{12} = U_{13} = \emptyset$ and $|U_{23}| = 1$. Then the graph $\mathbb{A}G(L)$ with vertices $u_1 \in U_1$, $u_2, I_2 \in U_2$, $u_3, I_3 \in U_3$ and $I_{23} \in U_{23}$ is planar, and so $\mathfrak{L}(\mathbb{A}G(L))$ is pictured in Figure 12, which is a projective graph. □

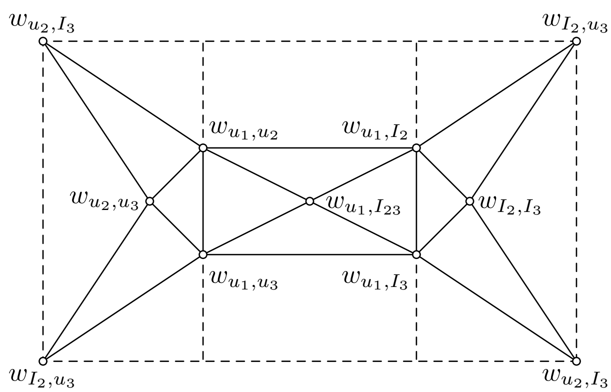


Figure 12.

In the following, we study the planarity and projectivity of $\mathfrak{L}(\mathbb{A}G(L))$, when $|A(L)| = 4$.

Lemma 2.4. *If $\mathfrak{L}(\mathbb{A}G(L))$ is planar or projective, then* $\left|\bigcup_{j=1}^{4} U_j\right| = 4$.

P r o o f. Suppose on the contrary that $\left|\bigcup_{j=1}^{4} U_j\right| \geqslant 5$. Then the graph $\mathbb{A}G(L)$ has a vertex of degree 4 which is not a cut vertex. Hence, by Theorem 1.1, $\mathfrak{L}(\mathbb{A}G(L))$ is not a planar graph, which is a contradiction. Also, on the contrary, consider that $\left|\bigcup_{j=1}^{4} U_j\right| = 5$ and $|U_1| = 2$. Then $\mathfrak{L}(\mathbb{A}G(L))$ contains a subgraph isomorphic to E_{20}, one of the graphs listed in [11], which is not a projective graph. It is again a contradiction. □

Theorem 2.5. *Suppose that $|A(L)| = 4$. Then $\mathfrak{L}(\mathbb{A}G(L))$ is a planar graph if and only if $U_{ij} = \emptyset$ and $|U_{ijk}| \leqslant 1$ for all $i, j, k \in \{1, 2, 3, 4\}$.*

P r o o f. First, assume that the graph $\mathfrak{L}(\mathbb{A}G(L))$ is planar. By Lemma 2.4, we have $\left|\bigcup_{j=1}^{4} U_j\right| = 4$. If there exists at least one element in U_{ij} for $i, j \in \{1, 2, 3, 4\}$, then one can easily check that the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a subdivision of $K_{3,3}$, which is not planar. Also, if one of the sets U_{ijk} has at least two elements for $i, j, k \in \{1, 2, 3, 4\}$, then the graph $\mathbb{A}G(L)$ has a vertex of degree 5. Hence the graph $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of K_5, which is impossible.

Conversely, suppose that $U_{12} = U_{13} = U_{23} = \emptyset$ and $|U_{123}| = |U_{124}| = |U_{134}| = |U_{234}| = 1$. The graph $\mathbb{A}G(L)$ with vertices $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, $I_{123} \in U_{123}$, $I_{124} \in U_{124}$, $I_{134} \in U_{134}$ and $I_{234} \in U_{234}$ is pictured in Figure 13.

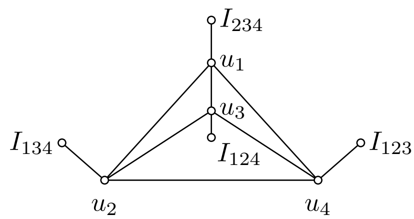


Figure 13.

Hence $\mathfrak{L}(\mathbb{A}G(L))$ is pictured in Figure 14, which is a planar graph. Therefore, in the case that $U_{ij} = \emptyset$ and $|U_{ijk}| \leqslant 1$ for all $i, j, k \in \{1, 2, 3, 4\}$, we have $\mathfrak{L}(\mathbb{A}G(L))$ is planar. □

In the sequel, suppose that $\left|\bigcup_{j=1}^{4} U_j\right| = 4$. We have the following situations.

(i) There exist $i, j \in \{1, 2, 3, 4\}$ such that $|U_{ij}| \geqslant 2$. Then $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of A_1, one of the listed graphs in [11], which is not a projective graph.

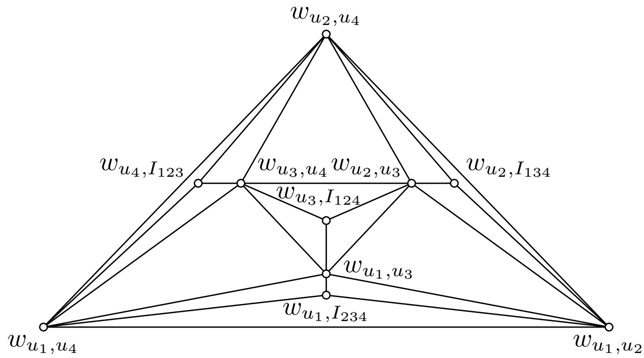


Figure 14.

(ii) There exist $i, i', j, j' \in \{1,2,3,4\}$ with $i \neq i'$, $j \neq j'$, such that $|U_{ij}| = |U_{i'j'}| = 1$. Then the contraction $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of D_{17}, one of the graphs listed in [11], which is not a projective graph.

