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Some Steffensen-type dynamic inequalities on time scales

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Abstract

We consider some new Steffensen-type dynamic inequalities on an arbitrary time scale by utilizing the diamond-α dynamic integrals, which are characterized as a combination of the delta and nabla integrals. These inequalities expand some known dynamic inequalities on time scales, bind together and broaden some integral inequalities and their discrete analogs.

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1 Introduction

The renowned integral Steffensen inequality [28] is written as

$$\int_{b-\lambda}^{b} \phi(t)\,dt \le \int_{a}^{b} \phi(t)\psi(t)\,dt \le \int_{a}^{a+\lambda} \phi(t)\,dt, \tag{1.1}$$

where u is nonincreasing, $\lambda = \int_a^b \psi(t)\,dt$ and $0 \le \psi(t) \le 1$ on $[a,b]$. It is simple to notice that inequalities (1.1) are reversed if u is nondecreasing.

Also we have

$$\sum_{k=n-\lambda_2+1}^{n} \phi(k) \le \sum_{k=1}^{n} \phi(k)\psi(k) \le \sum_{k=1}^{\lambda_1} \phi(k) \tag{1.2}$$

such that $0 \le \psi(k) \le 1$, $\lambda_1, \lambda_2 \in \{1,\dots,n\}$ with $\lambda_2 \le \sum_{k=1}^{n} \psi(k) \le \lambda_1$. Inequality (1.2) is known as discrete Steffensen's inequality [15].

Stefan Hilger started the hypotheses of time scales in his PhD thesis [16] so as to bring together discrete and continuous analysis (see [17]). From that point onward, this theory has gotten a ton of consideration. The book due to Bohner and Peterson [9] regarding the matter of time scales briefs and sorts out a lot of time scales calculus.

Over the previous decade, a reasonable number of dynamic inequalities on time scales has been proven by many analysts who were propelled by certain applications (see [1–4, 9–14, 18, 29]). A few researchers created different outcomes concerning fractional calculus on time scales to deliver related dynamic inequalities (see [5–7, 24]).

Anderson, in [8], extended Steffensen's inequality to times scale with nabla integrals as follows:

$$\int_{b-\lambda}^{b} \phi(t)\nabla t \le \int_{a}^{b} \phi(t)\psi(t)\nabla t \le \int_{a}^{a+\lambda} \phi(t)\nabla t, \tag{1.3}$$

where u is of one sign and nonincreasing, $0 \le \psi(t) \le 1$ for every $t \in [a,b]_{\mathbb{T}}$, $\lambda = \int_a^b \psi(t)\nabla t$, and $b-\lambda, a+\lambda \in [a,b]_{\mathbb{T}}$.

By employing diamond-α integrals, Ozkan and Yildirim [21] gave a generalization of inequality (1.3) of the form:

If the following

$$\int_{l}^{b} w(t)\diamondsuit_\alpha t \le \int_{a}^{b} \phi(t)\diamondsuit_\alpha t \le \int_{a}^{\eta} w(t)\diamondsuit_\alpha t \quad \text{if } u \ge 0, t \in [a,b]_{\mathbb{T}},$$
$$\int_{l}^{b} w(t)\diamondsuit_\alpha t \ge \int_{a}^{b} \phi(t)\diamondsuit_\alpha t \ge \int_{a}^{\eta} w(t)\diamondsuit_\alpha t \quad \text{if } u \le 0, t \in [a,b]_{\mathbb{T}},$$

hold, then

$$\int_{l}^{b} u(t)w(t)\diamondsuit_\alpha t \le \int_{a}^{b} u(t)v(t)\diamondsuit_\alpha t \le \int_{a}^{\eta} u(\tau)w(t)\diamondsuit_\alpha t, \tag{1.4}$$

where $0 \le \psi(t) \le w(t)$ for all $t \in [a,b]_{\mathbb{T}}$ with $l, \eta \in [a,b]_{\mathbb{T}}$.

Also in [21], the authors have given the following interesting result:

$$\begin{aligned}
&\int_{b-\lambda}^{b} \phi(t)w(t)\diamondsuit_\alpha t + \int_{a}^{b} \bigl|\bigl[\phi(t)-\phi(b-\lambda)\bigr]z(t)\bigr|\diamondsuit_\alpha t \\
&\quad \le \int_{a}^{b} \phi(t)\psi(t)\diamondsuit_\alpha t \\
&\quad \le \int_{a}^{a+\lambda} \phi(t)w(t)\diamondsuit_\alpha t - \int_{a}^{b} \bigl|\bigl[\phi(t)-\phi(a+\lambda)\bigr]z(t)\bigr|\diamondsuit_\alpha t,
\end{aligned}$$

with u is nonincreasing, $0 \le z(t) \le \psi(t) \le w(t) - z(t)$ for every $t \in [a,b]_{\mathbb{T}}$, $\int_{b-\lambda}^{b} w(t)\diamondsuit_\alpha t = \int_a^b \psi(t)\diamondsuit_\alpha t = \int_a^{a+\lambda} w(t)\diamondsuit_\alpha t$, and $b-\lambda, a+\lambda \in [a,b]_{\mathbb{T}}$.

