

YETTER-DRINFELD-LONG BIMODULES ARE MODULES

DAOWEI LU, Qufu, SHUANHONG WANG, Nanjing

Received December 9, 2015. First published March 3, 2017.

Abstract. Let H be a finite-dimensional bialgebra. In this paper, we prove that the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules, introduced by F. Panaite, F. Van Oystaeyen (2008), is isomorphic to the Yetter-Drinfeld category ${}_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ over the tensor product bialgebra $H \otimes H^*$ as monoidal categories. Moreover if H is a finite-dimensional Hopf algebra with bijective antipode, the isomorphism is braided. Finally, as an application of this category isomorphism, we give two results.

Keywords: Hopf algebra; Yetter-Drinfeld-Long bimodule; braided monoidal category

MSC 2010: 16T05, 18D10

1. INTRODUCTION

Panaite and Oystaeyen in [5] introduced the notion of L-R smash biproduct, with the L-R smash product introduced in [4] (or in [7]) and L-R smash coproduct introduced in [5] as multiplication and comultiplication, respectively. When an object A , which is both an algebra and a coalgebra, and a bialgebra H form a L-R-admissible pair (H, A) , $A \sharp H$ becomes a bialgebra with the smash product and smash coproduct, and the Radford biproduct is a special case. It turns out that A is in fact a bialgebra in the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules (introduced in [5]) with some compatible condition.

The aim of this paper is to show that the category $\mathcal{LR}(H)$ coincides with the Yetter-Drinfeld category over the bialgebra $H \otimes H^*$, in the case when H is finite-dimensional. Hence any object $M \in \mathcal{LR}(H)$ is just a module over the Drinfeld double $D(H \otimes H^*)$ (see [1]).

This work was supported by the NSF of China (No. 11371088) and the Fundamental Research Funds for the Central Universities (No. KYLX15_0109).

The paper is organized as follows. In Section 2, we recall the category $\mathcal{LR}(H)$. In Section 3, we give the main result of this paper.

Throughout this article, all the vector spaces, tensor products and homomorphisms are over a fixed field k . For a coalgebra C , we will use the Heyneman-Sweedler notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$ (summation omitted).

2. PRELIMINARIES

Let H be a bialgebra. The category $\mathcal{LR}(H)$ is defined as follows. The objects of $\mathcal{LR}(H)$ are vector spaces M endowed with H -bimodule and H -bicomodule structures (denoted by $h \otimes m \mapsto h \cdot m$, $m \otimes h \mapsto m \cdot h$, $m \mapsto m_{(-1)} \otimes m_{(0)}$, $m \mapsto m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle}$ for all $h \in H$, $m \in M$), such that M is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e.,

$$(2.1) \quad (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} = h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)},$$

$$(2.2) \quad (h \cdot m)_{\langle 0 \rangle} \otimes (h \cdot m)_{\langle 1 \rangle} = h \cdot m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle},$$

$$(2.3) \quad (m \cdot h_2)_{\langle 0 \rangle} \otimes h_1 (m \cdot h_2)_{\langle 1 \rangle} = m_{\langle 0 \rangle} \cdot h_1 \otimes m_{\langle 1 \rangle} h_2,$$

$$(2.4) \quad (m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} = m_{(-1)} \otimes m_{(0)} \cdot h.$$

The morphisms in $\mathcal{LR}(H)$ are H -bilinear and H -bilinear maps.

If H has a bijective antipode S , $\mathcal{LR}(H)$ becomes a strict braided monoidal category with the following structures: for all $M, N \in \mathcal{LR}(H)$, and $m \in M$, $n \in N$, $h \in H$,

$$\begin{aligned} h \cdot (m \otimes n) &= h_1 \cdot m \otimes h_2 \cdot n, \\ (m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} &= m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}, \\ (m \otimes n) \cdot h &= m \cdot h_1 \otimes n \cdot h_2, \\ (m \otimes n)_{\langle 0 \rangle} \otimes (m \otimes n)_{\langle 1 \rangle} &= m_{\langle 0 \rangle} \otimes n_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} n_{\langle 1 \rangle}, \end{aligned}$$

the braiding

$$c_{M,N}: M \otimes N \mapsto N \otimes M, \quad m \otimes n \mapsto m_{(-1)} \cdot n_{\langle 0 \rangle} \otimes m_{(0)} \cdot n_{\langle 1 \rangle},$$

and the inverse

$$c_{M,N}^{-1}: N \otimes M \mapsto M \otimes N, \quad n \otimes m \mapsto m_{\langle 0 \rangle} \cdot S^{-1}(n_{\langle 1 \rangle}) \otimes S^{-1}(m_{(-1)}) \cdot n_{\langle 0 \rangle}.$$

3. MAIN RESULT

In this section, we will give the main result of this paper.

