Quantum Indefinite Spacetime: Part II

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We adopt an operational mode of thinking to study spacetime fluctuations and their impacts on questions of quantum gravity. Particular emphasis is put on studying causal structure fluctuations. The study is guided by the principle of causal neutrality, which says that fundamental concepts and laws of physics should be stated without assuming a definite spacetime causal structure. Most traditional concepts and theories of physics violate this principle. A major theme of this thesis is to upgrade the traditional concepts and theories in accordance with the principle of causal neutrality.

Among other things, the thesis include works on the following. (1) An axiomatic derivation of the complex Hilbert space structure of quantum theory without assuming definite causal structure. (2) A so-called causally neutral quantum field theory (CNQFT) framework for algebraic quantum physics allowing indefinite causal structure. (3) A proposal that causal fluctuation regularizes quantum field ultraviolet divergences (UV) and that the UV regularizing correlation functions with causal fluctuations characterize the UV structure of physical states. (4) A study on quantitative measures of causality. (5) A so-called "correlation networks" framework to study quantum theory with indefinite causal structure, which generalizes previous frameworks. (6) New definitions of entanglement and entanglement measures that conform to the causal neutrality principle. (7) Ideas on finding a causally neutral analogue of Einstein's equation for quantum spacetime. (8) A simple theorem showing that the communication capacities of general correlations (with definite or indefinite causal structure) defined with respect to state transmission can be reduced to those of channels.

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Dedication

To my father Hui Jia, who shows me to keep a broad vision, and my mother Li Jin, who shows me to think independently.

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Preface

The title of the thesis is *Quantum Indefinite Spacetime: Part II*, because it is a sequel to my previous Bachelor's thesis, *Quantum Indefinite Spacetime* [1]. Both theses study effects of quantum uncertainty and indefiniteness for spacetime itself, and "quantum indefinite spacetime" means spacetime with quantum indefiniteness and uncertainty.

In the Bachelor's thesis, I presented a generalization of the concept of entanglement applicable to quantum indefinite spacetime, an idea (supported by some theorems derived using entropic inequalities) that spontaneous spacetime causal structure fluctuations provide an ultraviolet regularization mechanism, some arguments that black holes in quantum indefinite spacetime do not contain strict causal boundaries, and a speculative idea on relating spacetime causal structure fluctuations to the apparent accelerated expansion of the universe. The thesis ends with an outlook on three directions for further research: incorporating indefinite causal structure into a theory of quantum fields, into a theory of quantum gravity, and into a theory of communication.

The current thesis picks up on all these three directions and presents some recent developments. It covers some new ideas that have not appeared elsewhere, as well as ideas that had appeared in my other writings during the completion of the Master's program [2, 3, 4, 5, 6]. All the different pieces are connected together under what I call the "principle of causal neutrality". The principle says that "fundamental concepts and laws of physics should be stated without assuming a definite spacetime causal structure", and is explained in more detail in Section 1.5.5. Most traditional theories and concepts of physics do not obey this principle. The works presented in the two theses can be viewed as attempts to upgrade various traditional theories and concepts to be in accordance with the causal neutrality principle.

Many further interesting questions had arisen through the study (Chapter 6). Hopefully we can make progress on some of them in the near future, maybe in a further sequel "Quantum Indefinite Spacetime: Part III".

Chapter 1

Foundational considerations

We describe this thesis as *adopting an operational mode of thinking to study spacetime fluctuations and their impacts on questions of quantum gravity.* Particular emphasis is put on studying causal structure fluctuations.

Many questions arise upon reading the above description. What are spacetime fluctuations? Why study spacetime fluctuations? Why adopt an operational mode of thinking? What are the questions of quantum gravity to be considered? What are causal structure fluctuations? Why emphasize on causal structure fluctuations in particular? This chapter addresses these foundational questions.

1.1 The generality of spacetime fluctuations

We regard length[1], time, and causal structure as the most basic properties of spacetime. Spacetime fluctuations manifest themselves as fluctuations of of these basic properties of spacetime.

One simple reason to study spacetime fluctuations would be that they exist. There are several reasons to expect the general presence of spacetime fluctuations in nature:

1. The equivalence principle implies that all forms of matter gravitate [7]. Quantum matter with quantum uncertainty is expected to gravitate with uncertainty, leading to spacetime fluctuations.

2. Many thought experiments of length and time measurements (for an instance see Example 1 below) lead to the conclusion that there are fluctuations in these properties (e.g. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). These fluctuations in length and time usually also imply fluctuations in the causal structure (Section 1.5.3).

3. In quantum theory all dynamical variables exhibit uncertainties. Spacetime causal structure is dynamical in general relativity, so is expected to exhibit indefiniteness in a quantum theory of gravity [18, 19].

[1] By "length" we mean spatial length.

As an example for point 2, we review Ng and Van Dam's analysis of length fluctuation in an operational measurement procedure.

Example 1 (Ng and Van Dam's analysis of length fluctuation [11, 12]**).** One learns from general relativity that mathematical spacetime coordinates do not have physical meaning independent of operational procedures of distance and duration measurements. Consider the following operational procedure of measuring the length between spatial location A and B. Put a clock at A and a mirror at B. Arrange the mirror so that a light signal sent from A towards B will be reflected and reach back to A. By recording the light signal's time of departure t_i and time of arrival t_f at A using the clock, one gets an operational procedure that assigns $l = c(t_f - t_i)/2$ as the length between A and B.

Quantum uncertainties in the clock and mirror positions introduce an inaccuracy δl for the length. The uncertainties for the clock and the mirror are expected to be of the same order, so concentrate on the clock uncertainty. If the clock has a positional uncertainty δx when the signal leaves at t_i, the position-momentum uncertainty relation indicates an uncertainty of $\delta v = \delta p/m \sim \hbar/m\delta x$ in velocity. After time $t_f - t_i = 2l/c$ the positional uncertainty grows to $\sim \delta x + \hbar l/mc\delta x$, where m is the mass of the clock. The minimum of this uncertainty is reached at $\delta x \sim (\hbar l/mc)^{1/2}$, and one concludes that the final positional uncertainty for the clock obeys

$$\delta l^2 \gtrsim \frac{\hbar l}{mc}. \tag{1.1}$$

This result based on position-momentum uncertainty relation was previously obtained by Salecker and Wigner in 1958 [8]. It says that increasing the clock mass increases the accuracy of distance measurement.

On the other hand, an analysis based on general relativity indicates that a less massive clock is preferable. Consider a light-clock with a beam of light bouncing along the diameter of size d in a spherical cavity. The mass of the cavity is m. To achieve a distance resolution δl, the clock must tick fast enough so that $\delta l/c \gtrsim d/c$. For the clock to be readable, it must be not form a black hole so $d > r_S = 2Gm/c^2$. This implies

$$\delta l \gtrsim \frac{Gm}{c^2}. \tag{1.2}$$

Reference [11] sketches an alternative way to arrive at this relation by considering the clock mass induced spacetime curvature effects.

The overall fluctuation is obtained by multiplying eq. (1.1) and eq. (1.2)

$$\delta l \gtrsim (l l_P^2)^{1/3}, \tag{1.3}$$

where $l_P = (G\hbar/c^3)^{1/2}$ is the Planck length.

A question arises naturally in considering this analysis carefully. The analysis is based on a particular clock-mirror operational procedure of length measurement. The spherical cavity clock is also only one particular type of clock. How fundamental is the relation 1.3 for length fluctuation? Can it be beaten by other operational procedures or other types of clocks?

In addition, the analysis does not involve any *direct* consideration of the fluctuation of the gravitational field. The fluctuation arises from the position-momentum uncertainty of the matter that makes up the clock and the classical gravitational limitation on the size of the spherical cavity. Can this kind of fluctuation be regarded as a spacetime/gravitational fluctuation? We discuss this question next.

Before moving on to the next section, we review another analysis of spacetime fluctuation with a different flavour based on a classical background spacetime. This gives a hint that spacetime fluctuations may be studied in quite distinct ways.

Example 2 (Ford and Svaiter's analysis of lightcone fluctuation [20, 21])**.** Perturbation around a background classical spacetime metric induces perturbations of the geodesic distances. Consider the perturbation

$$g_{ab} \to g_{ab} + h_{ab} \tag{1.4}$$

of the background metric g_{ab} by h_{ab}. Then the geodesic distance between two points x and x' becomes

$$\sigma_0 \to \sigma = \sigma_0 + \sigma_1 + O(h_{ab}^2), \tag{1.5}$$

where σ_0 is the geodesic distance in the unperturbed metric, and σ_1 is the perturbation of first order in h_{ab}.

This perturbation impacts the field propagators. In flat spacetime background the unperturbed retarded propagator for a massless scalar field is

$$G^{(0)}_{\mathrm{ret}}(x - x') = \frac{\theta(t - t')}{4\pi}\delta(\sigma_0). \tag{1.6}$$

Near the perturbed lightcone, replace the propagator with

$$G_{\mathrm{ret}}(x - x') = \frac{\theta(t - t')}{4\pi}\delta(\sigma) = \frac{\theta(t - t')}{8\pi^2}\int_{-\infty}^{\infty} d\alpha e^{i\alpha\sigma_0} e^{i\alpha\sigma_1}. \tag{1.7}$$

Now quantize the perturbation to turn σ_1 into an operator. Take a squeezed vacuum state $|\psi\rangle$ so that σ_1 can be decomposed into positive and negative frequency parts $\sigma_1 = \sigma_1^+ + \sigma_1^-$ so that

$$\sigma_1^+ |\psi\rangle = 0, \quad \langle\psi| \sigma_1^- = 0. \tag{1.8}$$

Use the creation and annihilation operators σ_1^+ and σ_1^- to write

$$e^{i\alpha\sigma_1} = e^{i\alpha(\sigma_1^+ + \sigma_1^-)} = e^{i\alpha\sigma_1^-} e^{-\frac{1}{2}\alpha^2[\sigma_1^+, \sigma_1^-]} e^{i\alpha\sigma_1^+}, \tag{1.9}$$

where the Campbell-Baker-Haussdorff formula is used in the last step. The expectation value with respect to $|\psi\rangle$ can be derived using $[\sigma_1^+, \sigma_1^-] = \langle\sigma_1^2\rangle$

$$\left\langle e^{i\alpha\sigma_1} \right\rangle = e^{-\frac{1}{2}\alpha^2 \langle\sigma_1^2\rangle}. \tag{1.10}$$

Plug this in Equation (1.7) to obtain

$$G_{\mathrm{ret}}(x - x') = \frac{\theta(t - t')}{8\pi^2} \sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}} \exp\left\{ -\frac{\sigma_0^2}{2\langle\sigma_1^2\rangle} \right\}. \tag{1.11}$$

1.2 What fluctuations are fundamental to spacetime?

Can one draw conclusions about fundamental spacetime fluctuations from analyzing one particular operational procedure of measurement for the basic spacetime properties. Our answer is no. In fact, it was admitted in [12] that the analysis about length fluctuation in Example 1 was not conclusive enough to draw a definitive conclusion about the scaling of fundamental spacetime length fluctuations.[2] In our view a spacetime fluctuation can be regarded as fundamental only if it passes the test in the following proposition.

Proposition 1. A fluctuation of some basic property of spacetime is fundamental to spacetime if and only if it cannot be reduced in any operational procedures of measuring this property.

Here reducing a fluctuation means eliminating the amount of the fluctuation as quantified by some measure. For example, some fluctuation in the statistics may be due to the limitation of the sensitivity of the measurement device. A device with better sensitivity may be used to reduce the statistical fluctuation.

The fluctuations of the basic spacetime properties of length, time and causal structure will be characterized using some suitable measures, such as the standard

[2] In [12] the author establishes a connection to the holographic principle to add support to the analysis.

deviation for a set of data of numbers. The causal structure is usually expressed as an order relation but not numbers. However, there are measures of causality in the form of numbers (see Chapter 4), based on which the fluctuations of causal structure can be quantified.

The general idea behind Proposition 1 is that a feature fundamental to spacetime should be universal. It affects all operational procedures that give meaning to the basic property of spacetime.

Proposition 1 gives a necessary and sufficient condition for a fluctuation to be fundamental to spacetime. The condition is necessary, because if the fluctuation is *fundamental* to spacetime is should not be possible to reduce it using any operational procedure of measurement. The condition is sufficient. If the fluctuation contributes to all operational procedures of measurement, it should be regarded as fundamental to spacetime.

In the argument that the condition is sufficient, we took the operational perspective to conceive spacetime and its basic properties. If all operational procedures to determine some property have a common feature, we regard this common feature as fundamental. The operational approach to quantum gravity is assessed in Section 1.4.

1.3 Is the fluctuation gravitational?

Question: Suppose we identified some fluctuation as fundamental to spacetime. Is this fluctuation always gravitational?

A motivation for the question comes from Example 1. The uncertainty in the length is analyzed without considering quantum properties of the spacetime metric g^{ab}, but is based directly on the uncertainties of the matter (clock and mirror in this case). Should not an analysis of quantum gravitational fluctuations be based on the quantum properties of a quantized g_{ab} (or analogous entities)?

The answer is twofold. First, under the equivalence principle all forms of matter gravitate, and an analysis of uncertainty based on matter *is* effectively and indirectly an analysis of uncertainty based on gravity. Second, one can say that there are further uncertainties not accounted for in the analysis of Example 1. These come from the gravitational uncertainties not yet reflected in the matter of clock and mirror, e.g., from the quantum gravitational degrees of freedom in the empty space between the clock and the mirror. Related to this we note that Ford and Svaiter distinguish active ("spontaneous" fluctuations by quantum gravitational degrees of freedom) and passive (induced by quantum matter) spacetime fluctuations [20, 21]. Regarding the analysis of Example 1, one could make it more comprehensive by adding towards the spacetime fluctuations contributions from quantum gravitational degrees of freedom not reflected in the matter, as Amelino-Camelia did in his analysis [13]. To sum up our opinion, matter induced fundamental fluctuations of spacetime properties should be considered gravitational, but they may not account for all fundamental gravitational fluctuations.

This discussion on what counts as gravitational fluctuations highlights an important aspect of an operational approach to quantum gravity new to most traditional approaches. In most traditional approaches to quantum gravity, there is some *autonomous* entity that encodes quantum spacetime/gravity degrees of freedom (e.g., string, spin network, spin foam, causal set etc.). In these approaches spacetime/gravitational features such as fluctuations are completely determined by these entities. In this sense these entities autonomously govern spacetime/gravity.

In contrast, in an operational approach, spacetime/gravitational features may be inferred from properties and relationships of matter degrees of freedom. Fundamental statements about spacetime/gravity may be made on the basis of analyzing operational procedures without ever directly analyzing an entity that autonomously encode spacetime/gravitational degrees of freedom. A strategic discussion of the operational approach is the topic of the next section.

1.4 On the operational approach[3]

What do we mean by the "operational approach to quantum gravity" in this thesis? A cornerstone for the approach is Hardy's perspective on physical theories [18, 19]:

> A theory of physics, whatever else it does, must correlate recorded data.

All theories of physics must admit a description at the level of operations (which register data) and correlations. So must a satisfactory theory of quantum gravity that would resolve the problems of quantum gravity (Section 1.6). The operational approach to quantum gravity we talk about in this thesis is then the approach that treats operations and correlations as central concepts in an attempt to resolve open problems of quantum gravity.

Quantum gravity is a difficult subject full of dangers and pitfalls. It is interesting to observe while traditional approaches usually face the danger of postulating too much, the operational approach faces the danger of postulating too little. String theory [22], for example, postulates that the fundamental degrees of freedom of spacetime are strings. Many results also assume supersymmetry. The danger is that these

[3] I am aware that some readers are not impressed by operational approaches to physics and prefer to formulate physics without reference to agents. In fact, as far as physical events are concerned the essential aspect of an "operation" is exactly that a certain possibility (such as a physical variable taking a particular value) is realized out of the many possibilities. It is inessential whether the "operation" is performed by a human being or by nature itself as the agent. Therefore in this thesis we refer to "agents" and "operations" in a generalized sense, which incorporates operations performed by nature as the agent. Hence for those who have other preferred modes of thinking, most of this thesis can be understood without committing to an operational mode of thinking that involves agents. Whenever the word "operational outcome" appears, one can think of it as a physical variable. Whenever the word "operation" appears, one can think of it as having some physical variable taking some value(s). The "operation-correlation perspective" to be presented becomes a "variable-correlation perspective".

assumptions may not be fulfilled in Nature. The operational approach, on the other hand, can incorporate quantum features of spacetime without postulating about the microscopic degrees of freedom of spacetime (Chapters 2 and 3). This reduces the risk of making false assumptions, but faces the threat of not having enough material to solve problems.

Of course, finding unknown microscopic structure/degree of freedom of spacetime (if there is any) may be regarded as a fundamental problem of quantum gravity, and refraining from saying anything about it will not solve it. Yet quantum gravity is a special subject where empirical data that can falsify the tentative assumptions about the microscopic structure of spacetime are limited. Regarding other problems of quantum gravity, it is preferable to find their solutions independent of some tentative assumptions, if such solutions can be found.