The following inequality is a special case of the above inequality: if we put $z(t) = M$ and $w(t) = 1$, so

$$\begin{aligned}
&\int_{b-\lambda}^{b} \phi(t)\diamondsuit_\alpha t + M\int_{a}^{b} \bigl|\phi(t)-\phi(b-\lambda)\bigr|\diamondsuit_\alpha t \\
&\quad \le \int_{a}^{b} \phi(t)\psi(t)\diamondsuit_\alpha t \\
&\quad \le \int_{a}^{a+\lambda} \phi(t)\diamondsuit_\alpha t - M\int_{a}^{b} \bigl|\phi(t)-\phi(a+\lambda)\bigr|\diamondsuit_\alpha t,
\end{aligned}$$

$a, b \in \mathbb{T}_\kappa^\kappa$ with $a < b$, $\lambda = \int_a^b \psi(t)\diamondsuit_\alpha t$, and $0 \le M \le \psi(t) \le 1 - M$ for all $t \in [a,b]_{\mathbb{T}}$.

Since its establishment, Steffensen's inequality has played crucial roles in numerous fields of mathematics, particularly in mathematical analysis. In the past several decades, numerous speculations and refinements of Steffensen's inequality have been given by different authors. A few researchers have focused on Steffensen's inequality related to local and conformable fractional integrals (see [22, 25, 26, 30]). For a comprehensive review, we refer the interested reader to the books [19, 20] and the references cited in them.

This article is about to extend some Steffensen-type inequalities given in [23] to a general time scale, and build up some new generalizations of the diamond-α dynamic Steffensen inequality on time scales. As special cases of our outcomes, we recapture the integral inequalities presented in the above mentioned paper. Our outcomes additionally give several original discrete Steffensen's inequalities.

We get the unique Steffensen inequalities by utilizing the diamond-α integrals on time scales. For $\alpha = 1$, the diamond-α integral moves toward becoming delta integral and for $\alpha = 0$ it moves toward becoming nabla integral. An excellent review about the diamond-α calculus can be viewed in the paper [27].

2 Basics of time scales

For our convenience, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integers, and a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, we have $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. The interval $[a,b]_\mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Assume the function $\phi : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, then $\phi^\Delta(t) \in \mathbb{R}$, $\phi^\nabla(t) \in \mathbb{R}$ are said to be the delta derivative and nabla derivative of ϕ at t, respectively, if for any $\varepsilon > 0$ there exist a neighborhood U and a neighborhood V of t such that, for all $s \in U$ and $s \in V$ simultaneously, we have

$$\left|\left[\phi\left(\sigma(t)\right) - \phi(s)\right] - \phi^\Delta(t)\left[\sigma(t) - s\right]\right| \leq \varepsilon\left|\sigma(t) - s\right|$$

and

$$\left|\left[\phi\left(\rho(t)\right) - \phi(s)\right] - \phi^\nabla(t)\left[\rho(t) - s\right]\right| \leq \varepsilon\left|\rho(t) - s\right|.$$

Moreover, ϕ is said to be delta differentiable on $\mathbb{T}^\kappa$ if it is delta differentiable at every $t \in \mathbb{T}^\kappa$ and is said to be nabla differentiable on $\mathbb{T}_\kappa$ if it is nabla differentiable at each $t \in \mathbb{T}_\kappa$.

There is the following formula of delta integration by parts on time scales:

$$\int_a^b \psi^\Delta(t)\phi(t)\Delta t = \psi(b)\phi(b) - \psi(a)\phi(a) - \int_a^b \psi^\sigma(t)\phi(t)\Delta t, \tag{2.1}$$

the nabla integration by parts on time scales is given by

$$\int_a^b \psi^\nabla(t)\phi(t)\nabla t = \psi(b)\phi(b) - \psi(a)\phi(a) - \int_a^b \psi^\rho(t)\phi(t)\nabla t. \tag{2.2}$$

We will use the following relations between calculus on time scales $\mathbb{T}$ and either differential calculus on $\mathbb{R}$ or difference calculus on $\mathbb{Z}$. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} &\sigma(t) = \rho(t) = t, \qquad \mu(t) = \nu(t) = 0, \qquad \phi^{\Delta}(t) = \phi^{\nabla}(t) = \phi'(t), \\ &\int_a^b \phi(t)\Delta t = \int_a^b \phi(t)\nabla t = \int_a^b \phi(t)\,dt. \end{aligned} \tag{2.3}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(t) = t+1, \qquad \rho(t) = t-1, \qquad \mu(t) = \nu(t) = 1, \\ &\phi^{\Delta}(t) = \Delta\phi(t), \qquad \phi^{\nabla}(t) = \nabla\phi(t), \\ &\int_a^b \phi(t)\Delta t = \sum_{t=a}^{b-1} \phi(t), \qquad \int_a^b \phi(t)\nabla t = \sum_{t=a+1}^{b} \phi(t), \end{aligned} \tag{2.4}$$

where the forward and backward difference operators are denoted by Δ and ∇, respectively.