Lemma 3.1. *Let H be a finite-dimensional bialgebra. Then we have a functor $F: \mathcal{LR}(H) \rightarrow_{H \otimes H^*}^{\mathcal{H} \otimes \mathcal{H}^*} \mathcal{YD}$ given for any object $M \in \mathcal{LR}(H)$ and any morphism ϑ by*

$$F(M) = M \quad \text{and} \quad F(\vartheta) = \vartheta,$$

where $H \otimes H^*$ is a bialgebra with the tensor product and tensor coproduct.

Proof. For all $M \in \mathcal{LR}(H)$, first of all, define the left action of $H \otimes H^*$ on M by

$$(3.1) \quad (h \otimes f) \cdot m = \langle f, m_{\langle 1 \rangle} \rangle h \cdot m_{\langle 0 \rangle},$$

for all $h \in H$, $f \in H^*$ and $m \in M$. Then M is a left $H \otimes H^*$ -module. Indeed, for all $h, h' \in H$, $f, f' \in H^*$ and $m \in M$,

$$\begin{aligned} (h \otimes f)(h' \otimes f') \cdot m &= (hh' \otimes ff') \cdot m \\ &= \langle ff', m_{\langle 1 \rangle} \rangle hh' \cdot m_{\langle 0 \rangle} \\ &= \langle f, m_{\langle 1 \rangle 1} \rangle \langle f', m_{\langle 1 \rangle 2} \rangle h \cdot (h' \cdot m_{\langle 0 \rangle}) \\ &= \langle f, m_{\langle 0 \rangle \langle 1 \rangle} \rangle \langle f', m_{\langle 1 \rangle} \rangle h \cdot (h' \cdot m_{\langle 0 \rangle \langle 0 \rangle}) \\ &\stackrel{(2.2)}{=} \langle f, (h' \cdot m_{\langle 0 \rangle})_{\langle 1 \rangle} \rangle \langle f', m_{\langle 1 \rangle} \rangle h \cdot (h' \cdot m_{\langle 0 \rangle})_{\langle 0 \rangle} \\ &= \langle f', m_{\langle 1 \rangle} \rangle (h \otimes f) \cdot (h' \cdot m_{\langle 0 \rangle}) \\ &= (h \otimes f) \cdot ((h' \otimes f') \cdot m). \end{aligned}$$

And

$$(1 \otimes \varepsilon) \cdot m = \langle \varepsilon, m_{\langle 1 \rangle} \rangle m_{\langle 0 \rangle} = m,$$

as claimed. Next, for all $m \in M$, define the left coaction of $H \otimes H^*$ on M by

$$(3.2) \quad \varrho(m) = m_{[-1]} \otimes m_{[0]} = \sum m_{(-1)} \otimes h^i \otimes m_{(0)} \cdot h_i,$$

where $\{h_i\}_i$ and $\{h^i\}_i$ are dual bases in H and H^* . Then on the one hand,

$$(\Delta_{H \otimes H^*} \otimes \text{id})\varrho(m) = \sum m_{(-1)1} \otimes h_1^i \otimes m_{(-1)2} \otimes h_2^i \otimes m_{(0)} \cdot h_i.$$

Evaluating the right-hand side of the equation on $\text{id} \otimes g \otimes \text{id} \otimes h \otimes \text{id}$, we obtain

$$m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.$$

On the other hand,

$$\begin{aligned}
(\text{id} \otimes \varrho)\varrho(m) &= \sum m_{(-1)} \otimes h^i \otimes (m_{(0)} \cdot h_i)_{(-1)} \otimes h^j \otimes (m_{(0)} \cdot h_i)_{(0)} \cdot h_j \\
&\stackrel{(2.4)}{=} \sum m_{(-1)} \otimes h^i \otimes m_{(0)(-1)} \otimes h^j \otimes (m_{(0)(0)} \cdot h_i) \cdot h_j \\
&= \sum m_{(-1)1} \otimes h^i \otimes m_{(-1)2} \otimes h^j \otimes m_{(0)} \cdot h_i h_j.
\end{aligned}$$

Evaluating the right-hand side of the equation on $\text{id} \otimes g \otimes \text{id} \otimes h \otimes \text{id}$, we obtain

$$m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.$$

Since $g, h \in H$ were arbitrary, we have

$$(\Delta_{H \otimes H^*} \otimes \text{id})\varrho = (\text{id} \otimes \varrho)\varrho.$$

And since

$$(\varepsilon_{H \otimes H^*} \otimes \text{id})(\varrho(m)) = \varepsilon(m_{(-1)})m_{(0)} = m,$$

M is a left $H \otimes H^*$ -comodule.