The operational approach to quantum gravity is relatively new in comparison to traditional approaches.[4] At this stage it is not clear which problems of quantum gravity cannot be solved without introducing additional assumptions. Some day some detailed assumption about spacetime may be shown to be realized in Nature, refuting competitive assumptions and results based on them. There could be some problem that can only be solved by knowing the confirmed assumption, and the operational approach, without making the assumption, would not be able to offer a solution to this problem. Yet results on other problems offered by the operational approach will likely survive and help identify the minimal logical conditions for the resolution of certain problems, once we have a full theory of quantum gravity.

The operational approach can be said to have a "*worry less, live more*" character. By reducing the worries about making tentative assumptions it is expected to offer some long-lasting results. The operational approach is also free from some other common worries held by other approaches of quantum gravity [27, 28, 29], such as the need to be background independent, the need to be non-perturbative, the need to identify physically meaningful observables about quantum spacetime, the need to find a probability rule for quantum spacetime, and the problem of time. When no autonomous spacetime/gravitational entity is assumed, the needs to be background independent and non-perturbative are automatically fulfilled. When spacetime properties (such as causal structure) are inferred from physical operations, which have well-conceived probability rules, and inferred relationally among the operations, the physical observables and probabilities rules are clear, and there is no problem of time.

[4] Some related early works are [23, 24, 18, 19]. The distinction between "top down" and "bottom up" approaches to quantum gravity was stressed by Hardy already in 2005 [18]. See [25] for an exposition from an information theory perspective and the references therein for some previous works that can be said to belong to the operational approach. See [26] for some arguments for discarding certain familiar notions of physics motivated by operational consideration.

1.5 Causal fluctuations

We mentioned in Section 1.1 that we regard length, time, and causal structure as the most basic properties of spacetime. In quantum spacetime all these properties are expected to exhibit fluctuations. We have touched on length fluctuation in Example 1 (which can easily be turned into an analysis of temporal fluctuation [12]). Now we turn to causal fluctuations.

1.5.1 Primacy of causal fluctuations

One difficulty in studying fundamental spacetime length or time fluctuations is the multiplicity of operational procedures for measuring length or time. We would like to know some properties about fundamental spacetime length and time fluctuations such as whether they scale according to Example 1. Yet in view of Proposition 1, to establish such properties fundamental to spacetime fluctuations we need to demonstrate that they hold for all operational procedures. This is not an easy job, given the multiplicity of operational procedures for measuring length and time (using rods, light clocks, atomic clocks, astrophysical clocks, the scaling of forces, the scaling of field correlations [30], the scaling of field entanglement etc.).

On the other hand, there seems to be a preferred operational procedure to measure the causal structure among events – by transmitting signals. Indeed, one way to *define* the causal structure is through the capability of signal sending – causal connectedness is qualified by the possibility of sending signals. There could potentially be other operational procedures to measure causal structure, but the signalling procedure is so basic that it is plausible that properties about spacetime causal structure fluctuation derived from it hold for all other operational procedures, whence the properties themselves are fundamental to spacetime.

There are additional advantages for studying fundamental spacetime fluctuations in terms of causal structure rather than length or time. Causal structure is invariant under changes of reference frames[5], while spatial length and temporal duration are not. Moreover, familiar operational procedures of length/time measurement only apply to special cases. For example, the length measurements in Example 1 loses meaning when spacetime is evolving fast (e.g., swiftly expanding), and some stationarity property should be assumed for it to work [13]. As another example, the length/time measurements based on statistical correlations such as the field correlation function or entanglement seem to require many measurements to gather enough statistics, and the length/time measured can best be viewed as an average over an ensemble of many similar pairs or groups of events. On the other hand, the signal-sending procedure of causal structure measurement applies to arbitrary spacetime scenarios (e.g., no stationarity requirement), and applies to particular pairs or groups

[5]This is certainly true for classical spacetime, and so far we see no reason to challenge it for quantum spacetime.

of operational event.

There is a potential limitation that some physical object blocks the way of signal sending from A to B, so that although in terms of spacetime causal relation A is causally connected to B, signals from A cannot reach B. However, in principle there is always a way to send some signal despite the block. For example, gravitational waves cannot be shielded, so in principle they can be used to signals despite the block. With the existence of these "penetrating" signals as support, we assume that the spacetime causal structure can be identified with the operational causal structure. Implicit behind this assumption is that the penetrating signals are available for signal sending.

All these considerations taken together, causal structure seems to be a preferable basic property of spacetime in studying fundamental spacetime fluctuations. With this understanding, we turn to a more detailed investigation of causal structure and its fluctuations. For simplicity, we will often refer to the fluctuations of spacetime causal structure as "causal fluctuations" in the rest of the thesis.

1.5.2 Causal relation of what?

Causal relation is fundamentally causal relation of physical events. For classical spacetime the causal relation of physical events is reduced to the causal relation of sets on a spacetime manifold [31, 32]. The reasoning is that all physical events happen over certain spacetime regions. These regions correspond to sets on a spacetime manifold. The causal relation of these sets are determined by the mathematical properties of the spacetime manifold itself. In the end, to determine the spacetime causal relations of physical events, one only needs to study the causal relations of sets on a spacetime manifold.

In the special case that the sets are singletons, the causal relation reduces to be about points on the spacetime manifold. In GR literature, the points of the spacetime manifold are called "events" [33]. This notion of "event" should be distinguished from the "physical events" we talk about, which refers to physical happenings and is meaningful for quantum spacetime even if the point on spacetime manifold notion of event may lose relevance.

For quantum spacetime the manifold model of spacetime (differentiable manifold plus spacetime metric) is no longer available. The spacetime causal relation of physical events must be studied on a new basis. One option is to postulate a new model of spacetime (e.g., causal set [34], spin foam [35], quantum graphity [36] etc.) and reduce the spacetime causal relation of physical events to properties of the new spacetime model.

We adopt an alternative model-independent approach. We refrain from postulating any particular model of quantum spacetime, and study spacetime causal relation of physical events directly without "embedding" the physical events in a model of quantum spacetime. But what mathematical model is used for this study? Answer:

a model of physical events.[6] In the rest of this section we reflect on modelling physical events. As will be shown in details in Chapter 2, the correlations among variables associated with physical events encode causal relations.

What makes a physical event a physical event? There are different characterizations of physical events. For example, a physical event can correspond to the "(approximate) actuality (i.e., probability equal to, or close to, unity) of some property" [37]. As another example, Einstein conceived spacetime events as affirmations of *coincidences* [38]. For instance, for events regarding the motion of material points, Einstein thinks what we actually only observe are coincidences of material points, such as the material points of measuring instruments with other material points being measured.[7] Einstein's perspective on events as coincidences was developed by Westman and Sonego [39, 40], and Hardy [41] to formulate GR in manifestly diffeomorphism invariant ways and to pave ways for a quantum treatment of spacetime. In these cases the coincidences are for the values of physical scalar fields. Note that the fields are not conceived to live on an independent spacetime manifold, but actually themselves form spaces to locate physical events on. As a final example, Lloyd, drawing lessons from quantum information and computation, requires that different elementary spacetime events be associated with (fiducial) physical clocks/detectors with orthogonal quantum states [42].

The starting point of our conception of physical event is that any physical event must be associated some in principle possible data gathering. The data can be the actualized property, the coincidence of the material points or the values of physical scalar fields, or the evolution of a quantum state evolving into an orthogonal one. Conversely, the confirmation of all the defining criterion for events listed in the above examples, the actuality of properties, the coincidence of the material points or values of physical scalar fields, and the emergence of a distinguishable physical state, must be accompanied by some in principle possible data gathering. We therefore make the assertion that *physical events are associated with data gathering.*

This then ties in with the operational approach of Section 1.4. Data are recorded through operations, and causal relation of the data and the operations that record the data can be reflected in the correlations. Therefore it is possible to study the causal relation of physical events in physical theories that take operations and correlations as the basic concepts. We think will call physical events characterized by operations by the name *operational events*, and use the two terms physical events and operational events interchangeably in the rest of the thesis when no ambiguity arises.

There have been many previous works on thinking of events without a background spacetime structure and studying spacetime properties such as causal structure in terms of the correlations among events. For some recent developments, see, for instance, [41, 43, 25, 44, 45, 46].

[6] One may call this type of approach, à la Wheeler, a "model without model" for spacetime.

[7] The context of this conception of events was to eliminate spacetime conceived as independent of the matter content and promote the principle of general covariance.

There is a final point that not all physical/operational events are meant to be localized in spacetime. For example, think of the operational event of Alice starting to have breakfast on a typical day. This is not meant to take place at a definite time and place, and can easily (supposing Alice is not a very regular person in terms of when to start having breakfast) generate indefinite causal relation with respect to some other operational event, e.g., Alice's hand watch showing 8:00 am. This kind of indefinite causal relation tells us little about spacetime causal structure, because the operational event is not defined with respect to a reference frame that is supposed to characterize spacetime. In studying spacetime causal structure fluctuations, we focus on *spacetime physical events*/*spacetime operational events* defined with intention to characterize spacetime events (with respect to some spacetime reference frame such as the one in Example 1).

1.5.3 From length and time fluctuations to causal structure fluctuations

Length and time fluctuations can usually be viewed as causal fluctuations. Consider Example 1 for an illustration. Recall that in the thought experiment a light signal is sent from location A towards B, where there is a mirror that reflects the light signal back towards A. The times of the signal leaving A, t_i, and reaching back A, t_i, are recorded with a clock at A. A length $l = c(t_f - t_i)/2$ is assigned to be the distance between the spatial locations A and B, and an analysis yields a bound eq. (1.3) on the uncertainty in the length δl.

To see that this fluctuation in length implies a fluctuation in causal relation for some operations, define a first operation to consist of the receiving and reflecting of the signal at B. Without loss of generality we can assume that the signal is always sent out at A at a fixed time t_i reading on the clock. Then the positive δl implies that the final time t_f fluctuates. Now if we define a second operation to take place at A at some fixed time t according to the clock reading, however we choose the time we cannot guarantee that the signal from the first operation always reaches the second operation without uncertainty. In a realistic setting the second operation takes place in over an extended spatial region and during an extended time period. Provided the extensions are small enough there will still be an uncertainty in whether the signal will be received by the second operation.

In general, fluctuations in length and time in a measurement procedure based on signal sending always yields causal fluctuations based on the same signaling procedure. Fluctuation in length and time are measured against some physical reference system. This could be a clock as in Example 1, or rods and other references in other cases. The reference system can be used to define operations independently of the signal arrival. In the example above we used the clock reading to define a second operation independently of when the light signal arrives at location B. Now, fluctuations of signal arrival with respect to the reference system implies indefiniteness in whether the signal reaches the independently defined operation. Hence fluctuations in length and time based on signal sending are turned into fluctuations of causal structure.

One might think of only defining operations according to signal arrival. For instance, in the above example define the second operation to be conducted always when the light signal is received at A. Then it is true that the first operation is always causally connected to the second by the signal, but the point is that it is always possible to define some other fiducial operations defined according to some reference system reading, and such operations always has indefinite causal relation with some other operations used in the length/time measurement procedure. Requiring that operations be exclusively defined with respect to signal arrival is not reasonable.

One important point in the above discussion is that the extension of the operations matters. If the second operation is so extended that it surrounds the first operation and that the signal is always received by the second operation, then there is no fluctuation in the causal relation between the two operations. In this case the causal structure is trivial. In general we would like to gather as much information of spacetime as possible through properties of the operational events. Then it is reasonable to consider operational events with small extensions, because these operational events yield more information about the causal structure than those with larger extensions. For example, for a classical spacetime manifold causal relations are generally defined for spacetime regions [31, 32], but in discussions of spacetime causal structure attention is usually paid to pointlike spacetime events since they are more informative about the spacetime causal structure than regions. For quantum spacetime we also want to gather as much information about spacetime causal structure as possible, so we usually want to focus on operational events with small extensions, as small as possible so that the intended operation can still be conducted. By the above analysis operational events with smaller extension are also able to detect causal fluctuations more sensitively than operational events with larger extensions.

1.5.4 On the generality of causal fluctuations

In Section 1.5.3 we considered indefinite causal structure with respect to some particular signal sending procedure. Yet spacetime causal structure is *universal* and constrains all signal sending procedures. To consider fluctuations of spacetime causal structure we need a condition that deals with all signal sending procedures (e.g., not just light signals).

Two spacetime physical events should be regarded to have an indefinite causal relation if there is a signalling direction for which: 1) There is at least one procedure to signal from one to the other with a positive chance, and 2) There is no procedure to signal with a certain chance.

Is fundamental (in the sense of Section 1.2) spacetime causal structure fluctuation a generically present effect? Based on the last subsection and reasons mentioned in Section 1.1, it is plausible for the answer to be yes.

This has a profound implication, which we discuss next.

1.5.5 The principle of causal neutrality

If spacetime causal fluctuation is a generically present effect, a profound consequence is what we call the **principle of causal neutrality**.[8]

> Fundamental concepts and laws of physics should be stated without assuming a definite spacetime causal structure.

Physical theories should make sense under the generic presence of indefinite spacetime causal structure, and their fundamental concepts should not rely on the assumption of a definite spacetime causal structure. The theories and concepts should be neutral to whether there is a definite spacetime causal structure in the natural world.[9]

Many concepts and theories are "causally biased" to assume there is a definite spacetime causal structure. For example, entanglement is traditionally defined for systems on tensor product Hilbert spaces. Hence conventionally entanglement is only considered for causally disconnected systems. In the presence of causal fluctuation the systems' causal structure becomes indefinite, and hence the traditional concept of entanglement loses meaning. As an example for a theory that is causally biased, consider traditional quantum field theory in flat or curved spacetime. The assumption of a definite spacetime causal structure comes directly into the axioms (e.g., microcausality, spectral condition). In Chapter 3 and Section 5.1 we present some ideas on making quantum field theory and entanglement causally neutral.

Other theories and concepts that are not causally neutral are abundant. In fact, any theory/concept based on classical spacetime is likely not causally neutral, e.g., the speed of light, the notions of locality (no superluminal signalling, no superluminal causal influence, no spacelike causal influence, fields interact at local points in classical spacetime), the notions of spacetime/black hole horizons.

In the next section we list some basic questions of quantum gravity. Obtaining causally neutral analogues of the above concepts is crucial to making progress in addressing these questions. In our view, implementing the principle of causal neutrality

[8] To our knowledge Hardy [18, 19] first formulated a framework of general probabilistic theories (including quantum theories) that does not assume definite spacetime causal structure. The term "causally neutral" was used by Leifer and Spekkens in a different but closely related context to develop quantum theory as a generalization of classical probability theory [47, 48]. There the motivation is that inference in classical probability theory is independent of the causal structure of the conditioned and the conditioning variables. It was hoped that quantum theory can be reformulated in an analogous way.

[9] One might think that it suffices for the theories and concepts to apply only when there is indefinite causal structure and that asking them to be neutral – they also have to apply when there is definite causal structure – is too much. It is reasonable, though, that in special circumstances there is a definite spacetime causal structure and we want the theories and concepts to still apply in these circumstances.

to upgrade traditional concepts and theories of physics is a major task of quantum gravity.

1.6 Questions and opportunities

This thesis adopts an operational mode of thinking to study spacetime fluctuations and their impact on questions of quantum gravity, with particular emphasis on studying causal structure fluctuations. In comparison to other approaches, this is a road less traveled, and a lot of questions await investigation. In this section we discuss the challenges and the opportunities of this relatively new approach towards quantum gravity.

There are many questions a theory quantum gravity may solve. Among them are questions that are believed must be addressed by a theory of quantum gravity:

1. Tell how quantum matter gravitates.
2. Tell what actually happens at singularities of classical gravity.
3. Tell whether and how black holes radiate and evaporate, as well as how they process information.

The approach under discussion has the potential to say something valuable on all these questions. For example, in this thesis we inquire about the ultraviolet/short distance structure of field correlations with indefinite causal structure in details. This can lead to some chain reaction that leads to progress towards the above questions. The causal fluctuation UV regularization mechanism (Section 3.4) may resolve singularities and tell about black hole information processing. In addition, the regularization mechanism applied to field entanglement may lead to an analogue of Einstein's equation in quantum spacetime and address the question how quantum matter gravitates.

There are also questions that quantum gravity may or may not address. As examples: explain "dark energy": explain "dark matter"; offer alternatives to cosmic inflation; unify the four fundamental forces of nature; explain why spacetime is four dimensional etc. Unlike questions of the previous kind, these questions do not necessarily arise from deficiencies of general relativity and quantum theory, and hence their resolution may not come from quantum gravity.

These questions will not be the focus of this thesis, but the approach under discussion may have something to say about them as well. For example, see [1] for an idea on "dark energy" and cosmology.