We dedicate the rest of this section to the diamond-α calculus on time scales, and we recommend the paper [27] for further knowledge.

For any $t \in \mathbb{T}$, the diamond-α dynamic derivative of u at t is defined by

$$u^{\diamond_\alpha}(t) = \alpha u^{\Delta}(t) + (1-\alpha)u^{\nabla}(t), \quad 0 \le \alpha \le 1, \tag{2.5}$$

and denoted by $u^{\diamond_\alpha}(t)$, where $\mathbb{T}$ is a time scale, and u is a function that is delta and nabla differentiable on $\mathbb{T}$.

Now, it is time to discuss our main results.

3 Main results

Lemma 3.1 *Assume that*

(B1) *k is a positive $\diamond_\alpha$-integrable function on $[a,b]_{\mathbb{T}}$.*
(B2) *$\phi, \psi, h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are $\diamond_\alpha$-integrable functions on $[a,b]_{\mathbb{T}}$.*
(B3) *$[c,d]_{\mathbb{T}} \subseteq [a,b]_{\mathbb{T}}$ with $\int_c^d h(t)k(t)\diamond_\alpha t = \int_a^b \psi(t)k(t)\diamond_\alpha t$.*
(B4) *$z \in [a,b]_{\mathbb{T}}$.*

Then

$$\begin{aligned} \int_c^d \phi(t)h(t)\diamond_\alpha t - \int_a^b \phi(t)\psi(t)\diamond_\alpha t = {} & \int_c^a \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t \\ & + \int_c^d \left(\frac{\phi(t)}{k(t)} - \frac{\phi(z)}{k(z)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t \\ & + \int_d^b \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t. \end{aligned} \tag{3.1}$$

Proof By straightforward calculations, we get

$$\begin{aligned} &\int_c^d \phi(t)h(t)\diamond_\alpha t - \int_a^b \phi(t)\psi(t)\diamond_\alpha t \\ &\quad = \int_c^d k(t)\bigl[h(t)-\psi(t)\bigr]\frac{\phi(t)}{k(t)}\diamond_\alpha t - \left[\int_a^c \frac{\phi(t)}{k(t)}\psi(t)k(t)\diamond_\alpha t + \int_d^b \frac{\phi(t)}{k(t)}\psi(t)k(t)\diamond_\alpha t\right] \end{aligned}$$

$$\begin{aligned}
&= \int_a^c \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamondsuit_\alpha t + \int_c^d \left(\frac{\phi(t)}{k(t)} - \frac{\phi(z)}{k(z)}\right)k(t)\big[h(t) - \psi(t)\big]\diamondsuit_\alpha t \\
&\quad + \int_d^b \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamondsuit_\alpha t \\
&\quad + \frac{\phi(z)}{k(z)}\left[\int_c^d k(t)h(t)\diamondsuit_\alpha t - \int_a^c \psi(t)k(t)\diamondsuit_\alpha t\right. \\
&\quad \left. - \int_c^d \psi(t)k(t)\diamondsuit_\alpha t - \int_d^b \psi(t)k(t)\diamondsuit_\alpha t\right]. \qquad (3.2)
\end{aligned}$$

Consider

$$\int_c^d k(t)h(t)\diamondsuit_\alpha t = \int_a^b k(t)\psi(t)\diamondsuit_\alpha t,$$

therefore

$$\begin{aligned}
&\frac{\phi(z)}{k(z)}\left[\int_c^d k(t)h(t)\diamondsuit_\alpha t - \int_a^c \psi(t)k(t)\diamondsuit_\alpha t\right. \\
&\quad \left. - \int_c^d \psi(t)k(t)\diamondsuit_\alpha t - \int_d^b \psi(t)k(t)\diamondsuit_\alpha t\right] = 0. \qquad (3.3)
\end{aligned}$$

Our desired result follows directly from (3.2) and (3.3). □

Corollary 3.2 *Setting $\alpha = 1$ in Lemma 3.1, we get the delta form of inequality (3.1) by*

$$\begin{aligned}
\int_c^d \phi(t)h(t)\Delta t - \int_a^b \phi(t)\psi(t)\Delta t &= \int_c^a \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t \\
&\quad + \int_c^d \left(\frac{\phi(t)}{k(t)} - \frac{\phi(z)}{k(z)}\right)k(t)\big[h(t) - \psi(t)\big]\Delta t \\
&\quad + \int_d^b \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t. \qquad (3.4)
\end{aligned}$$

Corollary 3.3 *Letting $\alpha = 0$ in Lemma 3.1, we obtain the nabla version of (3.1) as follows:*

$$\begin{aligned}
\int_c^d \phi(t)h(t)\nabla t - \int_a^b \phi(t)\psi(t)\nabla t &= \int_c^a \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t \\
&\quad + \int_c^d \left(\frac{\phi(t)}{k(t)} - \frac{\phi(z)}{k(z)}\right)k(t)\big[h(t) - \psi(t)\big]\nabla t \\
&\quad + \int_d^b \left(\frac{\phi(z)}{k(z)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t. \qquad (3.5)
\end{aligned}$$

Corollary 3.4 *If $\mathbb{T} = \mathbb{R}$ in Corollary 3.2, then, with the help of relation (2.3), we recapture* [23, Lemma 2.1].