Finally,

$$\begin{aligned}
&[(h \otimes f)_1 \cdot m]_{[-1]}(h \otimes f)_2 \otimes [(h \otimes f)_1 \cdot m]_{[0]} \\
&= (h_1 \cdot m_{\langle 0 \rangle})_{[-1]} \langle f_1, m_{\langle 1 \rangle} \rangle (h_2 \otimes f_2) \otimes (h_1 \cdot m_{\langle 0 \rangle})_{[0]} \\
&= \sum \langle f_1, m_{\langle 1 \rangle} \rangle ((h_1 \cdot m_{\langle 0 \rangle})_{(-1)} h_2 \otimes h^i f_2) \otimes (h_1 \cdot m_{\langle 0 \rangle})_{(0)} \cdot h_i \\
&\stackrel{(2.1)}{=} \sum \langle f_1, m_{\langle 1 \rangle} \rangle h_1 m_{\langle 0 \rangle(-1)} \otimes h^i f_2 \otimes h_2 \cdot m_{\langle 0 \rangle(0)} \cdot h_i.
\end{aligned}$$

Evaluating the right-hand side of the equation on $\text{id} \otimes g \otimes \text{id}$, we obtain

$$\langle f, m_{\langle 1 \rangle} g_2 \rangle h_1 m_{\langle 0 \rangle(-1)} \otimes h_2 \cdot m_{\langle 0 \rangle(0)} \cdot g_1.$$

And

$$\begin{aligned}
&(h \otimes f)_1 m_{[-1]} \otimes (h \otimes f)_2 \cdot m_{[0]} \\
&= \sum (h_1 \otimes f_1)(m_{(-1)} \otimes h^i) \otimes (h_2 \otimes f_2) \cdot (m_{(0)} \cdot h_i) \\
&= \sum h_1 m_{(-1)} \otimes f_1 h^i \otimes \langle f_2, (m_{(0)} \cdot h_i)_{\langle 1 \rangle} \rangle h_2 \cdot (m_{(0)} \cdot h_i)_{\langle 0 \rangle}.
\end{aligned}$$

Evaluating the right-hand side of the equation on $\text{id} \otimes g \otimes \text{id}$, we obtain

$$\begin{aligned}
&h_1 m_{(-1)} \otimes \langle f, g_1(m_{(0)} \cdot g_2)_{\langle 1 \rangle} \rangle h_2 \cdot (m_{(0)} \cdot g_2)_{\langle 0 \rangle} \\
&\stackrel{(2.3)}{=} h_1 m_{(-1)} \otimes \langle f, m_{(0)\langle 1 \rangle} g_2 \rangle h_2 \cdot m_{(0)\langle 0 \rangle} \cdot g_1 \\
&= \langle f, m_{\langle 1 \rangle} g_2 \rangle h_1 m_{\langle 0 \rangle(-1)} \otimes h_2 \cdot m_{\langle 0 \rangle(0)} \cdot g_1.
\end{aligned}$$

Therefore M is a left-left Yetter-Drinfeld module over $H \otimes H^*$. It is straightforward to verify that any morphism in $\mathcal{LR}(H)$ is also a morphism in ${}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$. The proof is completed. \square

Lemma 3.2. *Let H be a finite-dimensional bialgebra. Then we have a functor $G: {}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD} \rightarrow \mathcal{LR}(H)$ given for any object $M \in {}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$ and any morphism θ by*

$$G(M) = M \quad \text{and} \quad G(\theta) = \theta.$$

Proof. We denote by ε^* the map ε_{H^*} defined by $\varepsilon_{H^*}(f) = f(1)$ for all $f \in H^*$. For any $M \in {}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$, denote the left $H \otimes H^*$ -coaction on M by

$$m \mapsto m_{[-1]} \otimes m_{[0]},$$

for all $m \in M$. Define the H -bimodule and H -bicomodule structures as follows:

$$\begin{aligned} (3.3) \quad & h \cdot m = (h \otimes \varepsilon) \cdot m, \\ & \varrho_L(m) = m_{(-1)} \otimes m_{(0)} = (\text{id} \otimes \varepsilon^*)(m_{[-1]}) \otimes m_{[0]}, \\ (3.4) \quad & m \cdot h = \langle (\varepsilon \otimes \text{id})m_{[-1]}, h \rangle m_{[0]}, \\ & \varrho_R(m) = m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} = \sum (1 \otimes h^i) \cdot m \otimes h_i, \end{aligned}$$

for all $h \in H$.