Chapter 2

Basic framework: operations and correlations

This chapter develops the operation-correlation perspective presented in Section 1.4. Section 2.1 specifies a basic framework, and Sections 2.2 to 2.6 illustrate how classical mechanics, ordinary quantum theory, local quantum physics/quantum field theory, and quantum theories with indefinite causal structure can be formulated within the basic framework. The chapter concludes with an original axiomatization of quantum theories without assuming definite causal structure based on [6] in Section 2.7.

2.1 Physical theories as theories of operations and correlations

The backbone of all the considerations in the rest of the chapter is Hardy's perspective on physical theories [18, 19][1]:

> A theory of physics, whatever else it does, must correlate recorded data.

There are other things a theory of physics can do, such as categorizing the constituents of the universe and offering a picture of reality, but at a minimum, whatever else the theory does, it must correlate recorded data. Because data are always recorded through some operations, this perspective on physics brings us to an operational mode of thinking, focusing particularly on the central concepts of *operation* and *correlation*.

2.1.1 Operation

An operation consists of some action and some observation. For example, the game of "throwing the paper ball into the basket" involves an operation that consists of

[1]Similar perspectives can be found from other authors. For example, one influential view by Rovelli is that mechanics is a theory of relations between variable (see [28, 49] and reference therein).

the action of picking up the paper ball and throwing it towards the basket, and the observation of seeing whether the paper ball goes into the basket.

Note that the action and observation do not have to occur in a definite sequence. There are operations with the observation preceding the action, and others with the action and the observation occurring simultaneously. It is helpful to simplify the situation by introducing the notion of "general action" to unify action and observation. A general action may be an action with a trivial observation (e.g., Alice throws the paper ball towards the basket and look into the sky without observing whether the ball falls in), a pure observation (e.g., another person Bob observes if Alice's ball falls in), or a combined action-observation (throw the ball and keep on observing where it flies).

Data is always gathered through the observation part of the general action. The trivial observation with only one possible outcome is still viewed to gather some data, even though this piece of data offers no nontrivial information to distinguish among more than one possibility.

An operation always refers to some physical objects. In the example above the relevant physical objects are the paper ball and the basket. In general, the relevant physical objects for an operation can be more complicated. For example, the operation of taking an orange and producing a cup of orange juice has the relevant physical object, the orange, going through different forms of existence (raw orange and orange juice). To be specific and talk about the different forms of existence, we speak of the relevant *physical system* of an operation. The physical system shows up as part of the mathematical description of an operation to specify what state of affairs are relevant for the operation. In the example above, we may take the operation to have two relevant physical systems: the state of the orange when it is raw and the state of the orange when it becomes juice. The physical system of an operation specifies a condition that enables the operation and/or a condition that checks the validity of an operation. Only when a paper ball and a basket is present can one play the game of throwing, and only when the orange is turned into juice (but not, say, a half peeled orange) is the operation valid in that context. We note that in some situations the data recorded also invokes physical systems to store the data. For example, in a paper ball throwing competition the result of whether Alice's ball lands in may be recorded on a piece of paper for further reference. This piece of data of either "yes" or "no" is classical. In other cases the data recorded may take the form of a quantum state or states on some type of systems.

To summarize, in a physical theory, a minimal description of an operation consists of a general action, a set of possible data gathered from the general action, and the relevant physical systems for the operation. More generally, there are situations where multiple choices for the operation are available. A general operation consists of a set of possible general actions, each with its own possible data set and its own relevant physical systems. We settle on this characterization of operations.

To symbolize an operation we adopt the following convention. A general action is denoted with capital letters in the form A. A physical system is denoted with lower-case letters in the form a. Sometimes we group systems together into a composite system. If the composite physical system a consists subsystems $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{n}$, we write $\mathsf{a} = \mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_\mathsf{n}$ and may use either the left side or the right side to refer to the composite system. The set of possible data is enumerated by letters i in a different font. These symbols $\mathsf{A}, \mathsf{a}, i$ can be combined to make explicit different pieces of information. For example, a general action A with system a is referred to as A_a, and its i-th data may be referred to as $\mathsf{A}_\mathsf{a}[i]$.

In this language, an operation $\mathcal{O}$ is described by an indexed set of objects $\{\mathsf{A}_\mathsf{a}[i]\}_{\mathsf{A},\mathsf{a},i}$, where it is understood that the sets of possible values a and i vary according to the choice of general action A. We write

$$\mathcal{O} = \{\mathsf{A}_\mathsf{a}[i]\}_{\mathsf{A},\mathsf{a},i}. \tag{2.1}$$

A familiar example of operation is the quantum instrument used in quantum theory [50]. A quantum instrument is a set of completely positive (CP) maps $\{\mathcal{E}[i]\}_i$ from some input state space $L(\mathcal{H}_{\mathsf{a}_1})$ (the space of bounded linear operators on the complex Hilbert space $\mathcal{H}_{\mathsf{a}_1}$) to some output state space $L(\mathcal{H}_{\mathsf{a}_2})$. The set of maps is required to sum up to a completely positive trace preserving map (channel). The quantum instrument describes a general action whose possible observational outcomes are i and whose physical system has two subsystems. The input subsystem a_1 is the one associated with the space $L(\mathcal{H}_{\mathsf{a}_1})$ and the output system a_2 is the one associated with the space $L(\mathcal{H}_{\mathsf{a}_2})$. We write the composite system of the operation as $\mathsf{a} = \mathsf{a}_1\mathsf{a}_2$. Then the operation takes the form $\{\mathcal{E}_\mathsf{a}[i]\}_{\mathsf{a},i}$, which is a special case of (2.1) with only one choice for the general action.

2.1.2 Correlation

The other basic concept of the framework is correlation.

Correlations for variables related with operations exist in various forms. They may be reflected in the probabilities for joint operational outcomes, in the correlation function for amplitudes of quantum fields, in the deterministic relation between time and other physical observables in classical mechanics etc. In general, we use the term "correlation" to refer to any object that allows us to increase the knowledge of some variables related to operations based on some other knowledge of variables related to operations.

In terms of mathematical description, a correlation can usually be captured in the form of a function

$$f : S \to V \tag{2.2}$$

from the set S of some variables related to operations to some other set of relevant variables V. The various examples of theories in the following sections will provide more detailed illustrations. For example, in classical mechanics S could be the space of observables and $f = 0$ encodes a correlation of the observables by determining a curve or surface in the space of observables. In a probabilistic theory, f could simply be a function from the set of joint observational outcomes S to the probabilities V. In quantum field theory, f could be a map from quantum fields S to the complex numbers $V = \mathbb{C}$ of the value of a field/observable correlation function.

In terms of physical interpretation, a theory usually comes with an explanation for why there exist correlations and for what physical mechanism mediates the correlation. Depending on particular theories, the correlation may be explained by some other agent-controlled operation (as in operational circuit models of quantum theory Section 2.3), by physical states that exist in nature (as in quantum field theory Section 2.4), by natural laws (as in classical mechanics Section 2.2), or by some other explanations.

2.1.3 Summary

We adopted Hardy's perspective that a physical theory must correlate data recorded through operations. A physical theory that takes operation and correlation as the central concepts is expected to provide mathematical descriptions and physical interpretations of the allowed operations and correlations. In explaining the physical mechanism of the correlation among variables related to operations, some infrastructure is usually provided to specify what configurations for the operations and correlations are allowed.

In the next several sections we illustrate these concepts by (re)formulating some important physical theories in this language.

2.2 Example: classical mechanics

In [28][2], Rovelli presented the following perspective.

> In a fully relativistic context, mechanics is a theory of correlations between partial observables.

The motivation is that time should not be a special variable in a fully relativistic theory. In general relativity, reference systems for space and time should be physical – can act on and be acted on by other physical systems. Therefore space and time are on equal footing with other dynamical observable quantities, and mechanics is a theory about the correlation of these dynamical and physical quantities, which are called "partial observables" in this context. A *partial observable* explicitly means "a physical quantity to which we can associate a (measuring) procedure leading to a

[2] See Chapter 3 of the book.

number" [51]. The connection of Rovelli's perspective on mechanics to the operation-correlation perspective is now obvious. Partial observables are variables related to operations. Mechanics is viewed as a theory about their correlations.

Concretely, in Rovelli's formulation mechanics is encoded the structure $(\mathcal{C}, \Gamma, f)$ based on the basic concepts of the event space $\mathcal{C}$, the space of motions Γ, and the evolution equation $f = 0$.

- Operations: The relevant concept regarding operations is the event space (also called the relativistic configuration space) $\mathcal{C}$ of the partial observables.

  Let t be the partial observable of time (e.g., the reading of a physical clock), and α be some other partial observables of interest. Suppose there is an operational procedure that allows one to observe the pair (t, α) together. Such a tuple of data constitutes an *event*. There is then a space $\mathcal{C}$ coordinatized by the partial observables t and α. This is space is called *event space*.

- Correlations: A correlation of the partial observables t and α can usually be expressed in the form of an evolution equation.

  Suppose we observe (t, α) of the same physical system for many different instances. In classical physics the data gathered usually fits on a curve or surface γ in the event space. Such a subset parametrized by the partial observables and realized by some actual physical system is called a *physical motion*. This physical motion expresses one way the partial observables are correlated.

  Now if we do the same for a different physical system, the physical motion may be a different curve/surfaces. It is possible that we find some physically meaningful quantities that take on different values for different physical motions to distinguish them. These quantities coordinatize another space Γ, called the *space of motions*. Each point in Γ forms a *state*, and encodes one correlation for the partial observable. One can think of the coordinates of the space Γ as some initial conditions or their generalizations, which when changed yield a different state.

- Evolution equation: It often happens that all the physical motions obey some same physical law, and in particular the physical law can often be captured by function of the form $f = 0$, where f is a function

  $$f : \Gamma \times \mathcal{C} \to V. \tag{2.3}$$

  Here V is some vector space. This kind of relation $f = 0$ is called an *evolution equation*. Once the coordinates on Γ is fixed, i.e., the state is fixed, the evolution equation yields a physical motion in $\mathcal{C}$.

2.3 Example: quantum theory

There exist many different operational formulations of quantum theory. Most of the formulations share the common structure that physical operations/observational outcomes are represented by some complex Hilbert space operators and probabilistic correlations are encoded in a generalized Born's rule.

In this section we present two formulations both having the important feature that a global state evolution is not a necessary part of the theory. Within these formulations, the essential predictions of quantum theory are regarding the correlations of operations, rather than the evolution of a global state. Echoing Rovelli's perspective of the last section, this is a welcoming feature for quantum gravity because of the general covariance of general relativity.

2.3.1 Quantum networks

The first formulation is Chiribella, D'Ariano, and Perinotti's theoretical framework for quantum network [52].

- Operations: Operations are circuits built out of composing elementary circuits.

  Elementary circuits consists of the most basic concepts of quantum theory such as quantum states, quantum channels, and POVMs. They are commonly described by completely positive (CP) maps from some (possibly trivial) input systems to some (possibly trivial) output systems. It is also required that the probabilities are non-negative and normalized (probabilities of different outcomes in the same experiment sum up to one).

  - Infrastructre: By feeding output systems into input systems of the same types, elementary circuits can be composed, resulting in other circuits. Similarly, the new circuits can be further composed with other circuits. The composition of circuits is restricted by two rules: 1) An input system is connected to an output system. 2) There cannot be causal cycles. These rules are enforced by backing up the circuit connection with directed acyclic graphs (DAGs). Each node corresponds to a circuit, and each arrow points from an output system of one circuit to the input system of another circuit.

  Fundamentally, a circuit thus obtained is a map with some input systems and output systems. When the elementary circuits used to obtain the circuit are CP maps, the resulting circuit is still CP. To keep track of the action of the circuit arising from composing a lot of elementary circuits can be complicated. Fortunately there is a simple mathematical description of the circuits and their compositions.

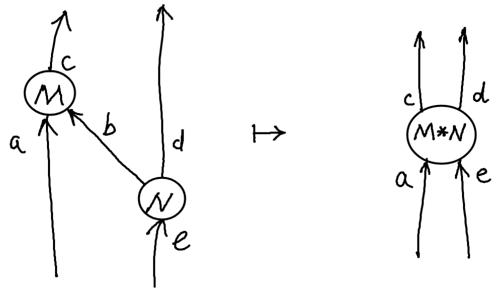

Figure 2.1: Diagrammatic expression for eq. (2.5).

The maps of the circuits can represented isomorphically as their Choi operators [53]. A completely positive map $\mathcal{M} : \mathcal{L}(\mathcal{H}^{a_1}) \to \mathcal{L}(\mathcal{H}^{a_2})$ can be represented isomorphically by its positive semidefinite "Choi operator" defined as:

$$M = (\mathcal{M} \otimes \mathbb{1}) |\Phi\rangle\langle\Phi| \in L(\mathcal{H}^{a_2} \otimes \mathcal{H}^{a_1}), \tag{2.4}$$

where $\mathbb{1}$ is the identity channel on system a_1, $|\Phi\rangle = \sum_i |ii\rangle \in \mathcal{H}^{a_1} \otimes \mathcal{H}^{a_1}$ is an unnormalized maximally entangled state in a canonical basis on two copies of system a_1.

Composition of circuits can be conducted at the Choi operators level through the link product. For two Choi operators M and N the *link product* is defined by [52]:

$$M * N := \mathrm{Tr}_{M\cap N}[(I_{N\setminus M} \otimes M^{T_{M\cap N}})(N \otimes I_{M\setminus N})]. \tag{2.5}$$

Here the subscripts label Hilbert spaces: M (N) denotes the Hilbert space on which the Choi operator M (N) acts, with unions and differences given the meaning of Hilbert space unions and differences. The superscript T denotes partial transpose.

- Correlations: This theory is a probabilistic theory and the correlation is reflected in the rule for calculating joint probabilities for the outcomes represented by certain circuits to occur. The probability calculus is actually provided by the link product.

  The link product can be applied iteratively until all the systems are connected to each other. The link product then yields a real number, the probability for the outcomes of the operators that show up in the link product to happen:

$$p(i, j, \cdots) = M_i * N_j \cdots . \tag{2.6}$$

2.3.2 Operator tensors

Hardy's operator tensors [54, 55] offer a similar operational formulation of quantum theory with a simpler composition rule.

- Operations: An operation $\mathsf{A}_{\mathsf{a}_1\mathsf{b}_2\cdots}^{\mathsf{c}_3\mathsf{d}_4\cdots}$ is associated with some input systems $\mathsf{a}_1, \mathsf{b}_2, \cdots$ and some output systems $\mathsf{c}_3, \mathsf{d}_4, \cdots$. Different letters distinguish different types of physical systems (such as photon systems and neutrino systems), and different numbers break the degeneracy when the same type of system shows up multiple times in an operation. Below, for notational simplicity we drop the numbers when there is no ambiguity.

    - Infrastructure: Different operations can be connected through "wires". The intuition is that the operations are performed by experimental apparatuses, which are connected by some wires that propagate causal correlations. The composition of operations follow the same rules as the quantum networks that an output system is only connected to an input system of the same type, and that there is no causal loop.

      A connection of a set of operations can be symbolically represented in the form $\mathsf{A}_{\mathsf{ab}}^{\mathsf{cd}}\mathsf{B}_{\mathsf{cd}}$, with input systems as subscripts, output systems as superscripts, and repeated scripts signifying wire connection. In this particular case the resulting operation is one with input systems a and b and no output systems.

  The quantum operations are mathematically represented as operators on the Hilbert spaces of the input and output systems and they obey some physicality conditions so that the probabilities calculated from them belong to $[0, 1]$. Specifically, each input system a is associated with a Hilbert space $\mathcal{H}_{\mathsf{a}}$ and each output system b is associated with a Hilbert space $\mathcal{H}^{\mathsf{b}}$. Composite systems have the standard tensor product structure. An operation $\mathsf{A}_{\mathsf{a}}^{\mathsf{b}}$ is associated with a *Hermitian* operator $\hat{A}_{\mathsf{a}}^{\mathsf{b}}$ on $\mathcal{H}_{\mathsf{a}} \otimes \mathcal{H}^{\mathsf{b}}$ and further obeys the following *physicality conditions*:

$$\hat{A}_{\mathsf{a}^T}^{b} \geq 0, \tag{2.7}$$

$$\hat{A}_{\mathsf{a}}^{\mathsf{b}} \hat{I}_{\mathsf{b}} \leq \hat{I}_{\mathsf{a}}, \tag{2.8}$$

  where $\hat{I}_{\mathsf{x}}$ is the identity operator on $\mathcal{H}_{\mathsf{x}}$. The first condition ensures the probabilities ≥ 0 and the second ensures that probabilities ≤ 1 (see below for the rule of calculating probability).

- Correlations: As a probabilistic theory the correlations are regarding joint probabilities for the operational outcomes to occur. A set of operations totally connected (all input and output system are connected by wires) is assigned a probability.