Corollary 3.5 *If $\mathbb{T}=\mathbb{Z}$ in Corollary 3.2, then, with the help of relation (2.4), inequality (3.4) becomes*

$$\begin{aligned}\sum_{t=c}^{d-1}\phi(t)h(t)-\sum_{a}^{b}\phi(t)\psi(t)\,dt&=\sum_{t=c}^{a-1}\left(\frac{\phi(z)}{k(z)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\\&\quad+\sum_{t=c}^{d-1}\left(\frac{\phi(t)}{k(t)}-\frac{\phi(z)}{k(z)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\\&\quad+\sum_{t=d}^{b-1}\left(\frac{\phi(z)}{k(z)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t).\end{aligned}$$

Theorem 3.6 *Let* (B1)–(B3) *of Lemma* 3.1,

(B5) *ϕ/k is nonincreasing, and*

(B6) $0\le\psi(t)\le h(t)$ $\forall t\in[a,b]_{\mathbb{T}}$

be satisfied, then the following inequalities hold:

$$\text{(i)}\quad \int_a^b\phi(t)\psi(t)\diamond_\alpha t\le\int_c^d\phi(t)\psi(t)\diamond_\alpha t+\int_a^c\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\diamond_\alpha t, \tag{3.6}$$

$$\text{(ii)}\quad \int_c^d\phi(t)\psi(t)\diamond_\alpha t-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t\le\int_a^b\phi(t)\psi(t)\diamond_\alpha t, \tag{3.7}$$

$$\begin{aligned}\text{(iii)}\quad \int_a^b\phi(t)\psi(t)\diamond_\alpha t&\le\int_c^d\phi(t)\psi(t)\diamond_\alpha t-\int_c^d\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t\\&\quad+\int_a^c\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\diamond_\alpha t\\&\le\int_c^d\phi(t)\psi(t)\diamond_\alpha t+\int_a^c\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\diamond_\alpha t,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\text{(iv)}\quad &\int_c^d\phi(t)\psi(t)\diamond_\alpha t-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t\\&\le\int_c^d\phi(t)\psi(t)\diamond_\alpha t+\int_c^d\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t\\&\quad-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t\\&\le\int_a^b\phi(t)\psi(t)\diamond_\alpha t.\end{aligned} \tag{3.9}$$

If ϕ/k is nondecreasing, then inequalities (3.6), (3.7), (3.8), *and* (3.9) *should be switched.*

Proof (i) Since ϕ/k is nonincreasing, k is positive, and $0\le\psi\le h$, we have

$$\int_c^d\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t\ge 0 \tag{3.10}$$

and

$$\int_d^b\left(\frac{\phi(d)}{k(d)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t\ge 0. \tag{3.11}$$

From (3.1), (3.10), and (3.11) with $z = d$, we obtain

$$\begin{aligned}&\int_c^d \phi(t)\psi(t)\diamond_\alpha t - \int_a^b \phi(t)\psi(t)\diamond_\alpha t - \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\diamond_\alpha t \\ &\quad= \int_c^d \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t + \int_d^b \left(\frac{\phi(d)}{k(d)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t \geq 0.\end{aligned}$$

This proves our claim.

(ii) Since ϕ/k is nonincreasing, k is positive, and $0 \leq \psi \leq h$, we have

$$\int_c^d \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t \geq 0 \tag{3.12}$$

and

$$\int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(c)}{k(c)}\right)\psi(t)k(t)\diamond_\alpha t \geq 0. \tag{3.13}$$

From (3.1) with $z = c$, (3.12), and (3.13), we have

$$\begin{aligned}&\int_a^b \phi(t)\psi(t)\diamond_\alpha t - \int_c^d \phi(t)\psi(t)\diamond_\alpha t - \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t \\ &\quad= \int_c^d \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\diamond_\alpha t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(c)}{k(c)}\right)\psi(t)k(t)\diamond_\alpha t \geq 0.\end{aligned}$$

This completes our proof.