Obviously M is a left H -module. And

$$\begin{aligned} (\Delta \otimes \text{id})\varrho_L(m) &= \Delta((\text{id} \otimes \varepsilon^*)(m_{[-1]})) \otimes m_{[0]} \\ &= (\text{id} \otimes \varepsilon^*)(m_{[-1]1})(\text{id} \otimes \varepsilon^*)(m_{[-1]2}) \otimes m_{[0]} \\ &= (\text{id} \otimes \varepsilon^*)(m_{[-1]})(\text{id} \otimes \varepsilon^*)(m_{[0][-1]}) \otimes m_{[0][0]} \\ &= (\text{id} \otimes \varrho_L)\varrho_L(m). \end{aligned}$$

The counit is straightforward. Thus M is a left H -comodule. For all $h, h' \in M$,

$$\begin{aligned} m \cdot hh' &= \langle (\varepsilon \otimes \text{id})m_{[-1]}, hh' \rangle m_{[0]} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]1}, h \rangle \langle (\varepsilon \otimes \text{id})m_{[-1]2}, h' \rangle m_{[0]} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]}, h \rangle \langle (\varepsilon \otimes \text{id})m_{[0][-1]}, h' \rangle m_{[0][0]} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]}, h \rangle m \cdot h' \\ &= (m \cdot h) \cdot h'. \end{aligned}$$

The unit is obvious. Thus M is a right H -module. Since

$$\begin{aligned} (\text{id} \otimes \Delta)\varrho_R(m) &= \sum (1 \otimes h^i) \cdot m \otimes h_{i1} \otimes h_{i2} \\ &= \sum (1 \otimes h^i h^j) \cdot m \otimes h^j \otimes h^i \\ &= (\varrho_R \otimes \text{id})\varrho_R(m), \end{aligned}$$

it follows that M is a right H -comodule. Moreover,

$$\begin{aligned} (h \cdot m) \cdot h' &= ((h \otimes \varepsilon) \cdot m) \cdot h' \\ &= \langle (\varepsilon \otimes \text{id})((h \otimes \varepsilon) \cdot m)_{[-1]}, h' \rangle ((h \otimes \varepsilon) \cdot m)_{[0]} \\ &= \langle (\varepsilon \otimes \text{id})[(h_1 \otimes \varepsilon) \cdot m]_{[-1]}(h_2 \otimes \varepsilon), h' \rangle ((h_1 \otimes \varepsilon) \cdot m)_{[0]} \\ &\stackrel{(2.1)}{=} \langle (\varepsilon \otimes \text{id})((h_1 \otimes \varepsilon)m_{[-1]}), h' \rangle (h_2 \otimes \varepsilon) \cdot m_{[0]} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]}, h' \rangle (h \otimes \varepsilon) \cdot m_{[0]} \\ &= h \cdot (m \cdot h'). \end{aligned}$$

Thus M is an H -bimodule. And

$$\begin{aligned} (\varrho_L \otimes \text{id})\varrho_R(m) &= \sum (\text{id} \otimes \varepsilon^*)((1 \otimes h^i) \cdot m)_{[-1]} \otimes ((1 \otimes h^i) \cdot m)_{[0]} \otimes h_i \\ &= \sum (\text{id} \otimes \varepsilon^*)((1 \otimes h_1^i) \cdot m)_{[-1]}(1 \otimes h_2^i) \otimes ((1 \otimes h_1^i) \cdot m)_{[0]} \otimes h_i \\ &\stackrel{(2.1)}{=} \sum (\text{id} \otimes \varepsilon^*)((1 \otimes h_1^i)m_{[-1]}) \otimes (1 \otimes h_2^i) \cdot m_{[0]} \otimes h_i \\ &= (\text{id} \otimes \varrho_R)\varrho_L(m). \end{aligned}$$

Thus M is an H -bicomodule.