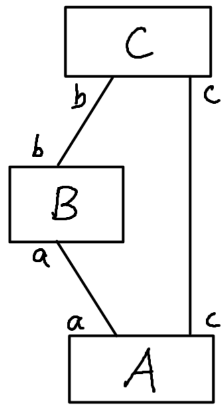

Figure 2.2: Diagrammatic expression for eq. (2.9).

The formula for the probability is of the form

$$p = \hat{A}^{\mathsf{ac}} \hat{B}_{\mathsf{a}}^{\mathsf{b}} \hat{C}_{\mathsf{bc}}, \tag{2.9}$$

The right hand side means that whenever two operators have a same script, multiply the operators at the system of the script and take the partial trace on that system. In the end all systems will be traced out and we are left with a number which is the probability. The above formula can be straightforwardly generalized to all totally connected sets of operations.

2.4 Example: local quantum physics/algebraic quantum field theory

Local quantum physics/algebraic quantum field theory [56, 57] has an observable algebra and the states as the basic concepts. The former represent operations, and the latter represent correlations. Hence the theory fits well with the operation-correlation perspective.

- Operations. The operations are built out of the algebra of observables.
- Correlations. The correlations are global states which are defined to be linear functionals from the algebra of observables to the complex numbers. The linear functional evaluates to the value of a correlation function.

  Through the GNS construction the operations can be represented as operators on a global infinite dimensional Hilbert space and the states can be represented as density operators on the Hilbert space.
- Probability. The correlation functions can be used to calculate the transition amplitudes, from which the probabilities of operational outcomes can be derived.

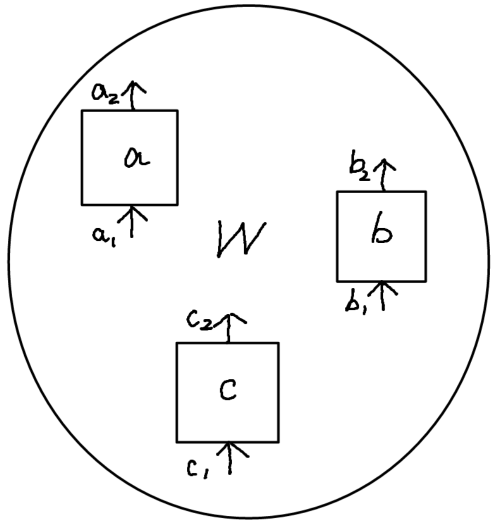

Figure 2.3: A process matrix setup

2.5 Example: process matrices

A very important lesson made obvious from the process matrix framework [58, 59, 60] is that statistical correlations among quantum operations can directly encode indefinite causal structure. This lesson is especially important for the present thesis because it opens the door for an operational description quantum spacetime causal structure by encoding the causal structure in the statistical correlations.

- Operations: Each operation indexed by a has an input Hilbert space $\mathcal{H}^{a_1}$ and an output Hilbert space $\mathcal{H}^{a_2}$. The allowed operations are quantum instruments on these input and output spaces.

- Correlations: The correlations are the processes. They are maps from operational outcomes to probabilities. The processes associated with the operations labelled by letters in the set $\mathcal{N} = \{a, b, \cdots, c\}$ are represented as matrices $W \in \mathcal{L}(\mathcal{H}^{a_1} \otimes \mathcal{H}^{a_2} \otimes \mathcal{H}^{b_1} \otimes \mathcal{H}^{b_2} \otimes \cdots \otimes \mathcal{H}^{c_1} \otimes \mathcal{H}^{c_2})$ through the use of Choi operators. To ensure that the probabilities are physical (non-negative and normalized), the process matrices obey [59]

$$\begin{aligned} W \geq& 0, && (2.10)\\ \operatorname{Tr} W =& d_O, && (2.11)\\ W =& L_V(W), && (2.12) \end{aligned}$$

  where d_O is the dimension of the joint output Hilbert space of the nodes, and L_V is a projector onto a subspace whose explicit form is given in Section 4.2.2.

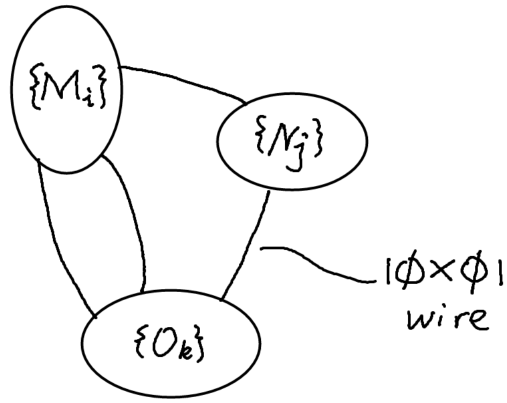

Figure 2.4: A quantum network without predefined time

The first condition ensures that probabilities (see below) are non-negative, and the next two conditions ensure that probabilities are normalized.

- Infrastructure: Since the causal structure is encoded in the process matrices rather than the way the operations are connected in any form of a graph, there is not a non-trivial infrastructure in the framework.

- Probabilities: For a joint outcome represented by quantum instrument elements (more precisely, their Choi operators) $M_a \in \mathcal{L}(\mathcal{H}^{a_1} \otimes \mathcal{H}^{a_2}), M_b \in \mathcal{L}(\mathcal{H}^{b_1} \otimes \mathcal{H}^{b_2}), \cdots, M_c \in \mathcal{L}(\mathcal{H}^{c_1} \otimes \mathcal{H}^{c_2})$ at the nodes $a, b, \cdots, c$, its probability of occurrence given that the correlation is W is

$$p = \mathrm{Tr}[(M_a \otimes \cdots M_c)W]. \tag{2.13}$$

Some explicit examples of process matrices encoding indefinite causal structure among the operations can be found in Section 3.4.1.

In Chapter 4 we present original works on a more general "correlation network" framework more suitable for studying quantum spacetime and more convenient to deal with the "tensor product issue" of the process matrices [61, 62].

2.6 Example: operational quantum theory without predefined time

The Oreshkov-Cerf theory of an operational quantum theory without predefined time [63]. In accord with the absence of a predefined time, the systems associated with an operation are not separated into input and output subsystems, and no causality condition (which would imply a time direction) is imposed on the operations.

- Operations: Using the notations of the original paper, an operation $\{M_i^{AB\cdots}\}_{i\in O}$ consists of a set of possible events indexed by the data set element $i \in O$. $A, B, \cdots$ are the physical systems associated with the operation, with corresponding Hilbert spaces $\mathcal{H}^A, \mathcal{H}^B, \cdots$ whose dimensions are $d^A, d^B, \cdots$. The events are represented by positive semidefinite operators $M_i^{AB\cdots}$ on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \cdots$.

  Operations come in equivalence classes. Two operations $\{M_i^{AB\cdots}\}_{i\in O}$ and $\{N_i^{AB\cdots}\}_{i\in O}$ that yield the same joint probabilities for all experimental setups (or circuits) belong to the same equivalence class. Similarly events come in equivalence classes. Two events $M_i^{AB\cdots}$ and $N_i^{AB\cdots}$ coming from different operations that yield the same joint probabilities with other events in all experimental setups (or circuits) belong to the same equivalence class.

  Events/operations in the same equivalence class have operators that differ by a constant factor. One way to avoid this ambiguity is to represent an equivalence class of events by specifying a pair of operators in the form $(M_i^{AB\cdots}, \overline{M}^{AB\cdots})$, where $\overline{M}^{AB\cdots} := \sum_{i\in O} M_i^{AB\cdots}$, and fixing a normalization convention, such as

  $$\operatorname{Tr} \overline{M}^{AB\cdots} = d^A d^B \cdots . \tag{2.14}$$

  The null operation $\{O^{AB\cdots}\}$ with trace zero is treated separately as a singular case.

  The normalization requirement (2.14) is weaker than what is usually imposed in ordinary quantum theory. Ordinary quantum theory is time-asymmetric in the sense that measurement outcomes represented by POVM elements sum up to the identity (or more generally, outcomes represented by quantum instrument elements sum up to a channel), but states in a preparation are only required to have their traces sum up to one. In a theory without predefined time this time-asymmetry should be absent, and in the Oreshkov-Cerf theory the time-asymmetry is eliminated by weakening the requirement on outcomes so that only a sum of trace condition (2.14) is imposed.

- Correlations: The correlation is encoded in the following formula for joint probabilities:

  $$p(i, j, \cdots | \{M_i^{\cdots}\}_{i\in O}, \{N_j^{\cdots}\}_{j\in Q}, \cdots ; \text{network}) = \frac{\operatorname{Tr}\Big[(M_i^{\cdots} \otimes N_j^{\cdots} \otimes \cdots) W_{\text{wires}}\Big]}{\operatorname{Tr}\Big[(\overline{M}^{\cdots} \otimes \overline{N}^{\cdots} \otimes \cdots) W_{\text{wires}}\Big]}. \tag{2.15}$$

  This is a special case of (2.20). The condition in the conditional probability specifies the relevant operations and the way they are connected ("network").

- The connection can be specified using a graph. The operations are located at the nodes. Each (sub)system of an operation is connected to a (sub)system of another operation with the same dimension using a "wire", which is an edge labelled by the system dimension. A wire tells which system interact with which, and is mathematically described as a pure bipartite entangled state $|\Phi\rangle\langle\Phi|$ whose precise form depends on the symmetry of the system. The operator W_{wires} is the tensor product of all these wire operators.

2.7 Principles for complex Hilbert space quantum theories

The framework presented in this chapter is intended to be very general as to incorporate various kinds of physical theories taking operations and correlations as the central concepts. As illustrated in Section 2.2, classical mechanics is included in the list of theories incorporated. To deal with questions of quantum gravity we are interested in a complex Hilbert space theory that allows indefinite causal structure. Can such a theory be axiomatized within the current framework? In this section we provide a set of axioms and derive the complex Hilbert space structure of quantum theories without assuming definite causal structure.

The task of identifying postulates and deriving the complex Hilbert space structure is made easy by the previous works of Wilce and Barnum [64, 65] (see also [66] and references therein for a comprehensive account of the approach and [67] for a related work based on category theories). The original postulates and derivations in their work are for theories with definite causal structure. Yet we show that the same general strategy of using the Jordan algebra structure to arrive at the complex Hilbert space works in a framework with indefinite causal structure.

The derivation given here is restricted to theories with finite dimensions. To give a derivation for theories with infinite dimensions is an open question.

This work falls within the subject of axiomatizing/reconstructing quantum theory. There has been a body of illuminating previous work on this topic (see, e.g., [68, 69, 70, 71, 54, 72, 64, 65, 73, 67] and the book [74]). The works listed above commonly assume definite causal structure, either at the level of the general framework so that all theories in the landscape have definite causal structure, or at the level of the postulates so that the quantum theory that is singled out has definite causal structure. The novelty of the work presented here is that definite causal structure is not assumed, making it more relevant for quantum gravity.

The content of this section is based on [6]. The following is only intended as a brief overview of the postulates and derivations. The technical details are skipped and will be found in the original paper.

2.7.1 Probabilistic theories

For the purpose of deriving the structure of quantum theory we restrict attention to probabilistic theories. The main function of a probabilistic theory is to calculate probabilities for allowed operations to register certain data. In general, the probabilities to be calculated take the form of conditional probabilities. When a conditional probability is well-defined[3], a probabilistic theory is expected to offer a method to calculate it.

In general the conditional probabilities are of the form $p(i, j, \cdots, k|\text{cond}) \in \mathbb{R}$, where $i, j, \cdots, k$ is a possible set of data to be registered from a set of general actions, and cond encode the prerequisite conditions for the probability to make sense. The conditions contain the choice of general action for each operation, and further conditions to make the probabilities well-defined. In this probabilistic theory setting a *correlation* specifically refers to a map from a set of data to the set of real numbers, offering information on the conditional probabilities.

Conventionally, *absolute probability* are used for probabilities. The conditional probabilities of the form $p(i, j, \cdots, k|\text{cond}) \in \mathbb{R}$ obey

$$p(i, j, \cdots, k|\text{cond}) \geq 0 \tag{2.16}$$

$$\sum_{i,j,\cdots,k} p(i, j, \cdots, k|\text{cond}) = 1, \tag{2.17}$$

where the sum is over possible data to be recorded from the set general actions. These imply

$$1 \geq p(i, j, \cdots, k|\text{cond}) \geq 0. \tag{2.18}$$

There is an alternative option of using *probability weights*. The probability weights $w(i, j, \cdots, k|\text{cond}) \in \mathbb{R}$ are only required to obey

$$\infty > w(i, j, \cdots, k|\text{cond}) \geq 0. \tag{2.19}$$

These probability weights are meaningful in comparison with each other, which saves the need for normalization. For any pair $w(i|\text{cond})$ and $w(j|\text{cond})$ of probability weights (Here for simplicity we used one letter i or j to represent a list of observational outcomes.), if $w(j|\text{cond}) \neq 0$, then the prediction is that the data i is $r = w(i|\text{cond})/w(j|\text{cond})$ times as likely to be recorded as j. If $w(j|\text{cond}) = 0$, a comparison of probability weights in terms of the ratio $r = w(i|\text{cond})/w(j|\text{cond})$ should not be made, and physical meaning is that the data j is predicted never to be recorded.

[3] See [18, 19] for a discussion on the non-triviality of the requirement that the probabilities are well defined.

When $0 < \sum_{i,j,\cdots,k} w(i,j,\cdots,k|\text{cond}) < \infty$, where the sum is over all possible outcome for the set of general actions, normalization can be conducted and the absolute probabilities can be obtained from the relative probabilities as

$$p(i,j,\cdots,k|\text{cond}) = \frac{w(i,j,\cdots,k|\text{cond})}{\sum_{i,j,\cdots,k} w(i,j,\cdots,k|\text{cond})}. \tag{2.20}$$

The case of $0 = \sum_{i,j,\cdots,k} w(i,j,\cdots,k|\text{cond})$ should not appear in a physically meaningful setup, since among all possible outcomes some outcome should happen. In a physically meaningful setup and for finitely many outcomes, $0 < \sum_{i,j,\cdots,k} w(i,j,\cdots,k|\text{cond}) < \infty$ always holds, and the absolute probabilities can always be obtained from the probability weights. Whereas the absolute probabilities are unique, the probability weights may be rescaled by the same factor without changing the physical content. This means that two theories using probability weights may give physically equivalent predictions even when the exact values for the probability weights of the same outcomes do not agree. The case of a diverging $\sum_{i,j,\cdots,k} w(i,j,\cdots,k|\text{cond})$ may appear when infinitely many outcomes are allowed by a theory. Then one needs to specify a separate rule to convert probability weights to absolute probabilities, if one still wants to do the conversion. As far as the derivation of the complex Hilbert space structure of this paper goes we do not need to worry about this case, since the number of outcomes will be assumed to be finite.

So far we have been talking about operations as an abstract concept without embedding them in a mathematical model. We will now introduce a basic postulate to endow the operations (along with correlations) with some additional mathematical structure. Under this postulate, observational data will become vector spaces elements, and the map of correlations will become (multi)linear functionals over such vector spaces.