The proof of (iii), (iv) is similar to (i), (ii) of Theorem 3.6, respectively. Details are omitted. □

Corollary 3.7 *Substituting* $\alpha = 1$ *and* $\alpha = 0$ *in Theorem* 3.6(i), (ii), (iii), (iv) *simultaneously, we achieve the following delta and nabla versions of inequalities* (3.6), (3.7), (3.8), *and* (3.9), *respectively*:

$$\text{(v)}\quad \int_a^b \phi(t)\psi(t)\Delta t \leq \int_c^d \phi(t)\psi(t)\Delta t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\Delta t, \tag{3.14}$$

$$\text{(vi)}\quad \int_c^d \phi(t)\psi(t)\Delta t - \int_a^c \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t \leq \int_a^b \phi(t)\psi(t)\Delta t, \tag{3.15}$$

$$\begin{aligned}\text{(vii)}\quad \int_a^b \phi(t)\psi(t)\Delta t &\leq \int_c^d \phi(t)\psi(t)\Delta t - \int_c^d \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\Delta t \\ &\quad+ \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\Delta t \\ &\leq \int_c^d \phi(t)\psi(t)\Delta t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\Delta t,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\text{(viii)}\quad &\int_c^d \phi(t)\psi(t)\Delta t - \int_a^c \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t \\ &\quad\leq \int_c^d \phi(t)\psi(t)\Delta t + \int_c^d \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)k(t)\bigl[h(t)-\psi(t)\bigr]\Delta t\end{aligned}$$

$$-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t$$
$$\leq\int_a^b\phi(t)\psi(t)\Delta t, \tag{3.17}$$

(ix) $$\int_a^b\phi(t)\psi(t)\nabla t\leq\int_c^d\phi(t)\psi(t)\nabla t+\int_a^c\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\nabla t,$$

(x) $$\int_c^d\phi(t)\psi(t)\nabla t-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t\leq\int_a^b\phi(t)\psi(t)\nabla t,$$

(xi) $$\int_a^b\phi(t)\psi(t)\nabla t\leq\int_c^d\phi(t)\psi(t)\nabla t-\int_c^d\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)k(t)\big[h(t)-\psi(t)\big]\nabla t$$
$$+\int_a^c\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\nabla t$$
$$\leq\int_c^d\phi(t)\psi(t)\nabla t+\int_a^c\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\nabla t,$$

(xii) $$\int_c^d\phi(t)\psi(t)\nabla t-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t$$
$$\leq\int_c^d\phi(t)\psi(t)\nabla t+\int_c^d\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)k(t)\big[h(t)-\psi(t)\big]\nabla t$$
$$-\int_a^c\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t$$
$$\leq\int_a^b\phi(t)\psi(t)\nabla t.$$

Corollary 3.8 *If* $\mathbb{T}=\mathbb{R}$ *in Corollary* 3.7(v), (vi), (vii), (viii), *then with the help of relation* (2.3), *we recapture* [23, Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4], *respectively.*

Corollary 3.9 *If* $\mathbb{T}=\mathbb{Z}$ *and applying* (2.4), *then inequalities* (3.14), (3.15), (3.16), *and* (3.17), *respectively, give*

$$\sum_{t=a}^{b-1}\phi(t)\psi(t)\leq\sum_{t=c}^{d-1}\phi(t)\psi(t)+\sum_{t=a}^{c-1}\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t), \tag{3.18}$$

$$\sum_{t=c}^{d-1}\phi(t)\psi(t)-\sum_{t=a}^{c-1}\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\leq\sum_{t=a}^{b-1}\phi(t)\psi(t), \tag{3.19}$$

$$\sum_{t=a}^{b-1}\phi(t)\psi(t)\leq\sum_{t=c}^{d-1}\phi(t)\psi(t)t-\sum_{t=c}^{d-1}\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)k(t)\big[h(t)-\psi(t)\big]$$
$$+\sum_{t=a}^{c-1}\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t)$$
$$\leq\sum_{t=c}^{d-1}\phi(t)\psi(t)+\sum_{t=a}^{c-1}\left(\frac{\phi(t)}{k(t)}-\frac{\phi(d)}{k(d)}\right)\psi(t)k(t), \tag{3.20}$$

$$\sum_{t=c}^{d-1}\phi(t)\psi(t)-\sum_{t=a}^{c-1}\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)$$

$$\begin{aligned}
&\le \sum_{t=c}^{d-1} \phi(t)\psi(t) + \sum_{t=c}^{d-1} \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)} \right) k(t)\big[h(t) - \psi(t)\big] \\
&\quad - \sum_{t=a}^{c-1} \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)} \right) \psi(t)k(t) \\
&\le \sum_{t=a}^{b-1} \phi(t)\psi(t).
\end{aligned} \tag{3.21}$$

Theorem 3.10 *Let* (B1)–(B3) *and*

(B7) ϕ/k *is nonincreasing in the* Δ *and* ∇ *sense,*

be fulfilled.