We now prove (2.1). For all $h \in H$, $m \in M$,

$$\begin{aligned} (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} &= ((h_1 \otimes \varepsilon) \cdot m)_{(-1)} h_2 \otimes ((h_1 \otimes \varepsilon) \cdot m)_{(0)} \\ &= (\text{id} \otimes \varepsilon^*)((h_1 \otimes \varepsilon) \cdot m)_{[-1]}(h_2 \otimes \varepsilon) \otimes ((h_1 \otimes \varepsilon) \cdot m)_{[0]} \\ &\stackrel{(2.1)}{=} (\text{id} \otimes \varepsilon^*)((h_1 \otimes \varepsilon)m_{[-1]}) \otimes (h_2 \otimes \varepsilon) \cdot m_{[0]} \\ &= h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}. \end{aligned}$$

We now prove (2.2):

$$\begin{aligned} (h \cdot m)_{\langle 0 \rangle} \otimes (h \cdot m)_{\langle 1 \rangle} &= ((h \otimes \varepsilon) \cdot m)_{\langle 0 \rangle} \otimes ((h \otimes \varepsilon) \cdot m)_{\langle 1 \rangle} \\ &= \sum (1 \otimes h^i)(h \otimes \varepsilon) \cdot m \otimes h_i \\ &= \sum (h \otimes \varepsilon)(1 \otimes h^i) \cdot m \otimes h_i \\ &= h \cdot m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle}. \end{aligned}$$

We now prove (2.3): On the one hand,

$$\begin{aligned} (m \cdot h_2)_{\langle 0 \rangle} \otimes h_1(m \cdot h_2)_{\langle 1 \rangle} &= \langle (\varepsilon \otimes \text{id})m_{[-1]}, h_2 \rangle m_{[0]\langle 0 \rangle} \otimes h_1 m_{[0]\langle 1 \rangle} \\ &= \sum \langle (\varepsilon \otimes \text{id})m_{[-1]}, h_2 \rangle (1 \otimes h^i) \cdot m_{[0]} \otimes h_1 h_i. \end{aligned}$$

Evaluating the right-hand side on $\text{id} \otimes f$ for all $f \in H^*$, we have

$$\begin{aligned} &\langle (\varepsilon \otimes \text{id})m_{[-1]}, h_2 \rangle (1 \otimes f_2) \cdot m_{[0]} f_1(h_1) \\ &= \langle (\varepsilon \otimes \text{id})(1 \otimes f_1)m_{[-1]}, h \rangle (1 \otimes f_2) \cdot m_{[0]} \\ &\stackrel{(2.1)}{=} \langle (\varepsilon \otimes \text{id})(((1 \otimes f_1) \cdot m)_{[-1]}(1 \otimes f_2)), h \rangle ((1 \otimes f_1) \cdot m)_{[0]}. \end{aligned}$$

On the other hand,

$$\begin{aligned} m_{\langle 0 \rangle} \cdot h_1 \otimes m_{\langle 1 \rangle} h_2 &= \sum ((1 \otimes h^i) \cdot m) \cdot h_1 \otimes h_i h_2 \\ &= \sum \langle (\varepsilon \otimes \text{id})((1 \otimes h^i) \cdot m)_{[-1]}, h_1 \rangle ((1 \otimes h^i) \cdot m)_{[0]} \otimes h_i h_2. \end{aligned}$$

Evaluating the right-hand side on $\text{id} \otimes f$, we have

$$\begin{aligned} &\langle (\varepsilon \otimes \text{id})((1 \otimes f_1) \cdot m)_{[-1]}, h_1 \rangle ((1 \otimes f_1) \cdot m)_{[0]} f_2(h_2) \\ &= \langle (\varepsilon \otimes \text{id})(((1 \otimes f_1) \cdot m)_{[-1]}(1 \otimes f_2)), h \rangle ((1 \otimes f_1) \cdot m)_{[0]}. \end{aligned}$$

Hence $(m \cdot h_2)_{\langle 0 \rangle} \otimes h_1(m \cdot h_2)_{\langle 1 \rangle} = m_{\langle 0 \rangle} \cdot h_1 \otimes m_{\langle 1 \rangle} h_2$, since f was arbitrary.

We now prove (2.4):

$$\begin{aligned} &(m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]}, h \rangle (\text{id} \otimes \varepsilon^*)(m_{[0]_{[-1]}}) \otimes m_{[0]_{[0]}} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]1}, h \rangle (\text{id} \otimes \varepsilon^*)(m_{[-1]2}) \otimes m_{[0]} \\ &= (\text{id} \otimes h)m_{[-1]} \otimes m_{[0]} \\ &= \langle (\varepsilon \otimes \text{id})m_{[-1]2}, h \rangle (\text{id} \otimes \varepsilon^*)(m_{[-1]1}) \otimes m_{[0]} \\ &= m_{(-1)} \otimes m_{(0)} \cdot h, \end{aligned}$$

where in the third equality, $(\text{id} \otimes h)m_{[-1]}$ means that the second factor of $m_{[-1]}$ acts on h .