The motivation comes from the probabilistic mixing of general actions. Let $\mathcal{O} = \{\mathsf{A}_\mathsf{a}[i]\}_{\mathsf{A},\mathsf{a},i}$ contain A_a and B_a as two choices for the general action associated with the same physical system a. Provided both general actions distinguish finitely many possible outcomes, without loss of generality we can suppose they have the same total number of outcomes (adding void outcomes that are never triggered to the general action with the smaller number of outcomes if needed). Suppose a theory predicts $w(i|\text{cond},\mathsf{A})$ should A be chosen as the general action to be performed, and $w(i|\text{cond},\mathsf{B})$ should B be chosen as the general action to be performed. Probabilistically mixing A and B means performing A with probability weight w_A and B with probability weight w_B. Under such a mixing $\{\mathsf{A},\mathsf{B};w_\mathsf{A},w_\mathsf{B}\}$ the predictions for the outcomes are expected to be

$$w(i|\text{cond},\{\mathsf{A},\mathsf{B};w_\mathsf{A},w_\mathsf{B}\}) = w_\mathsf{A}\overline{w}_\mathsf{B}w(i|\text{cond},\mathsf{A}) + w_\mathsf{B}\overline{w}_\mathsf{A}w(i|\text{cond},\mathsf{B}), \tag{2.21}$$

where $\overline{w}_\mathsf{A} = \sum_i w(i|\text{cond},\mathsf{A})$, and $\overline{w}_\mathsf{B} = \sum_i w(i|\text{cond},\mathsf{B})$. This is analogous to

$$p(i|\text{cond},\{\mathsf{A},\mathsf{B};p_\mathsf{A},p_\mathsf{B}\}) = p_\mathsf{A}p(i|\text{cond},\mathsf{A}) + p_\mathsf{B}p(i|\text{cond},\mathsf{B}) \tag{2.22}$$

for ordinary probabilities, where A is performed with probability p_{A} and B is performed with probability p_{B}. In (2.21) there are the extra $\bar{w}_{\mathsf{A}}$ and $\bar{w}_{\mathsf{B}}$ factors. Analogous factors are not present for (2.22) because $\bar{p}_{\mathsf{A}} := \sum_i p(i|\mathrm{cond}, \mathsf{A}) = 1 = \bar{p}_{\mathsf{B}} := \sum_i p(i|\mathrm{cond}, \mathsf{B})$. Equation (2.22) can be arrived at from (2.21) using $p(i|\mathrm{cond}, \mathsf{A}) := w(i|\mathrm{cond}, \mathsf{A})/\bar{w}_{\mathsf{A}}$ and $p(i|\mathrm{cond}, \mathsf{B}) := w(i|\mathrm{cond}, \mathsf{B})/\bar{w}_{\mathsf{B}}$, $p_{\mathsf{A}} := w_{\mathsf{A}}/(w_{\mathsf{A}} + w_{\mathsf{B}})$, $p_{\mathsf{B}} := w_{\mathsf{B}}/(w_{\mathsf{A}} + w_{\mathsf{B}})$, $p(i|\mathrm{cond}, \{\mathsf{A}, \mathsf{B}; w_{\mathsf{A}}, w_{\mathsf{B}}\}) := w(i|\mathrm{cond}, \{\mathsf{A}, \mathsf{B}; w_{\mathsf{A}}, w_{\mathsf{B}}\})/\sum_i w(i|\mathrm{cond}, \{\mathsf{A}, \mathsf{B}; w_{\mathsf{A}}, w_{\mathsf{B}}\})$, and noting that

$$\sum_i w(i|\mathrm{cond}, \{\mathsf{A}, \mathsf{B}; w_{\mathsf{A}}, w_{\mathsf{B}}\}) = \sum_i w_{\mathsf{A}}\bar{w}_{\mathsf{B}} w(i|\mathrm{cond}, \mathsf{A}) + w_{\mathsf{B}}\bar{w}_{\mathsf{A}} w(i|\mathrm{cond}, \mathsf{B}) \tag{2.23}$$

$$= (w_{\mathsf{A}} + w_{\mathsf{B}})\bar{w}_{\mathsf{A}}\bar{w}_{\mathsf{B}}. \tag{2.24}$$

Theories in which equation (2.21) holds have a certain linear structure for the correlation as a map from the outcomes to the probability weights. It suggests that the recorded data on the same physical system be represented as elements in a vector space, with real numbers such as $w_{\mathsf{A}}\overline{w}_{\mathsf{B}}$ and $-w_{\mathsf{B}}\overline{w}_{\mathsf{A}}$ forming the field for the vector space, and the correlations as multilinear maps from these vector spaces to the probability weights. We realize this suggestion as a postulate.

Postulate 1 (Linearity)**.** Recorded data for general actions with the same relevant physical system are represented as positive cone elements in an ordered vector space with some trivial data as an order unit. Correlations are represented as positive multilinear functionals on such spaces.

Here an *ordered vector space* is a real vector space V endowed with a convex cone V^+ such that V^+ spans V, and that $V^+ \cap -V^+ = \{0\}$. V^+ is called the *positive cone* of V. An *order unit* of an ordered vector space is an element $u \in V^+$ so that for any $v \in V$, there is an $a > 0$ such that $au - v \in V^+$.

The ordered vector space of Postulate 1 is called an *operational space*, and is denoted in the form $\mathfrak{O}_{\mathsf{a}}$, where a is the relevant physical system. The dimension of the space is denoted d_{a}. The positive cone is denoted $\mathfrak{O}_{\mathsf{a}}^+$. It contains the elements that represent physical data. Each $\mathsf{A}_{\mathsf{a}}[i]$ is represented by an element of $\mathfrak{O}_{\mathsf{a}}^+$. We refer to these vector space elements using the same symbols $\mathsf{A}_{\mathsf{a}}[i]$ for the observational outcomes when no ambiguity arises. When it is clear from the context we often suppress the labels $[i]$ and refer to the vector space elements in the form A_{a} for simplicity.

The correlations as positive multilinear functionals on $\mathfrak{O}_{\mathsf{a}}$, $\mathfrak{O}_{\mathsf{b}}, \cdots \mathfrak{O}_{\mathsf{c}}$ are denoted in the form $\mathsf{D}^{\mathsf{ab}\cdots\mathsf{c}}$ with the physical systems in the superscript to be distinguished from the recorded data with the system in the subscript:

$$\begin{aligned} \mathsf{D}^{\mathsf{ab}\cdots\mathsf{c}} : \mathfrak{O}_{\mathsf{a}} \times \mathfrak{O}_{\mathsf{b}} \times \cdots \times \mathfrak{O}_{\mathsf{c}} &\to \mathbb{R}, \\ (\mathsf{A}_{\mathsf{a}}[i], \mathsf{B}_{\mathsf{b}}[j], \cdots, \mathsf{C}_{\mathsf{c}}[k]) &\mapsto w(i, j, \cdots, k|\mathrm{cond}). \end{aligned} \tag{2.25}$$

The vector space generated by the correlations is called a *correlation space* and is denoted $\mathfrak{C}^{\mathsf{ab}\cdots\mathsf{c}}$. The dimension of the correlation space is denoted $c_{\mathsf{ab}\cdots\mathsf{c}}$.

2.7.2 Subsystem structures

As the last part to specify the basic framework for probabilistic theories with operations and correlations, we discuss the subsystem structure for composite physical systems. We assume two very basic properties for the operational spaces of composite systems. A system a with $d_{\mathsf{a}} = \dim \mathfrak{O}_{\mathsf{a}} = 1$ is called a *trivial system*. The space of a trivial system supports only one linearly independent vector, which describes a trivial data. We assume that for a trivial system a, $\mathfrak{O}_{\mathsf{ab}} \cong \mathfrak{O}_{\mathsf{b}}$ as ordered vector spaces for all b.

The second basic property we assume is that any operational space $\mathfrak{O}_{\mathsf{ab}}$ with two subsystems contain all the product elements while preserving linear independence, i.e., if $\mathsf{A}_{\mathsf{a}} \in \mathfrak{O}_{\mathsf{a}}$ and $\mathsf{B}_{\mathsf{b}} \in \mathfrak{O}_{\mathsf{b}}$, then there is an element $\mathsf{A}_{\mathsf{a}}\mathsf{B}_{\mathsf{b}} \in \mathfrak{O}_{\mathsf{ab}}$ so that if A_{a} and A'_{a} are linearly independent in $\mathfrak{O}_{\mathsf{a}}$ and B_{b} and B'_{b} are linearly independent in $\mathfrak{O}_{\mathsf{b}}$, then $\mathsf{A}_{\mathsf{a}}\mathsf{B}_{\mathsf{b}}$, $\mathsf{A}'_{\mathsf{a}}\mathsf{B}_{\mathsf{b}}$, $\mathsf{A}_{\mathsf{a}}\mathsf{B}'_{\mathsf{b}}$ and $\mathsf{A}'_{\mathsf{a}}\mathsf{B}'_{\mathsf{b}}$ are all linearly independent in $\mathfrak{O}_{\mathsf{ab}}$. This implies that $d_{\mathsf{a}}d_{\mathsf{b}} \leq d_{\mathsf{ab}}$.

There is a similar basic property we assume for the correlations that pertain to two operational spaces. Suppose C^{a} is a correlation pertaining to $\mathfrak{O}_{\mathsf{a}}$ itself and D^{b} is a correlation pertaining to $\mathfrak{O}_{\mathsf{b}}$. Then we assume that there is a correlation $\mathsf{C}^{\mathsf{a}}\mathsf{D}^{\mathsf{b}}$ pertaining to $\mathfrak{O}_{\mathsf{ab}}$ so that $\mathsf{C}^{\mathsf{a}}\mathsf{D}^{\mathsf{b}}(\mathsf{A}_{\mathsf{a}}\mathsf{B}_{\mathsf{b}}) = \mathsf{C}^{\mathsf{a}}(\mathsf{A}_{\mathsf{a}})\mathsf{D}^{\mathsf{b}}(\mathsf{B}_{\mathsf{b}})$, i.e., the probability weights multiply.

2.7.3 Comments on the framework

The framework just presented family-resembles other frameworks used in previous axiomatic works, but have some notable differences. First of all no assumption of definite causal structure is imposed on the current framework. Moreover, correlations carrying non-trivial physical information but not generated by operations is allowed in the current framework. This is in contrast with the circuit models [75, 70, 54], where the operations carry non-trivial physical correlation and the "wires" do not. Some theories are more naturally described in the current framework. For example, as mentioned, the global state of quantum field theory is not prepared by an operation and is more suitably viewed as encoding the correlation of operations. Another example is the process matrices that allow correlations with indefinite causal structure [58, 59, 60]. It is found that the process matrices cannot be parallel-composed without constraints [61, 62]. This would appear unnatural if the process matrices are viewed as operations, but natural if they are viewed as correlations among operations.

Another difference lies in the graphical representation of using hypergraphs instead of graphs. Graphical reasoning had been important in previous axiomatic works and works on operational theories in general (see, e.g., [75, 70, 54, 76], and [77] and reference therein). If one chooses to work with the current framework, the natural pictorial tool is the hypergraph, rather than the graph, which is widely used in other models (e.g., [52, 70, 54, 75, 63]). Roughly speaking a hypergraph is a generalized graph that allows edges to connect to other integer numbers of nodes rather than just two.

The generalized edge is called a "hyperedge". We can associate the nodes of a hypergraph to operations/outcomes and the hyperedges to the correlations, connecting the nodes they correlate. The implications of using hypergraphs instead of graphs for probabilistic theories remains to be explored.

2.7.4 Postulates

We now list the other postulates that are used to single out the complex Hilbert space structure. Since the derivation of the complex Hilbert space structure below uses dimension counting arguments and lemmas that work for finite dimensional spaces, we restrict attention to operations with finite dimensional operational spaces.

The next postulate refers to the dimension of operational spaces.

Postulate 2 (Dimension)**.** An operational space whose physical system has two subsystems has the same dimension as the correlation space over these two systems, and as the transformation spaces between these two systems.

Here the notion of transformation is a generalization of the notion of completely positive maps in quantum theory. Recall that in quantum theory a transformation is required to be completely positive so that physical states get mapped to physical states even if the transformation acts partially on a subsystem. In the more general framework complete positivity is generalized in an obvious way so that a transformation is a linear map that takes physical elements (positive cone elements) to physical elements even for partial actions on a subsystem, while acting on product elements in a local way. The transformations on the same systems as linear maps can be summed linearly to generate a vector space, with the transformations forming a convex cone that gives the vector space an ordered vector space structure. The dimension of the vector space with input operational space $\mathfrak{O}_{\mathsf{a}}$ and output operational space $\mathfrak{O}_{\mathsf{b}}$ is denoted $t_{\mathsf{a},\mathsf{b}}$. In this notation, Postulate 2 says that for arbitrary $\mathfrak{O}_{\mathsf{a}}$ and $\mathfrak{O}_{\mathsf{b}}$, $d_{\mathsf{ab}} = c_{\mathsf{ab}} = t_{\mathsf{a},\mathsf{b}} = t_{\mathsf{b},\mathsf{a}}$.

A commonly seen interpretation of a transformation is that it takes states from a previous time to a latter time. This interpretation does not hold at the most general level. For example, a transformation can be a supermap that transforms an operation (which may be a transformation rather than a state) to another operation that extends from an earlier time to a later time [52]. The above definition of transformations is intended to offer a basic mathematical characterization at the very general and abstract level, and leaves it to particular theories to specify what a transformation corresponds to in the physical world.

One can interpret the Postulate 2 as allowing the operations enough degrees of freedom to potentially realize all two system correlations and mathematically possible transformations. The correlations of two operations include both those arising from naturally and those controlled by agents. The latter type of correlation must interact with the two relevant systems, and is controlled by the agents through some operations containing the two systems as subsystems. The postulate says that as far as the degrees of freedom of the vector spaces are concerned, the operations have as many degrees of freedom as the set of all possible correlations, including the type arising from nature. A similar interpretation applies to the transformations.

We move on from discussing operational space elements transform into each other to how they correlate with each other. Without further constraints the framework allows weird theories such as one in which data recorded from any two operations on different systems are not correlated. In a universe described by this theory little inference can be made.

In a universe described by quantum theory, on the other hand, strong correlations are available. For example, one measurement operation on an d-dimensional system a can be strongly correlated with a measurement operation on another d-dimensional system a', if the two systems share a much entangled state. One characterization for the state to enable strong correlation is that it is a maximal-rank entangled state, i.e., in a Schmidt decomposition $\left|\Phi^{\mathsf{aa}'}\right\rangle = \sum_{i,i'} w_{ii'} \left|ii'\right\rangle$, $w_{ii'} \neq 0$ for any i. This establishes a correlation between the measurement outcomes expressed by the same matrices on the two systems. For a measurement outcome A_a expressed by the POVM element $\sum_{i,j} a_{ij} \left|i\middle\rangle\!\middle\langle j\right|$, we denote the corresponding measurement outcome on a' with the same matrix, $\sum_{i,j} a_{ij} \left|i'\middle\rangle\!\middle\langle j'\right|$, by $\mathsf{A}'_{\mathsf{a}'}$. More generally, we extend this notation to all matrices of the operational space of measurement outcomes. Then the correlation $\Phi^{\mathsf{aa}'} = \left|\Phi^{\mathsf{aa}'}\middle\rangle\!\middle\langle\Phi^{\mathsf{aa}'}\right|$ as a functional mapping to joint outcome probabilities obeys: 1) $\Phi^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{A}'_{\mathsf{a}'}) > 0$ for all nonzero A_a in the operational space of measurement outcomes. 2) $\Phi^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{B}'_{\mathsf{a}'}) = \Phi^{\mathsf{aa}'}(\mathsf{B}_\mathsf{a}, \mathsf{A}'_{\mathsf{a}'})$ for all $\mathsf{A}_\mathsf{a}, \mathsf{B}_\mathsf{a}$ in the operational space of measurement outcomes. 3) If A_a is such that $\Phi^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{B}'_{\mathsf{a}'}) \geq 0$ for all physical elements $\mathsf{B}'_{\mathsf{a}'}$, then A_a is a physical element.

The strongest form of correlation we can hope for is that from the outcomes of one operation we can infer unambiguously the outcomes of the paired operation. 1) is weaker implies that correlated outcomes appear together with some positive chance. 2) follows from the choice of basis $\{|i\rangle\}$ and $\{|i'\rangle\}$ so that $\left|\Phi^{\mathsf{aa}'}\right\rangle$ takes a symmetric form. 3) holds because the entangled state is "strong" so as to have maximal-rank. When there are further subsystems in the two systems, we can further choose maximal-rank entangled states so that it factorizes into maximal-rank entangled states across correlated pairs of subsystems.

These conditions can be abstracted and stated in the general framework for probabilistic theories. Universes described by theories where such correlations exist give physicists some handles to probe correlations among different systems. Note that the correlations obeying the above conditions are "strong" only in certain senses. They in no way single out quantum entanglement correlations exclusively, since even a classical probabilistic theory contains correlations that obey the conditions.

Next we state a postulate to distinguish theories that fulfill the above conditions in their abstracted form. Some technical definitions are needed to formalize the conditions. We need the notion of a "copy" of operational spaces. An *order-isomorphism*

f between ordered vector spaces V and W is a positive, invertible linear map having a positive inverse, where positive means $f(V^+) \subseteq W^+$. If two operational spaces $\mathfrak{O}_\mathsf{a}$ and $\mathfrak{O}_\mathsf{b}$ share an order-isomorphism, we say that they are *copies* of each other. We use primes on physical systems and vectors to signify copies (e.g., $\mathfrak{O}_{\mathsf{a}'}$ for the copy of $\mathfrak{O}_\mathsf{a}$, and $\mathsf{A'}_{\mathsf{a}'}$ for the "copy" of A_a under the order-isomorphism).