(i) *If*

$$\begin{aligned}
&\int_c^{\sigma(x)} k(t)\psi(t)\Delta t \le \int_c^{\sigma(x)} k(t)h(t)\Delta t, \quad c \le x \le d, \\
&\int_c^{\rho(x)} k(t)\psi(t)\nabla t \le \int_c^{\rho(x)} k(t)h(t)\nabla t, \quad c \le x \le d, \\
&\int_{\sigma(x)}^{b} k(t)\psi(t)\Delta t \ge 0, \quad d \le x \le b, \\
&\int_{\rho(x)}^{b} k(t)\psi(t)\nabla t \ge 0, \quad d \le x \le b,
\end{aligned}$$

then

$$\int_a^b \phi(t)\psi(t)\diamond_\alpha t \le \int_c^d \phi(t)h(t)\diamond_\alpha t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)} \right) \psi(t)k(t)\diamond_\alpha t. \tag{3.22}$$

(ii) *If*

$$\begin{aligned}
&\int_{\sigma(x)}^{d} k(t)\psi(t)\Delta t \le \int_{\sigma(x)}^{d} k(t)h(t)\Delta t, \quad c \le x \le d, \\
&\int_{\rho(x)}^{d} k(t)\psi(t)\nabla t \le \int_{\rho(x)}^{d} k(t)h(t)\nabla t, \quad c \le x \le d, \\
&\int_a^{\sigma(x)} k(t)\psi(t)\Delta t \ge 0, \quad a \le x \le c, \\
&\int_a^{\rho(x)} k(t)\psi(t)\nabla t \ge 0, \quad a \le x \le c,
\end{aligned}$$

then

$$\int_c^d \phi(t)h(t)\diamond_\alpha t - \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)} \right) \psi(t)k(t)\diamond_\alpha t \le \int_a^b \phi(t)\psi(t)\diamond_\alpha t. \tag{3.23}$$

Proof (i) Utilizing (3.4) and delta integration by parts formula on time scales, we get

$$\begin{aligned}
&\int_c^d \phi(t)h(t)\Delta t + \int_a^b \phi(t)\psi(t)\Delta t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\Delta t \\
&= \left[-\int_c^d \left(\int_c^{\sigma(x)} k(t)\big[h(t)-\psi(t)\big]\Delta t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\Delta}\Delta x\right] \\
&\quad \times \left[-\int_d^b \left(\int_{\sigma(x)}^b \psi(t)k(t)\Delta t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\Delta}\Delta x\right] \geq 0.
\end{aligned}$$

In a similar manner, using (3.5) and nabla integration by parts formula on time scales, we have

$$\begin{aligned}
&\int_c^d \phi(t)h(t)\nabla t + \int_a^b \phi(t)\psi(t)\nabla t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\nabla t \\
&= \left[-\int_c^d \left(\int_c^{\rho(x)} k(t)\big[h(t)-\psi(t)\big]\nabla t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\nabla}\nabla x\right] \\
&\quad \times \left[-\int_d^b \left(\int_{\rho(x)}^b \psi(t)k(t)\nabla t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\nabla}\nabla x\right] \geq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_c^d \phi(t)h(t)\diamond_\alpha t + \int_a^b \phi(t)\psi(t)\diamond_\alpha t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\diamond_\alpha t \\
&= \alpha\int_c^d \phi(t)h(t)\Delta t + (1-\alpha)\int_c^d \phi(t)h(t)\nabla t \\
&\quad + \alpha\int_a^b \phi(t)\psi(t)\Delta t + (1-\alpha)\int_a^b \phi(t)\psi(t)\nabla t \\
&\quad + \alpha\int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\Delta t \\
&\quad + (1-\alpha)\int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\nabla t \geq 0.
\end{aligned}$$

Hence, (3.22) holds.

(ii) Using (3.4) and delta integration by parts, we have

$$\begin{aligned}
&\int_a^b \phi(t)\psi(t)\Delta t - \int_c^d \phi(t)h(t)\Delta t + \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t \\
&= \left[-\int_a^c \left(\int_a^{\sigma(x)} \psi(t)k(t)\Delta t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\Delta}\Delta x\right] \\
&\quad \times \left[-\int_c^d \left(\int_{\sigma(x)}^d k(t)\big[h(t)-\psi(t)\big]\Delta t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\Delta}\Delta x\right] \geq 0.
\end{aligned}$$

Now, (3.5) and nabla integration by parts yield

$$\begin{aligned}
&\int_a^b \phi(t)\psi(t)\nabla t - \int_c^d \phi(t)h(t)\nabla t + \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t \\
&= \left[-\int_a^c \left(\int_a^{\rho(x)} \psi(t)k(t)\nabla t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\nabla} \nabla x\right] \\
&\quad \times \left[-\int_c^d \left(\int_{\rho(x)}^d k(t)\bigl[h(t) - \psi(t)\bigr]\nabla t\right)\left(\frac{\phi(x)}{k(x)}\right)^{\nabla} \nabla x\right] \geq 0,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_a^b \phi(t)\psi(t)\diamond_\alpha t - \int_c^d \phi(t)h(t)\diamond_\alpha t + \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\diamond_\alpha t \\
&= \alpha \int_a^b \phi(t)\psi(t)\Delta t + (1-\alpha) \int_a^b \phi(t)\psi(t)\nabla t \\
&\quad - \alpha \int_c^d \phi(t)h(t)\Delta t - (1-\alpha) \int_c^d \phi(t)h(t)\nabla t \\
&\quad + \alpha \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t \\
&\quad + (1-\alpha) \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t \geq 0,
\end{aligned}$$