Therefore $M \in \mathcal{LR}(H)$. It is straightforward to verify that any morphism in ${}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$ is also a morphism in $\mathcal{LR}(H)$. The proof is completed. \square

Theorem 3.3. *Let H be a finite-dimensional bialgebra. Then we have a monoidal category isomorphism*

$$\mathcal{LR}(H) \cong_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}.$$

Moreover, if H is a Hopf algebra with bijective antipode S , they are isomorphic as braided monoidal categories. Consequently

$$\mathcal{LR}(H) \cong_{D(H \otimes H^*)} \mathcal{M},$$

where $D(H \otimes H^)$ is the Drinfeld double of $H \otimes H^*$.*

Proof. It is easy to see that the functor $F: \mathcal{LR}(H) \rightarrow_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ is monoidal and that $F \circ G = \text{id}$ and $G \circ F = \text{id}$. And for all $M, N \in \mathcal{LR}(H)$, and $m \in M$, $n \in N$,

$$\begin{aligned} m_{[-1]} \cdot n \otimes m_{[0]} &\stackrel{(3.2)}{=} \sum (m_{(-1)} \otimes h^i) \cdot n \otimes m_{(0)} \cdot h_i \\ &\stackrel{(3.1)}{=} \sum m_{(-1)} \cdot n_{\langle 0 \rangle} \otimes m_{(0)} \cdot n_{\langle 1 \rangle}. \end{aligned}$$

The proof is completed. □

Corollary 3.4. *(A, H) is an L-R-admissible pair if and only if $(A, H \otimes H^*)$ is an admissible pair (introduced in [6]) satisfying the condition (1.14) in [5].*

By the isomorphism in Theorem 3.3, we can obtain the following result of [2] directly.

Proposition 3.5. *Let H be a finite-dimensional Hopf algebra. The canonical braiding of $\mathcal{LR}(H)$ is pseudosymmetric if and only if H is commutative and cocommutative.*

Proof. From [3], the canonical braiding of $_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ is pseudosymmetric if and only if $H \otimes H^*$ is commutative and cocommutative. By the bialgebra structure of $H \otimes H^*$, the proof is completed. □

Acknowledgement. The authors are grateful to the referee for his/her valuable comments and suggestions which improved the presentation of this paper.

References

- [1] *C. Kassel*: Quantum Groups. Graduate Texts in Mathematics 155, Springer, New York, 1995. [zbl](#) [MR](#) [doi](#)
- [2] *F. Panaite, M. D. Staic*: More examples of pseudosymmetric braided categories. *J. Algebra Appl.* *12* (2013), Paper No. 1250186, 21 pages. [zbl](#) [MR](#) [doi](#)
- [3] *F. Panaite, M. D. Staic, F. Van Oystaeyen*: Pseudosymmetric braidings, twines and twisted algebras. *J. Pure Appl. Algebra* *214* (2010), 867–884. [zbl](#) [MR](#) [doi](#)
- [4] *F. Panaite, F. Van Oystaeyen*: L-R-smash product for (quasi-)Hopf algebras. *J. Algebra* *309* (2007), 168–191. [zbl](#) [MR](#) [doi](#)
- [5] *F. Panaite, F. Van Oystaeyen*: L-R-smash biproducts, double biproducts and a braided category of Yetter-Drinfeld-Long bimodules. *Rocky Mt. J. Math.* *40* (2010), 2013–2024. [zbl](#) [MR](#) [doi](#)
- [6] *D. E. Radford*: The structure of Hopf algebras with a projection. *J. Algebra* *92* (1985), 322–347. [zbl](#) [MR](#) [doi](#)
- [7] *L. Zhang*: L-R smash products for bimodule algebras. *Prog. Nat. Sci.* *16* (2006), 580–587. [zbl](#) [MR](#) [doi](#)

Authors' addresses: Daowei Lu (corresponding author), Department of Mathematics, Jining University, No. 1 Xingtian Road, Qufu 273155, Shandong, P. R. of China, e-mail: ludaowei620@126.com; Shuanhong Wang, School of Mathematics, Southeast University, No. 2 Sipailou, Nanjing 210096, Jiangsu, P. R. of China, e-mail: shuanhwang@seu.edu.cn.