An operational space $\mathfrak{O}_\mathsf{a}$ is said to have a *pairing* if there is a copy $\mathfrak{O}_{\mathsf{a}'}$ and a correlation $\mathsf{C}^{\mathsf{aa}'}$ on the two spaces so that $\mathsf{C}^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{A}'_{\mathsf{a}'}) > 0$ for all nonzero $\mathsf{A}_\mathsf{a} \in \mathfrak{O}_\mathsf{a}$. The pairing is said to be *symmetric* if $\mathsf{C}^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{B}'_{\mathsf{a}'}) = \mathsf{C}^{\mathsf{aa}'}(\mathsf{B}_\mathsf{a}, \mathsf{A}'_{\mathsf{a}'})$ for all $\mathsf{A}_\mathsf{a}, \mathsf{B}_\mathsf{a} \in \mathfrak{O}_\mathsf{a}$. The pairing is said to be *distinguishing* if whenever an operational space element yields only physical (non-negative) probability weights through the correlation, the element is physical, i.e., whenever A_a is such that $\mathsf{C}^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{B}'_{\mathsf{a}'}) \geq 0$ for all $\mathsf{B}'_{\mathsf{a}'} \in \mathfrak{O}^+_{\mathsf{a}'}$, $\mathsf{A}_\mathsf{a} \in \mathfrak{O}^+_\mathsf{a}$. A *factorizably symmetric distinguishing pairing* is such that it factorizes for operational spaces with factors while preserving the symmetric and distinguishing properties, i.e., for $\mathfrak{O}_\mathsf{a} = \mathfrak{O}_{\mathsf{a}_1\mathsf{a}_2}$, $\mathsf{A}_\mathsf{a} = \mathsf{A}_{\mathsf{a}_1}\mathsf{A}_{\mathsf{a}_2}$, and $\mathsf{B}_\mathsf{a} = \mathsf{B}_{\mathsf{a}_1}\mathsf{B}_{\mathsf{a}_2}$, $\mathsf{C}^{\mathsf{aa}'}(\mathsf{A}_\mathsf{a}, \mathsf{B}'_{\mathsf{a}'}) = \mathsf{C}_1^{\mathsf{a}_1\mathsf{a}'_1}(\mathsf{A}_{\mathsf{a}_1}, \mathsf{B}'_{\mathsf{a}'_1})\mathsf{C}_2^{\mathsf{a}_2\mathsf{a}'_2}(\mathsf{A}_{\mathsf{a}_2}, \mathsf{B}'_{\mathsf{a}'_2})$ factorizes into two pairings $\mathsf{C}_1^{\mathsf{a}_1\mathsf{a}'_1}$ and $\mathsf{C}_2^{\mathsf{a}_2\mathsf{a}'_2}$ such that both are symmetric and distinguishing.

Postulate 3 (Pairing)**.** Each operational space has at least one factorizably symmetric distinguishing pairing.

The next postulate refers to the notion of homogeneity. An ordered vector space V is *homogeneous* if $\mathrm{Aut}(V)$, the group of order-automorphisms on V, acts transitively on the interior of V_+.

Postulate 4 (Homogeneity)**.** Operational spaces are homogeneous.

The postulate intuitively says that inside an operational space any region looks locally like any other. For example, the qubit space of ordinary quantum theory is homogeneous, as there is no preferred direction or region inside the space.

The previous postulates already offer strong constraints on the compatible theories. In particular, it can be shown that the theories are restricted to self-dual (A finite-dimensional ordered vector space V is *self-dual* if it has an inner product such that a belongs to the positive cone V^+ iff $\langle a, b\rangle \geq 0$ for all $b \in V_+$.) and homogeneous spaces, so that only the self-adjoint parts of real, complex, quaternionic, 3-by-3 octonions matrix algebras, spin factors, and their direct sums are allowed [78, 79, 80]. A most general theory fulfilling the above postulates appears to be direct sum of the different types of the systems just listed. However, as long as a single quantum qubit shows up in the combination, the theory must be exclusively complex Hilbert space quantum (see the Barnum-Wilce Theorem below). To rule out the only possibility against this (a qubit does not show up in the combination), we assume:

Postulate 5 (Qubit)**.** There exists a qubit.

2.7.5 Derivation

The task of deriving the complex Hilbert space structure is made fairly simple thanks to the previous works of Barnum and Wilce [81], Koecher [78], Vinberg [79], and Jordan, von Neumann and Wigner [80]. The relevance of these results is condensed in the Barnum-Wilce Theorem [81]:

Theorem (Barnum-Wilce)**.** For a homogeneous and factorizably self-dual probabilistic theory, if it obeys tomographic locality and contains a qubit, then all its systems are self-adjoint parts of complex matrix algebras.

Factorizably self-dual means that the self-dualizing inner product factors on two subsystems, i.e., $\langle \mathsf{A_a B_b}, \mathsf{X_a Y_b} \rangle = \langle \mathsf{A_a}, \mathsf{X_a} \rangle \langle \mathsf{B_b}, \mathsf{Y_b} \rangle$. Tomographic locality is concisely characterized as $d_{\mathsf{ab}} = d_{\mathsf{a}} d_{\mathsf{b}} = c_{\mathsf{ab}} = c_{\mathsf{a}} c_{\mathsf{b}}$.

The task is show that the conditions of the theorem are satisfied. Taking Postulate 1 as a fundamental one that all further deductions are based on, homegeneity and the existence of a qubit are guaranteed by Postulates 4 and 5. Factorizable self-duality is a consequence of Postulates 3. Tomographic locality is a consequence of Postulate 2. Therefore by the Barnum-Wilce Theorem, a theory obeying the postulates must be based on the familiar complex Hilbert space structure.

To point some directions for further works, we note that the postulates are compatible with more than one quantum theory, including quantum theories with explicit indefinite causal structure (e.g., [82, 58, 83, 63, 84]), and ordinary formulations of quantum theory with definite causal structure (definite causality can be imposed as a further postulate). An interesting question is if one among these many compatible theories describes nature best.

Another interesting question is to identify postulates that derive infinite dimensional quantum theory such as quantum field theory without assuming definite causal structure. A first step towards solving this question is to actually formulate a quantum field theory with indefinite causal structure. We turn to this topic in the next chapter.

Chapter 3

Causally neutral quantum field theory[1]

The Standard Model of particle physics is built on quantum field theory (QFT) in flat spacetime. Black hole thermodynamics is built on QFT in curved spacetime (QFTCS). These theories assume classical spacetime. Yet quantum spacetime effects are believed to be crucially relevant (e.g., for the ultraviolet regularizations of the Standard Model, and the transplanckian problem of Hawking radiation [85]).

Quantum field theory incorporating quantum spacetime effects had been studied, for example QFT with spin networks [28], and QFT with causal sets [86]. What has not been done is to study QFT with manifest indefinite spacetime causal structure. We present a framework for this kind of study in the present chapter.

A natural name for the framework could be "QFT on quantum spacetime", but this name has been used for noncommutative geometry spacetime [87]. We instead call the framework "causally neutral quantum field theory" (CNQFT). Table 3.1 compares CNQFT with ordinary QFT and highlight some important differences.

After a review of ordinary QFT in Section 3.1 as an orientation, the framework is presented in Section 3.2, followed by Section 3.3 on related works. One application of the framework is the proposal that causal fluctuations provide a UV regularization for field entanglement (Section 3.4). These are followed by some discussions on major open questions in Section 3.5 and Section 3.6.

The CNQFT framework to be presented below develops the ideas sketched out in an essay [4], with some technical updates such as the use of the free product algebra and of non-post-selected states rather than transition amplitude states (see Section 3.2).

[1]I am very grateful to my supervisors Lucien Hardy and Achim Kempf for various valuable discussions on the topics of this chapter (also of the whole thesis, but especially this chapter). Some ideas I originally had are rather crude in hindsight, and my supervisors' questions such as on the choice of the algebra and on the applicability of the GNS construction pointed out important directions to improve the ideas.

Table 3.1: A comparison between ordinary QFT and CNQFT

QFT	CNQFT
$(\mathcal{M}, g_{ab})$ - spacetime manifold	N/A
$\mathcal{A}$ - field/observable algebra	$\mathfrak{A} = \star_{i \in I} \mathfrak{A}_i$ - free product algebra
$[\phi_1, \phi_2] = 0$ for spacelike separation	$[\phi_1, \phi_2] = 0$ generically
$\omega : \mathcal{A} \to \mathbb{C}$ - states	$\omega : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ - generalized states
linear functional	bilinear functional
$\omega(a^* a) \geq 0$	$\omega(a^*, a) \geq 0$
$\omega(e) = 1$	$\omega(e, e) = 1$
ω does not encode causal structure	ω encodes causal structure

3.1 QFT in background spacetime

This section reviews QFT in flat and curved classical spacetime in their axiomatic formulations and orients the introduction of the new framework in the next section.

QFT in flat spacetime and QFT in curved spacetime are significantly different in terms of their axioms. The reasons for the differences are nicely summarized in Hollands and Wald's works on axiomatic formulations of QFTCS [88, 89].[2] In the following, we review their summary.

QFT in flat spacetime as usually taught in introductory courses falls under the Wightman axioms [91], which can be summarized as follows:

1. States are unit rays in a global Hilbert space $\mathcal{H}$ carrying a unitary representation of the Poincaré group.

2. (Spectrum condition) The 4-momentum operator P^μ (generator of Poincaré translations) acting on $\mathcal{H}$ has eigenvalues in the closed future light cone.

3. (The vacuum) There is a unique Poincaré invariant state, which is understood to be the vacuum.

4. The quantum fields are operator-valued distributions on a dense domain $D \subset \mathcal{H}$. D is invariant under the actions of both the Poincaré group elements and the fields and their adjoints.

5. The fields transform Poincaré covariantly.

6. For spacelike separations the fields either commute or anticommute.

[2]See also [90] for some informative reviews about some recent progresses in QFTCS, and AQFT in general.

These axioms don't generalize directly to curved spacetime [88]. There is no global symmetry for a general curved spacetime. One cannot speak of Poincaré invariance/covariance, and of the six Wightman axioms, only the sixth can be preserved without modification.

Hollands and Wald propose to proceed to curved spacetime (assuming the spacetime is *globally hyperbolic*) as follows [88, 89] (see also [92]).

- In order to not assume a preferred vacuum state and a preferred Hilbert space from the outset, use the algebraic approach to QFT [93, 56]. The field operators are now viewed as algebraic elements and the states are now viewed as functionals on the algebra. One can formulate physics on this algebraic structure without committing to particular Hilbert spaces and particular choices of vacuum states.

- The absence of a spectrum condition based on total energy-momentum can be dealt with by restricting the singularity structure of the correlation functions (microlocal spectrum condition).

- One can generalize "Poincaré invariance/covariance" conditios to "general invariance/covariance" conditions (local and covariant fields [94]). Roughly speaking its content is that a causality preserving isometric embedding of spacetime region induces a natural isomorphism of the quantum field algebra [88].

The axioms for QFT in flat spacetime is updated as follows for QFT in curved spacetime.

i) **Adopt the algebraic approach.** 1, 4, 5 are replaced by formulating the theory in the algebraic approach, requiring that the quantum fields be local and covariant, and using the GNS construction to find Hilbert space representations.

ii) **Take advantage of microlocal analysis and the correlation singularity structure.** 2 is replaced by the microlocal spectrum condition.

iii) **Assign more duties to the state.** 3 requires a more complicated update. It is replaced by the requirement that there exists an operator product expansion (OPE) for the products of quantum fields in the short-distance limit, and properties imposed on the OPE.

iv) **Microcausality stays.** 6 stays the same.

It can be said that a thorough update is made when one moves from flat to curved classical spacetime. Axioms 1-5 are all modified, and only 6 is preserved. As we will see in more detail in the following Section 3.2, from curved classical spacetime to quantum indefinite spacetime some further updates are needed.

The lesson of i), adopting the algebraic approach, is kept, although with an essential modification. We will use an algebra (free product algebra) that has a trivial algebraic structure. The algebra in no way reflect spacetime causal relations. Instead, spacetime causal relations is encoded in the generalized state.

Regarding ii), the microscopic UV structure of the correlations in the new framework is discussed in Section 3.4. We are inclined to think that causal fluctuations regularize the ultraviolet divergence of the correlation function, so that the singularity structure of QFT on classical spacetime is not preserved. Analysis based on the singularity structure needs to be replaced something new, which is unknown to us at present.

The general lesson of iii), assigning more duties to the state, characterizes the new framework, although the duties are not assigned through OPE. The algebra is made trivial to avoid a definite causal structure, and the generalized state steps in to incorporate indefinite causal structure. Selection criteria for physical theories are likely to be imposed on the generalized state.

Regarding iv), due to having an algebra with a trivial algebraic structure, microcausality is dropped.

3.2 Causaly neutral QFT (CNQFT)

The task is to find a suitable framework that incorporates indefinite causal structure in studies of QFT. We first give a strategic discussion on fulfilling the task, before going into the technical details.

Ordinary algebraic QFT (AQFT) has three basic elements, an algebra that contains the observables, states defined as linear functionals on the algebra, and the spacetime manifold. From the operation-correlation perspective, the algebra is used to describe operations, the state is used to describe correlations. The spacetime manifold is used to impose conditions on the algebra and the states to express ideas such as locality. There are two facts about the spacetime manifold relevant for our task of incorporating indefinite causal structure: 1. The spacetime is a third object that needs to be specified independently in addition to the algebra and the states. 2. The spacetime manifold has definite causal structure.

We certainly want to avoid item 2 to incorporate indefinite causal structure. The question is whether we still want to follow item 1 and specify spacetime with indefinite causal structure as an independent object. The "yes" option is made by theories such as loop quantum gravity (LQG) and causal set theory. The "no" option would need to encode spacetime properties such as indefinite causal structure in the algebra and/or the state.

We choose the second option. First, we have concrete ideas on how to incorporate indefinite causal structure in the operation-correlation paradigm (e.g., for finite dimensional systems this can be done using the process matrix correlations.), and it is unclear how to do so through independent spacetime object such as LQG states or causal sets. Second, the second option has the advantages of an operational approach discussed in Section 1.4.

An immediate question follows: What about the algebra? In traditional AQFT it is spacetime as a third object that offers the algebra its non-trivial algebraic structure[3]. How is the algebraic structure to be determined now?

Our answer is that in the new framework the algebra has no non-trivial algebraic structure (This will be made precise when we introduce the free product algebra in the next subsection.). Spacetime relations instead are reflected in the generalized states. This appears to go against the whole tradition of AQFT, but since the algebra and the states are dual to each other (in the functional analytic sense), it is nothing too exotic to move structures from one to the other. All in all, it is the interaction of the algebra and the states (the evaluation of the states on the algebra) that leads to quantities we have direct empirical access to such as probabilities and expectation values. As long as a structure is reflected in this interaction, it is not essential whether the structure is completely encoded in the algebra or in the states.

This is the strategic plan. As soon as one attempts to realize it into details, one encounters the question what to make of the global time evolution carried by the algebraic element in traditional AQFT. This question is illustrated in the following example.

3.2.1 A motivating example: the quantum switch

Consider the ordinary QFT correlation

$$\langle\psi|\,\tilde{x}\tilde{y}\,|\psi\rangle\,, \tag{3.1}$$

with $|\psi\rangle$ a vector state, and the operators $\tilde{x}, \tilde{y}$ belonging to the ordinary QFT algebra $\tilde{\mathfrak{A}}$. For example, these can be (smeared) field operators.

It is important that in ordinary QFT, the algebraic elements are understood to carry time evolution in themselves. For example, $\tilde{x} = U_1^* x U_1$, where U_1 is the time evolution unitary from the time of the state ψ to the time of $\tilde{x}$. Similarly $\tilde{y} = U_2^* y U_2$. For intuition, one may think of the tilde elements as "Heisenberg picture" operators, and the "untilde" elements, i.e., elements without tilde, as "Schrödinger picture" operators.[4] Using the untilde elements, Equation (3.1) becomes

$$\langle\psi|\,(U_1^* x U_1)(U_2^* y U_2)\,|\psi\rangle = \langle\psi_1|\,x U_{1,2} y\,|\psi_2\rangle\,, \tag{3.2}$$

[3] In traditional AQFT based on C*-algebras, the von Neumann algebra factors of open regions are believed to be universally hyperfinite type III_1 [56]. This is the "trivial part" of the algebraic structure (Note that within the factor itself there can be "non-trivial" algebraic relation such as between non-commuting observables.). The "non-trivial" part is the relations of the factors in the net of the global algebra. This is specified by the spacetime manifold.

[4] This is only for intuition but cannot be taken too seriously, because the Heisenberg and Schrödinger pictures suggest the presence of a global unitary evolution, which we will eliminate in the present framework in the end. The real point is that the tilde operators contains information about the causal/dynamical correlation among each other, which the untilde operators do not contain.

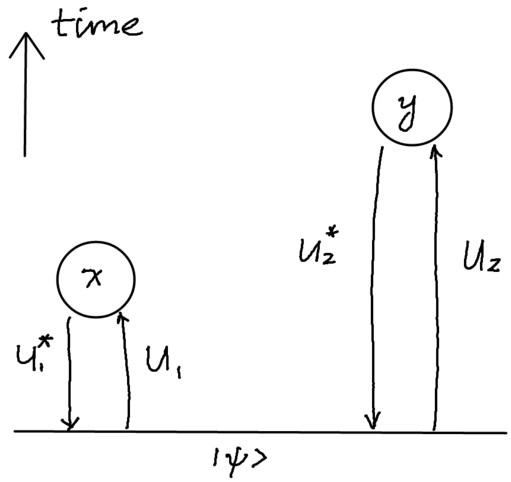

Figure 3.1: Ordinary QFT correlation function with the untilde elements. The state at a fixed reference time must be evolved by the global time evolution before being acted on by the tilde operators. Alternatively, the tilde operators must be evolved conjugatedly before acting on the state.

where $|\psi_1\rangle = U_1 |\psi\rangle$, $|\psi_2\rangle = U_2 |\psi\rangle$, and $U_{1,2} = U_1 U_2^*$. The structure is illustrated in Figure 3.1.