from which (3.23) is satisfied. □

Corollary 3.11 *Setting* $\alpha = 1$ *and* $\alpha = 0$ *in Theorem* 3.10(i), (ii) *simultaneously, we obtain the delta and nabla versions of inequalities* (3.22) *and* (3.23), *respectively, as follows*:

$$\text{(i)} \quad \int_a^b \phi(t)\psi(t)\Delta t \leq \int_c^d \phi(t)h(t)\Delta t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\Delta t, \tag{3.24}$$

$$\text{(ii)} \quad \int_c^d \phi(t)h(t)\Delta t - \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\Delta t \leq \int_a^b \phi(t)\psi(t)\Delta t, \tag{3.25}$$

$$\text{(iii)} \quad \int_a^b \phi(t)\psi(t)\nabla t \leq \int_c^d \phi(t)h(t)\nabla t + \int_a^c \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)\nabla t,$$

$$\text{(iv)} \quad \int_c^d \phi(t)h(t)\nabla t - \int_d^b \left(\frac{\phi(c)}{k(c)} - \frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\nabla t \leq \int_a^b \phi(t)\psi(t)\nabla t.$$

Corollary 3.12 *If* $\mathbb{T} = \mathbb{R}$ *in Corollary* 3.11, *then, with the help of* (2.3), (i), (ii) *recover* [23, Theorem 2.5, Theorem 2.6], *respectively.*

Corollary 3.13 *If* $\mathbb{T} = \mathbb{Z}$ *in Corollary* 3.11, *then, with the help of relation* (2.4), *inequalities* (3.24) *and* (3.25) *turn into*

$$\sum_{t=a}^{b-1} \phi(t)\psi(t) \leq \sum_{t=c}^{d-1} \phi(t)h(t) + \sum_{t=a}^{c-1} \left(\frac{\phi(t)}{k(t)} - \frac{\phi(d)}{k(d)}\right)\psi(t)k(t)$$

and

$$\sum_{t=c}^{d-1}\phi(t)h(t)-\sum_{t=d}^{b-1}\left(\frac{\phi(c)}{k(c)}-\frac{\phi(t)}{k(t)}\right)\psi(t)k(t)\leq\sum_{t=a}^{b-1}\phi(t)\psi(t),$$

respectively.

The following theorem can be obtained by taking $c = a$ and $d = a + \lambda$ in Theorem 3.10.

Theorem 3.14 *Let* (B1)–(B3), (B7) *hold.*

(i) *If* λ *is defined by* $\int_a^{a+\lambda} h(t)k(t)\diamondsuit_\alpha t = \int_a^b \psi(t)k(t)\diamondsuit_\alpha t$,

$$\int_a^{\sigma(x)} k(t)\psi(t)\Delta t \leq \int_a^{\sigma(x)} k(t)h(t)\Delta t, \quad a\leq x\leq a+\lambda,$$
$$\int_a^{\rho(x)} k(t)\psi(t)\nabla t \leq \int_a^{\rho(x)} k(t)h(t)\nabla t, \quad a\leq x\leq a+\lambda,$$
$$\int_{\sigma(x)}^{b} k(t)\psi(t)\Delta t \geq 0, \quad a+\lambda\leq x\leq b,$$

and

$$\int_{\rho(x)}^{b} k(t)\psi(t)\nabla t \geq 0, \quad a+\lambda\leq x\leq b,$$

then

$$\int_a^b \phi(t)\psi(t)\diamondsuit_\alpha t \leq \int_a^{a+\lambda}\phi(t)h(t)\diamondsuit_\alpha t.$$

(ii) *If* λ *is given by* $\int_{b-\lambda}^{b} h(t)k(t)\diamondsuit_\alpha t = \int_a^b \psi(t)k(t)\diamondsuit_\alpha t$,

$$\int_{\sigma(x)}^{b} k(t)\psi(t)\Delta t \leq \int_{\sigma(x)}^{b} k(t)h(t)\Delta t, \quad b-\lambda\leq x\leq b,$$
$$\int_{\rho(x)}^{b} k(t)\psi(t)\nabla t \leq \int_{\rho(x)}^{b} k(t)h(t)\nabla t, \quad b-\lambda\leq x\leq b,$$
$$\int_a^{\sigma(x)} k(t)\psi(t)\Delta t \geq 0, \quad a\leq x\leq b-\lambda,$$

and

$$\int_a^{\rho(x)} k(t)\psi(t)\nabla t \geq 0, \quad a\leq x\leq b-\lambda,$$

then

$$\int_{b-\lambda}^{b}\phi(t)h(t)\diamondsuit_\alpha t \leq \int_a^b \phi(t)\psi(t)\diamondsuit_\alpha t.$$