(iii) There exist $i, i', j \in \{1,2,3,4\}$ with $i \neq i', j$ such that $|U_{ij}| = |U_{i'j}| = 1$. Then $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of D_{17}, one of the graphs listed in [11], which is not a projective graph.

(iv) For all $1 \leqslant i, j, k \leqslant 4$, $|U_{ijk}| \leqslant 1$ and $U_{ij} = \emptyset$. Then, by Theorem 2.5, the graph $\mathfrak{L}(\mathbb{A}G(L))$ is planar, which is not projective.

(v) There exist i, j, k, with $1 \leqslant i, j, k \leqslant 4$ such that $|U_{ijk}| \geqslant 4$. Then $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of K_7, which is not projective.

(vi) There exist unique $i, i', j, k \in \{1,2,3,4\}$ with $i \neq i', j, k$ such that $2 \leqslant |U_{ijk}| \leqslant 3$ and $|U_{i'ij}| = |U_{i'ik}| = |U_{i'jk}| = 1$. Then the graph $\mathbb{A}G(L)$, with vertices $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, $I_{123}, I'_{123}, I''_{123} \in U_{123}$, $I_{124} \in U_{124}$, $I_{134} \in U_{134}$ and $I_{234} \in U_{234}$ is planar. Therefore the graph $\mathfrak{L}(\mathbb{A}G(L))$, which is pictured in Figure 15, is projective.

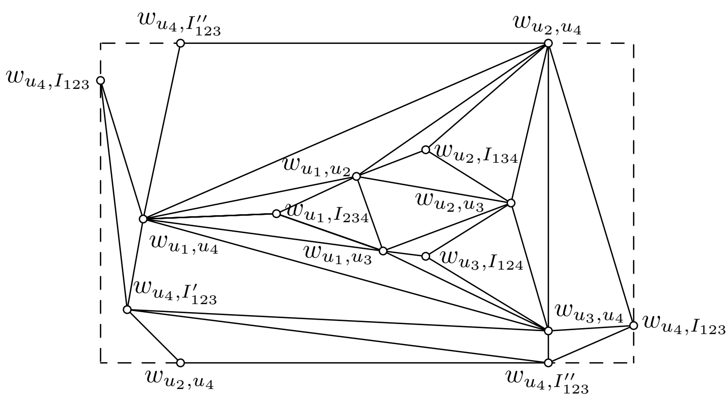


Figure 15.

(vii) There exist $i, i', j, k \in \{1,2,3,4\}$ with $i \neq i', j, k$ such that $|S_{ijk}| = |S_{i'jk}| = 2$. Then $\mathfrak{L}(\Gamma_2(L))$ contains a copy of A_1, one of the listed graphs in [11], which is not a projective graph.

(viii) There exist $i, j, j', k, k' \in \{1,2,3,4\}$ with $i, j \neq j', k \neq k'$ such that $|U_{ij}| = 1$ and $|U_{ij'k'}| = 2$. Then the contraction of $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of B_1, one of the listed graphs in [11], which is not a projective graph.

(ix) There exist i, j, k, with $1 \leqslant i, j, k \leqslant 4$, $|U_{ij}| = |U_{ijk}| = 1$. Then $\mathfrak{L}(\mathbb{A}G(L))$ contains a copy of E_{19}, one of the graphs listed in [11], which is not a projective graph.

(x) There exist unique i, i', j, j' with $\{i', j'\} = \{1,2,3,4\} \setminus \{i, j\}$ such that $|U_{ij}| = |U_{ii'j'}| = |U_{ji'j'}| = 1$. Then the graph $\mathbb{A}G(L)$, with vertices $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, $u_4 \in U_4$, $I_{12} \in U_{12}$, $I_{134} \in U_{134}$ and $I_{234} \in U_{234}$ is planar. Therefore the graph $\mathfrak{L}(\mathbb{A}G(L))$, which is pictured in Figure 16, is projective.

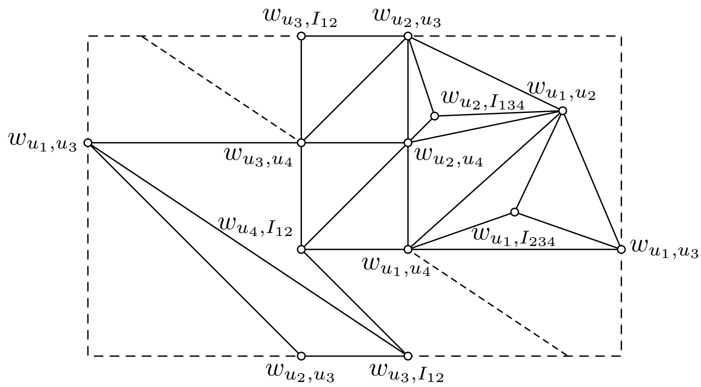


Figure 16.

As a consequence of the above discussion and Lemma 2.4, we state the necessary and sufficient conditions for the projectivity of the graph $\mathfrak{L}(\mathbb{A}G(L))$, when the size of $A(L)$ is equal to 4.

Theorem 2.6. *Suppose that $|A(L)| = 4$. Then $\mathfrak{L}(\mathbb{A}G(L))$ is a projective graph if and only if $\left|\bigcup_{j=1}^{4} U_j\right| = 4$ and one of the following conditions holds:*

(i) *There exist unique $i \neq i', j, k$ with $1 \leqslant i, i', j, k \leqslant 4$ such that $2 \leqslant |U_{ijk}| \leqslant 3$ and $|U_{i'ij}| = |U_{i'ik}| = |U_{i'jk}| = 1$.*

(ii) *There exist unique i, i', j, j' with $\{i', j'\} = \{1,2,3,4\} \setminus \{i, j\}$ such that $|U_{ij}| = |U_{ii'j'}| = |U_{ji'j'}| = 1$.*

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