That the ordinary QFT algebraic element implicitly carry time evolution is the reason why we need a new framework for indefinite causal structure. A global time evolution imposes a definite causal order. It should not be there for a framework for indefinite causal structure.

For example, ordinary QFT with global time evolutions does not incorporate the following example of a "quantum switch" (Figure 3.2). The quantum switch expresses two operators in a "superposition" of causal order. It is originally devised on finite dimensional systems to outperform circuits with definite causal structure in information processing tasks ([82, 95, 96, 97, 98, 99]), and there have been claims of its experimental realization in the laboratory (e.g. [100, 101, 102]). The following transition amplitude from the initial state $|\psi\rangle$ to the final state $|\phi\rangle$ can be stated for finite or infinite dimensional systems using the untilde operators.

$$\langle\phi| \, v \Big(\frac{1}{\sqrt{2}} |0\rangle\langle 0| \otimes U_{v,x} x U_{x,y} y U_{y,u} + \frac{1}{\sqrt{2}} |1\rangle\langle 1| \otimes U'_{v,y} y U'_{y,x} x U'_{x,u} \Big) u \, |\psi\rangle \, . \tag{3.3}$$

On the other hand, it is quite unclear how to express transition amplitudes or probabilities associated to the quantum switch using the tilde operators that carry global time evolutions, especially given that a U's and its corresponding U''s can differ.

Given this difficulty to incorporate the quantum switch or more general physical processes with indefinite causal structure into QFT using the traditionally adopted tilde operators, we are naturally led to the option of achieving this using the untilde

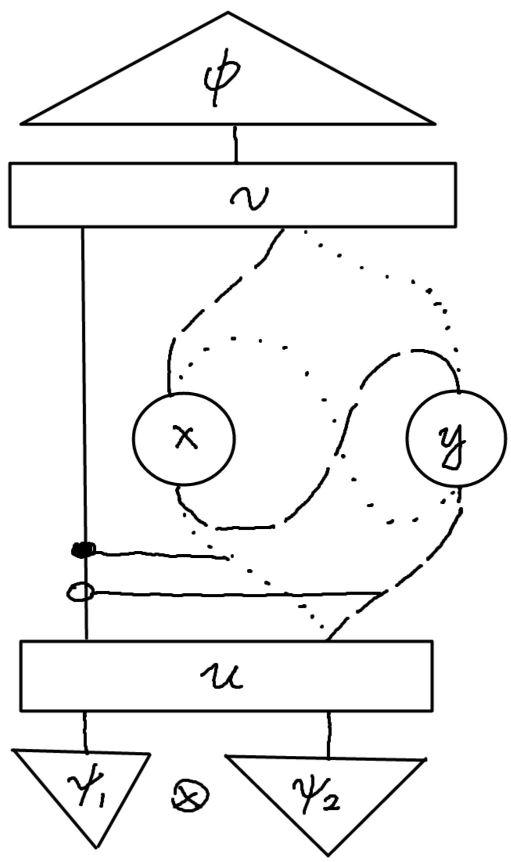

Figure 3.2: A QFT "quantum switch". The initial state $\psi = \psi_1 \otimes \psi_2$ factors into a qubit controlling part and the rest. When $|\psi_1\rangle = |0\rangle$, ψ_2 goes through y and then x. When $|\psi_1\rangle = |1\rangle$, ψ_2 goes through x and then y. For $|\psi_1\rangle$ in a superposition $|\psi_2\rangle$ goes through x and y in a quantum superposition of order.

elements which do not carry global time evolutions in themselves. The rest of this section realizes this option.

3.2.2 The free product algebra

We want to use the untilde type of elements for the algebra of the new framework. Yet an algebra needs to have a product. We need x and y to interact with each other to form elements such as xy and yx. The product of the tilde elements of ordinary QFT reflects a definite causal order. For example, $\tilde{x}\tilde{y}$ means applying $\tilde{y}$ first and then $\tilde{x}$. How to have an algebraic product without associating a definite causal order, or any causal structure at all to it?

One solution is provided by the free product algebra, previously used by Raasakka in his spacetime-free algebraic quantum theory [43]. The free product algebra imposes no non-trivial algebraic relations across the original factors of algebras. We adopt the free product algebra as the algebra for the new framework of QFT.

There are two ways to define the free product algebra. The definition right below spells out the generators of the algebra and gives a constructive definition. The definition to be presented afterwards is through a universal property that tells the fundamental reason to adopt the free product algebra for our purpose of not imposing any non-trivial algebraic relation.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two unital *-algebras. Their *free product *-algebra* (or free product algebra in short) $\mathfrak{A}_1 \star \mathfrak{A}_2$ is a unital *-algebras linearly generated by finite sequences of elements $x_1 x_2 \cdots x_n$, where $x_k \in \mathfrak{A}_1$ or $x_k \in \mathfrak{A}_2$ for all k. A sequence of elements of the form $x_1 x_2 \cdots x_n$ is also called a “word”. In the free product algebra, the product of $x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_n \in \mathfrak{A}_1 \star \mathfrak{A}_2$ is simply the concatenation of the words: $(x_1 x_2 \cdots x_n) \cdot (y_1 y_2 \cdots y_n) = x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n$. The *-operation of $\mathfrak{A}_1 \star \mathfrak{A}_2$ is simply given by $(x_1 x_2 \cdots x_n)^* = x_n^* x_{n-1}^* \cdots x_1^*$.

Two equivalence relations are imposed on elements of $\mathfrak{A}_1 \star \mathfrak{A}_2$. First, the unit elements of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A} := \mathfrak{A}_1 \star \mathfrak{A}_2$ are identified, i.e., $e_{\mathfrak{A}_1} \sim e_{\mathfrak{A}_2} \sim e_{\mathfrak{A}}$. In the rest of this chapter we use e to denote the unit. Second, if in $x_1 x_2 \cdots x_n$ two neighboring elements x_k, x_{k+1} belong to the same $\mathfrak{A}_i$, they can be contracted into a single element $x' = x_k x_{k+1} \in \mathfrak{A}_i$ according to the product rule of $\mathfrak{A}_i$. In other words, we impose the equivalence relation $x_1 x_2 \cdots x_k x_{k+1} \cdots x_n \sim x_1 x_2 \cdots x' \cdots x_n$. This implies that any element of $\mathfrak{A}_1 \star \mathfrak{A}_2$ can be written as a word with a sequence of letters coming alternatively from the two original algebras. The operation of forming the free product is associative and commutative. Given multiple unital *-algebras $\mathfrak{A}_i$ indexed by $i \in I$, we can form a joint free product algebra $\star_{i\in I}\mathfrak{A}_i$ by iteration.

The reason to use the free product algebras is its universal property, that $\mathfrak{A} = \star_{i\in I}\mathfrak{A}_i$ with unital *-homomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{A}$ is the unique unital *-algebra satisfying the following condition. Given any *-algebra B and unital *-homomorphisms $\phi_i : \mathfrak{A}_i \to B$, there exists a unique unital *-homomorphism $\Phi : \mathfrak{A} \to B$ so that $\phi_i = \Phi \circ \psi_i$ (In short, the free product is the categorical coproduct.). This suits the need to have a global algebra that imposes no non-trivial algebraic relations for elements from the original individual algebras.[5]

Another conceivable option for the global algebra is to form a tensor product of the individual algebras. This can first of all be reproduced by the free product algebra by the universal property, so the free product algebra is more general. In addition, imposing a tensor product structure may be too strong a starting point for a field theory in view of the subtleties of von Neumann algebras regarding the tensor product structure (e.g., Tsirelson's problems). Furthermore, one might be interested in generating an algebra without the tensor product structure from two original algebras, e.g., take two subalgebras from the same spacetime region so that they are not meant to commute in the generated algebra, whence a tensor product structure is inappropriate. For these reasons we adopt the free product algebra rather than the tensor product algebra.

We stress that the original factors $\mathfrak{A}_i$ used to form the free product does not necessarily correspond to algebras attached to spacetime regions. By this we mean that the factor algebras do not have to be attached to "regions" in any sense, and when they do, the regions do not have to be "spacetime" regions. We keep the structure adaptable so that a free product can be taken whenever there are original factor algebras that are distinguishable among each other. For instance, the original factors may be associated to regions in Hardy's operational space [41, 46], which is coordinatized by physical fields rather than the virtual spacetime coordinates.

3.2.3 An example: the ordinary "two-point" correlation

We illustrate the use of the free product algebra with a simple example. This also brings forward the next challenge we will deal with – a naive generalization of the traditional definition of the state for the untilde elements does not work.

For simplicity we consider the original factors to be von Neumann algebras of bounded operators. This is a standard practice for AQFT and avoids the complication of unbounded operators with their domains issues.

First consider the ordinary QFT "two-point" correlation of Equation (3.1). This is based on the tilde elements. Suppose $\tilde{x}$ and $\tilde{y}$ are operators associated with two different spacetime regions, each associated with a von Neumann subalgebra of the global algebra $\tilde{\mathfrak{A}}$ so that $\tilde{x} \in \tilde{\mathfrak{A}}_1$ and $\tilde{y} \in \tilde{\mathfrak{A}}_2$. Then the free product $\tilde{\mathfrak{A}}_\star = \tilde{\mathfrak{A}}_1 \star \tilde{\mathfrak{A}}_2$ is generated by elements of the form $\tilde{a}\tilde{b} \cdots \tilde{c}$, where each individual letter belongs to either $\tilde{\mathfrak{A}}_1$ or $\tilde{\mathfrak{A}}_2$. Define the state by linear extension as

$$\tilde{\omega} : \tilde{\mathfrak{A}}_\star \to \mathbb{C}, \quad \tilde{\omega}(\tilde{a}\tilde{b} \cdots \tilde{c}) = \langle \psi |\, \tilde{a}\tilde{b} \cdots \tilde{c}\, | \psi \rangle \,. \tag{3.4}$$

[5] In fact it is possible to specify a further structure to use the amalgamated free product algebra when there are common elements among the original *-algebras. This structure may be useful in some applications, but for the purpose of this thesis it is not used so we do not expand on it.

Note that inside $\tilde{\omega}$, $\tilde{a}\tilde{b}\cdots\tilde{c}$ is a sequence of elements, and is an element of the free product algebra $\tilde{\mathfrak{A}}_\star$, whereas sandwiched between $\langle\psi|$ and $|\psi\rangle$, $\tilde{a}\tilde{b}\cdots\tilde{c}$ is a product of the individual elements in the original ordinary QFT algebra $\tilde{\mathfrak{A}}$.

This state $\tilde{\omega}$ reproduces the physics of the original ordinary QFT correlation restricted to the two regions. As an ordinary state (a positive normalized linear functional), it has a GNS representation $\left|\tilde{\Omega}\right\rangle$ in some Hilbert space $\mathcal{H}_{\tilde{\omega}}$, with the algebraic elements $\tilde{a}, \tilde{b}, \cdots$ represented by $\tilde{A}, \tilde{B}, \cdots$ as operators on $\mathcal{H}_{\tilde{\omega}}$. If in the original theory $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ are associated with causally connected regions and $\langle\psi|\,[\tilde{x},\tilde{y}]\,|\psi\rangle \neq 0$, then $[\tilde{X},\tilde{Y}] \neq 0$ since

$$\left\langle\tilde{\Omega}\right|[\tilde{X},\tilde{Y}]\left|\tilde{\Omega}\right\rangle = \langle\psi|\,[\tilde{x},\tilde{y}]\,|\psi\rangle \neq 0. \tag{3.5}$$

Can one apply a similar procedure to use the free product algebra for the untilde elements? This would bring us one step closer towards incorporating indefinite causal structure. Consider Equation (3.2). Suppose the untilde elements in each factor $i = 1, 2$ form their own algebra $\mathfrak{A}_i$. Then we consider the free product $\mathfrak{A} = \mathfrak{A}_1 \star \mathfrak{A}_2$ generated by elements of the form $ab\cdots c$, where each individual letter belongs to either $\mathfrak{A}_1$ or $\mathfrak{A}_2$. Tentatively define the state by lineary extension according to

$$\omega : \mathfrak{A} \to \mathbb{C}, \quad \omega(ab\cdots c) = \langle\psi_1|\,(a\cdots)U_{1,2}(b\cdots)\,|\psi_2\rangle\,, \tag{3.6}$$

where the first factor $(a\cdots)$ collects all the elements from $\mathfrak{A}_1$ in the order of their appearance, and the second factor $(b\cdots)$ collects all the elements from $\mathfrak{A}_2$ in the order of their appearance. (Each factor can in fact be reduced to a single element in $\mathfrak{A}_1$ or $\mathfrak{A}_2$.)

It turns out this is not a state, because it is not positive. (Recall that ω is positive if $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{A}$.) For example, let $a = xy \in \mathfrak{A}$ with $x \in \mathfrak{A}_1$ and $y \in \mathfrak{A}_2$. Then $\omega(a^*a) = \langle\psi_1|\,x^*xU_{1,2}y^*y\,|\psi_2\rangle$. This need not be a real number, so $\omega(a^*a) \geq 0$ cannot hold in general. The reason that $\omega(a^*a) \geq 0$ holds for ordinary QFT is that in a Hilbert space $\omega(a^*a)$ is represented in the form $\langle\psi|\,A^*A\,|\psi\rangle$, which equals $\|A\,|\psi\rangle\|^2$. This in turn is possible because the time evolutions are absorbed in the algebraic elements. When the time evolutions are left out of the algebraic elements, they have to show up in between the algebraic elements such as in $\omega(a^*a) = \langle\psi_1|\,x^*xU_{1,2}y^*y\,|\psi_2\rangle$, which does not obtain the form $\langle\psi|\,A^*A\,|\psi\rangle$.

This naive trial to define states on the free product algebra for the untilde elements fails to give us states with the usually expected property of positivity. If the free product algebra for the untilde elements is the way to go for incorporating indefinite causal structure, we need some new idea.

3.2.4 State for transition amplitudes without positivity?

One idea is to give up the requirement for positivity. The physical reason for positivity is that probabilities are positive. Ordinary QFT states have a probabilistic interpretation. For a projection operator a of an observable, $\omega(a^*a) = \langle\psi| A^*A |\psi\rangle = \mathrm{tr}[A |\psi\rangle\langle\psi| A^*]$ expresses the probability for observing the outcome represented by a. A state for the tilde operator of the form $\omega(a) = \langle\psi_1| xU_{1,2}y |\psi_2\rangle$ is understood as a complex transition amplitude, there does not seem to be a probabilistic interpretation, and $\omega(a^*a)$ certainly is not supposed to yield a probability.

One can then try to define define the "transition amplitude states" abstractly as linear functionals

$$\omega : \mathfrak{A} \to \mathbb{C} \tag{3.7}$$

without imposing any positivity requirement. Since the transition amplitude is not a probability, there is no normalization requirement directly imposed on ω either, although one should require $|\omega(a)| \leq 1$ for a physical transition amplitudes.

The transition amplitude states can incorporate indefinite causal structure. For instance, $\omega(a)$ defined according to Equation (3.3) incorporates a superposition of causal order.

A new framework that incorporates indefinite causal structure should also incorporate definite causal structure as a special case. How does one recover the ordinary QFT "two-point" correlations such as Equation (3.4)? This question led us to the proposal in the following subsection, which defines states on the free product algebra in a way closer to ordinary QFT states.

3.2.5 The generalized states

A difference between the transition amplitude states and the ordinary QFT states is that an amplitude is not a probability. To turn an amplitude such as $\langle\psi_1| xU_{1,2}y |\psi_2\rangle$ into a probability, we can multiply by its complex conjugate

$$\langle\psi_2| y^*U_{2,1}x^* |\psi_1\rangle\langle\psi_1| xU_{1,2}y |\psi_2\rangle , \tag{3.8}$$

where $U_{2,1} = U^*_{1,2}$. Another difference between the transition amplitude states and the ordinary QFT states is that the former post-select some final vector, whereas the latter do not. To not post-select, we can sum over the final states in a basis:

$$\sum_{|\psi_1\rangle} \langle\psi_2| y^*U_{2,1}x^* |\psi_1\rangle\langle\psi_1| xU_{1,2}y |\psi_2\rangle = \langle\psi_2| y^*U_{2,1}x^* \sum_{|\psi_1\rangle} |\psi_1\rangle\langle\psi_1| xU_{1,2}y |\psi_2\rangle . \tag{3.9}$$

Here $\sum_{|\psi_1\rangle} |\psi_1\rangle\langle\psi_1|$ equals the identity operator. We refrained from substituting the identity operator to make the "conjugate multiplication" structure manifest.