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References

1. Abdeldaim, A., El-Deeb, A.A.: On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations. Appl. Math. Comput. **256**, 375–380 (2015). MR 3316076
2. Abdeldaim, A., El-Deeb, A.A., Agarwal, P., El-Sennary, H.A.: On some dynamic inequalities of Steffensen type on time scales. Math. Methods Appl. Sci. **41**(12), 4737–4753 (2018). MR 3828354
3. Agarwal, R., Bohner, M., Peterson, A.: Inequalities on time scales: a survey. Math. Inequal. Appl. **4**(4), 535–557 (2001). MR 1859660
4. Agarwal, R., O'Regan, D., Saker, S.: Dynamic Inequalities on Time Scales. Springer, Cham (2014). MR 3307947
5. Anastassiou, G.A.: Foundations of nabla fractional calculus on time scales and inequalities. Comput. Math. Appl. **59**(12), 3750–3762 (2010). MR 2651850
6. Anastassiou, G.A.: Principles of delta fractional calculus on time scales and inequalities. Math. Comput. Model. **52**(3–4), 556–566 (2010). MR 2658507
7. Anastassiou, G.A.: Integral operator inequalities on time scales. Int. J. Difference Equ. **7**(2), 111–137 (2012). MR 3000811
8. Anderson, D.R.: Time-scale integral inequalities. JIPAM. J. Inequal. Pure Appl. Math. **6**(3), Article ID 66 (2005). MR 2164307
9. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser Boston, Boston (2001). MR 1843232
10. El-Deeb, A.A.: Some Gronwall–Bellman type inequalities on time scales for Volterra–Fredholm dynamic integral equations. J. Egypt. Math. Soc. **26**(1), Article 1 (2018)
11. El-Deeb, A.A., Cheung, W.-S.: A variety of dynamic inequalities on time scales with retardation. J. Nonlinear Sci. Appl. **11**(10), 1185–1206 (2018). MR 3845535
12. El-Deeb, A.A., Elsennary, H.A., Cheung, W.-S.: Some reverse Hölder inequalities with Specht's ratio on time scales. J. Nonlinear Sci. Appl. **11**(4), 444–455 (2018). MR 3780318
13. El-Deeb, A.A., Elsennary, H.A., Nwaeze, E.R.: Generalized weighted Ostrowski, trapezoid and Grüss type inequalities on time scales. Fasc. Math. **60**, 123–144 (2018). MR 3846782
14. El-Deeb, A.A., Xu, H., Abdeldaim, A., Wang, G.: Some dynamic inequalities on time scales and their applications. Adv. Differ. Equ. **2019**, 130 (2019). MR 3934717
15. Evard, J.-C., Gauchman, H.: Steffensen type inequalities over general measure spaces. Analysis **17**(2–3), 301–322 (1997). MR 1486370
16. Hilger, S.: Ein makettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. PhD thesis (1988)
17. Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math. **18**(1–2), 18–56 (1990). MR 1066641
18. Li, W.N.: Some new dynamic inequalities on time scales. J. Math. Anal. Appl. **319**(2), 802–814 (2006). MR 2227939
19. Mitrinović, D.S.: Analytic Inequalities. Springer, New York (1970). MR 0274686
20. Mitrinović, D.S., Pečarić, J.E., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer Academic, Dordrecht (1993). MR 1220224
21. Ozkan, U.M., Yildirim, H.: Steffensen's integral inequality on time scales. J. Inequal. Appl. **2007**, Article ID 46524 (2007). MR 2335973
22. Pečarić, J., Perić, I., Smoljak, K.: Generalized fractional Steffensen type inequalities. Eurasian Math. J. **3**(4), 81–98 (2012). MR 3040688
23. Pečarić, J., Perušić, A., Smoljak, K.: Cerone's generalizations of Steffensen's inequality. Tatra Mt. Math. Publ. **58**, 53–75 (2014). MR 3242543
24. Sahir, M.: Dynamic inequalities for convex functions harmonized on time scales. J. Appl. Math. Phys. **5**, 2360–2370 (2017)
25. Sarikaya, M.Z., Tunc, T., Erden, S.: Generalized Steffensen inequalities for local fractional integrals. Int. J. Anal. Appl. **14**(1), 88–98 (2017)
26. Sarikaya, M.Z., Yaldiz, H., Budak, H.: Steffensen's integral inequality for conformable fractional integrals. Int. J. Anal. Appl. **15**(1), 23–30 (2017)

27. Sheng, Q., Fadag, M., Henderson, J., Davis, J.M.: An exploration of combined dynamic derivatives on time scales and their applications. Nonlinear Anal., Real World Appl. **7**(3), 395–413 (2006). MR 2235865
28. Steffensen, J.F.: On certain inequalities between mean values, and their application to actuarial problems. Scand. Actuar. J. **1918**(1), 82–97 (1918)
29. Tian, Y., El-Deeb, A.A., Meng, F.: Some nonlinear delay Volterra–Fredholm type dynamic integral inequalities on time scales. Discrete Dyn. Nat. Soc. **2018**, Article ID 5841985 (2018). MR 3847518
30. Tunç, T., Sarıkaya, M.Z., Srivastava, H.M.: Some generalized Steffensen's inequalities via a new identity for local fractional integrals. Int. J. Anal. Appl. **13**(1), 98–107 (2016)