These considerations motivate us to define states in the CNQFT framework as follows. A *generalized state* is a bilinear functional

$$\omega : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C} \tag{3.10}$$

satisfying $\omega(a^*, a) \geq 0$ and $\omega(e, e) = 1$. Here $\mathfrak{A}$ is a unital *-algebra taken to be a free product algebra.

In ordinary QFT a state $\omega' : \mathfrak{A} \to \mathbb{C}$ is a linear functional from a *-algebra or a C^*-algebra to the complex numbers, and obey the conditions $\omega'(a^* a) \geq 0$ and $\omega'(e) = 1$. $\omega(a^*, a) \geq 0$ and $\omega(e, e) = 1$ are analogues of these conditions.

The state ω is called "generalized" because in contrast to states in ordinary QFT the generalized states carry information about the dynamical correlations of the algebras. We show next how ordinary QFT correlation functions can be recovered from the generalized states next.

The condition $\omega(a^*, a) \geq 0$ implies two useful properties for ω.

$$\omega(a^*, b) = \overline{\omega(b^*, a)} \qquad \text{[conjugate symmetry]}, \tag{3.11}$$

$$|\omega(a^*, b)|^2 \leq \omega(a^*, a)\omega(b^*, b). \qquad \text{[Cauchy-Schwarz]} \tag{3.12}$$

Sketch of proof: $\omega((\lambda a + b)^*, \lambda a + b) \geq 0$ for arbitrary $\lambda \in \mathbb{C}, a, b \in \mathfrak{A}$. The LHS equals $|\lambda|^2 \omega(a^*, a)^2 + \bar{\lambda}\omega(a^*, b) + \lambda\omega(b^*, a) + \omega(b^*, b)^2$. That the imaginary part vanishes for arbitrary λ implies "conjugate symmetry", which in turn implies the second and third terms sum to $2 \operatorname{Re} \lambda\omega(b^*, a)$. Then turn the inequality into a quadratic polynomial for λ and derive that the discriminant must obey "Cauchy-Schwarz" for the inequality to hold.

Ordinary correlation functions

Equation (3.9) allows us to reproduce the ordinary QFT "two-point" correlation function. In this case the two original algebras are $\mathfrak{A}_1$ and $\mathfrak{A}_2$. The free product $\mathfrak{A} = \mathfrak{A}_1 \star \mathfrak{A}_2$ is generated by terms of the form $a = xy \cdots z$, where each individual element belongs to either $\mathfrak{A}_1$ or $\mathfrak{A}_2$. Define the state according to Equation (3.9) by

$$\begin{aligned} &\omega : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}, \\ &\omega(b, a) = \langle\psi_2| (y_b \cdots) U_{2,1}(x_b \cdots) \sum_{|\psi_1\rangle} |\psi_1\rangle\langle\psi_1| (x_a \cdots) U_{1,2}(y_a \cdots) |\psi_2\rangle, \end{aligned} \tag{3.13}$$

where we grouped elements of a and b according to their original factors. The factor $(x_a \cdots)$ collects all the elements of a coming from $\mathfrak{A}_1$ in the order of their appearance, the factor $(y_a \cdots)$ collects all the elements of a coming from $\mathfrak{A}_2$ in the order of their appearance, and similarly for b. (Each factor can in fact be reduced to a single element in $\mathfrak{A}_1$ or $\mathfrak{A}_2$.) It is easy to check that the three conditions in the definition of the generalized state hold. The ordinary QFT "two-point" correlation function (3.1) $\langle\psi| \tilde{x}\tilde{y} |\psi\rangle = \langle\psi| (U_1^* x U_1)(U_2^* y U_2) |\psi\rangle = \langle\psi_1| x U_{1,2} y |\psi_2\rangle$ is recovered as $\omega(e, a)$ with $a = xy$.

In general, an n-point correlation function can be recovered analogously by introducing more entries such as x and y, along with more U's to connect the entries.

The quantum switch

The quantum switch with indefinite causal structure described by the transition amplitude

$$\langle\phi| v\Big(\frac{1}{\sqrt{2}}|0\rangle\langle 0| \otimes U_{v,x}xU_{x,y}yU_{y,u} + \frac{1}{\sqrt{2}}|1\rangle\langle 1| \otimes U'_{v,y}yU'_{y,x}xU'_{x,u}\Big)u|\psi\rangle \tag{3.14}$$

of (3.3) can be incorporated similarly. This time there are four original algebras with elements of the form x, y, u and v. Denote the amplitude (3.14) by $\mathcal{A}(|\phi\rangle, x, y, u, v)$. For $b, a \in \mathfrak{A} = \star_{i=1,2,3,4}\mathfrak{A}_i$, define the generalized state ω by

$$\omega(b,a) = \sum_{|\phi\rangle} \overline{\mathcal{A}}(|\phi\rangle, x_b \cdots, y_b \cdots, u_b \cdots, v_b \cdots) \tag{3.15}$$

$$\times \mathcal{A}(|\phi\rangle, x_a \cdots, y_a \cdots, u_a \cdots, v_a \cdots), \tag{3.16}$$

where the line over $\mathcal{A}$ denotes complex conjugate, and similar to the example above we grouped elements of a and b according their original factors. The stars in the first factor are used to compensate for the complex conjugation of the amplitude. The defining properties of the generalized state clearly hold.

3.2.6 The GNS construction[6]

The examples above are defined referring to Hilbert space operators and vectors. However, in the general framework the algebra is an abstract *-algebra and the states are functionals on this abstract algebra – there is no reference to Hilbert spaces. If a state is given abstractly, the standard way in an algebraic approach to find a Hilbert space representation is through the GNS construction [103, 104]. The GNS construction was originally for C^*-algebras, which applies to algebraic elements which in the end are represented as bounded operators [56]. A version of the GNS construction for *-algebras can be found in, e.g., [105]. This version is suitable for algebraic elements which in the end are represented as unbounded operators (e.g., field operators). The latter version is a bit more technically complicated because of the domain restrictions of unbounded operators. In the following we present the GNS construction for CNQFT with C^*-algebras for simplicity. The *-algebras construction can be carried out analogously following [105].

The idea is to define an inner product according to the suggestion of the expression "$\langle a|b\rangle = \omega(a^*, b)$" and obtain a Hilbert space accordingly. First note that the free product algebra $\mathfrak{A}$ is a vector space over $\mathbb{C}$. Given $a \in \mathfrak{A}$, we denote its corresponding vector space element by $|a\rangle$.

The map $\langle\cdot|\cdot\rangle : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$, $\langle a|b\rangle := \omega(a^*, b)$ obeys $\langle a|b\rangle = \overline{\langle b|a\rangle}$, since $\omega(a^*, b) = \overline{\omega(b^*, a)}$. Further, it is linear in the first argument, and is positive semidefinite, since

[6] I thank Matti Raasakka for some valuable correspondences that clarified the use of the GNS representation in a spacetime-free approach to algebraic quantum physics.

$\omega(a^*, a) \geq 0$. These satisfy the requirement of an inner product, except that it is not necessarily positive definite. One way to proceed is to "mod out" the set $\mathcal{N}_\omega := \{a \in \mathfrak{A} : \omega(a^*, a) = 0\}$ and work with the quotient space. For this we need $\mathcal{N}_\omega$ to be a vector space. This turns out to be true, since if $a, b \in \mathcal{N}_\omega$, then $\omega((a+b)^*, a+b) = \omega(a^*, b) + \omega(b^*, a) = 0$ by Equation (3.11) and Equation (3.12). Take $\mathfrak{A}/\mathcal{N}_\omega$ to be the new vector space under consideration. The equivalence class of $|a\rangle$ is denoted $|[a]\rangle$. The map $\langle\cdot|\cdot\rangle : \mathfrak{A}/\mathcal{N}_\omega \times \mathfrak{A}/\mathcal{N}_\omega \to \mathbb{C}, \langle[a]|[b]\rangle := \omega(a^*, b)$ is well-defined by the Cauchy-Schwarz inequality (3.12), and is an inner product. A completion in the norm topology yields a Hilbert space $\mathcal{H}_\omega$.

There is a representation $\pi_\omega : \mathfrak{A} \to \mathcal{L}(\mathcal{H}_\omega)$ of actions of the the algebraic elements on the Hilbert space defined on a dense domain $\mathfrak{A}/\mathcal{N}_\omega$ by $\pi_\omega(a)\,|[b]\rangle := |[ab]\rangle$. In order for this to be well-defined, we need $\mathcal{N}_\omega$ to be a left ideal. In ordinary QFT this can be derived taking advantage of the algebra with global time evolutions encoded. To proceed in the present framework we need an assumption.

- **Working assumption**[7]: We restrict attention to those $\mathfrak{A}$ and ω so that if $a \in \mathcal{N}_\omega$, then $\omega((ba)^*, ba) = 0$ for all $b \in \mathfrak{A}$.

This assumption makes $\mathcal{N}_\omega$ a left ideal, whence $\pi_\omega(a)\,|[b]\rangle := |[ab]\rangle$ is a well-defined representation. For notational simplicity we sometimes denote $\pi_\omega(a), \pi_\omega(b), \cdots$ by the corresponding capital letters $A, B, \cdots$. Note that in contrast to the GNS construction for ordinary QFT, this is not a *-representation, i.e., $\pi_\omega(a)^* = \pi_\omega(a^*)$ does not hold in general. The reason is, again, that the algebra $\mathfrak{A}$ no longer encodes global time evolutions.

Denote $|[e]\rangle$ as $|\Omega\rangle$. Then it holds that

$$\omega(a^*, b) = \langle[ae]|[be]\rangle = \langle[e]|\,\pi_\omega(a)^*\pi_\omega(b)\,|[e]\rangle = \langle\Omega|\,\pi_\omega(a)^*\pi_\omega(b)|\Omega\rangle. \tag{3.17}$$

$|\Omega\rangle$ can be regarded as a Hilbert space representation of the state ω. Recall that ordinary QFT n-point correlation functions can be recovered from $\omega(e, a)$. This equation then yields a Hilbert space representation recovery of ordinary QFT n-point correlation functions.

Like in ordinary AQFT, a generalized state ω gives rise to a folium of generalized states on the Hilbert space $\mathcal{H}_\omega$. These are the vector states ω_Ψ for $|\Psi\rangle \in \mathcal{H}_\omega$ defined by

$$\omega_\Psi(a^*, b) = \langle\Psi|\,\pi_\omega(a)^*\pi_\omega(b)|\Psi\rangle, \tag{3.18}$$

and the more general states ω_ρ for positive trace class operators ρ in the set of bounded operators $\mathcal{B}(\mathcal{H}_\omega)$ defined by

$$\omega_\rho(a^*, b) = \mathrm{tr}[\rho\pi_\omega(a)^*\pi_\omega(b)]. \tag{3.19}$$

[7] Further investigations are needed to determine how restrictive the working assumption is.

3.2.7 On the causal structure[8]

In the new framework, the causal structure is not reflected in the algebraic commutation relations, either in the free product algebra or in the operator algebra induced by the GNS representation, in sharp contrast to ordinary QFT. We believe the lesson learned is that the imaginary part of the correlation function reflects causal structure in a more fundamental way.

To see that the commutation relations of the operators here do not reflect the causal structure, consider for instance the ordinary "two-point" correlation function (3.13). Let $a_1 = x$ and $a_2 = y$, where $x \in \mathfrak{A}_1$ and $y \in \mathfrak{A}_2$. By the definition of ω, $\omega(c, a_1 a_2 b) = \omega(c, a_2 a_1 b)$ for arbitrary b and c. Hence in the GNS representation $\langle c^* | A_1 A_2 | b \rangle = \langle c^* | A_2 A_1 | b \rangle$, i.e., $A_1 A_2 = A_2 A_1$ on a dense domain of $\mathcal{H}_\omega$. This holds true even when the regions of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are causally related. The reason for the difference from ordinary QFT is that we switched to the untilde elements which do not carry time evolution across the regions.

Unlike the algebraic commutation relations, the imaginary part of the correlation function still offers information about the causal structure. When it holds that the commutator of two operators is proportional to the identity, it makes no difference whether one looks at the algebraic commutation relation or the imaginary part of the correlation function. For example, this is true for ordinary QFT of a free scalar field ϕ when we consider $[\phi(x), \phi(y)] = c\mathbb{1}$, where x and y are spacetime coordinates, and c is a complex number. In this case

$$\operatorname{Im} \langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \frac{1}{2i}(\langle \Omega | \phi(x)\phi(y) | \Omega \rangle - \langle \Omega | \phi(x)\phi(y) | \Omega \rangle^*) \tag{3.20}$$

$$\propto \langle \Omega | [\phi(x), \phi(y)] | \Omega \rangle = c. \tag{3.21}$$

We saw that in the new framework of CNQFT the algebraic commutation relation no longer carries information about the causal structure. The imaginary part of the correlation function still does. For example, in the "two-point" function example above, the imaginary part of the correlation function must be the same as in ordinary QFT, since the generalized state is defined through the ordinary QFT "two-point" correlation function. Therefore it does not have to vanish for causally related regions.

On general grounds it is reasonable to look for information about the causal structure in the the evaluation of the generalized state. Of course all the information there is about the causal structure must be contained in the generalized state, since apart from the free product algebra (which by definition does not contain information about the causal structure across its factors), the generalized state is the only other object to look for such information.

[8] I thank Fabio Costa for a valuable discussion that improved my understanding of the commutator in QFT.

3.2.8 On time symmetry

Is the mathematical framework of CNQFT time symmetric?

The mathematical framework of CNQFT is not time asymmetric. This is in contrast to operational quantum theories with causality, and to operational quantum theories with indefinite causal structure but with local causality conditions.

We will not think that the mathematical framework of CNQFT is time symmetric either. It is perhaps suitable to say that the framework does not have a predefined time [63]. There is not a global notion of time at the basic level of the framework, although such a notion may be introduced on top of the framework, whence the framework will be time symmetric.

Equation (3.9) was used to motivated the definition of the generalized state. We motivated the summation in Equation (3.9), reproduced here

$$\sum_{|\psi_1\rangle} \langle\psi_2|\, y^* U_{2,1} x^* \,|\psi_1\rangle\!\langle\psi_1|\, x U_{1,2} y \,|\psi_2\rangle = \langle\psi_2|\, y^* U_{2,1} x^* \sum_{|\psi_1\rangle} |\psi_1\rangle\!\langle\psi_1|\, x U_{1,2} y \,|\psi_2\rangle \,,$$

by summing over $|\psi_1\rangle$ as the future state. This does not imply the mathematical framework of CNQFT is time asymmetric. One can view $|\psi_1\rangle$ as the past state, and $|\psi_2\rangle$ as the future state. Then Equation (3.9) can be interpreted as containing a sum over the past state (the unitary evolution does not have a preferred time direction either).

This particular state is defined with respect to global unitary evolutions, but other states do not have to. Hence there is no global notion of time imposed for the framework.

In the abstract definition of ω, there is a condition $\omega(e, e) = 1$ motivated by the normalization of probability. This condition, as an analogy to the condition $\omega'(e) = 0$ for ω' as an ordinary QFT state, does not impose a time asymmetry to the extend that $\omega'(e) = 0$ does not impose a time asymmetry to ordinary QFT.

In theories such as the process matrix framework [58], normalization of probability on the process matrix imposes a time asymmetry. The source of the asymmetry is the local causality condition in the local laboratories. There is no analogous local causality condition in the present framework.

3.3 Related works

There have been several related previous works on removing fixed classical background spacetimes and/or incorporating quantum spacetime effects into QFT.

In devising the framework presented above we gathered a few pieces of elements from previous works. The general algebraic approach to QFT and the use of the GNS construction of course follow from Haag and Kastler's classic work [93]. The brilliant idea of using the free product algebra to emancipate the algebra from assuming a preassigned spacetime structure is due to Raasakka's spacetime-free quantum theory

[43]. The possibility of encoding indefinite causal structure in general probabilistic correlations was pointed out in Hardy's ground-breaking works [18, 19]. A concrete proposal to incorporate indefinite causal structure in quantum probabilistic correlations is offered by the by-now quite popular process matrix formalism [58, 59, 60] (Section 2.5)[9].

Although all the fundamental ideas are drawn from the works mentioned above, when combined together these ideas lead to the present framework which differs in some essential ways from all these previous works. It differs from traditional AQFT in that the algebra does not encode spacetime structure. It differs from Raasakka's spacetime-free quantum theory in considering the "untilde" (or loosely speaking "Schrödinger picture") type of elements for the algebra even at the GNS representation level, so that even for the GNS representations the operators do not have commutation relations that reflect spacetime causal structure, in contrast to Raasakka's proposal. It differs from operational probabilistic frameworks incorporating indefinite causal structure, including the works of Hardy [18, 19], Chiribella, D'Ariano, Perinotti and Valiron [82], Oreshkov, Costa, and Brukner [58], and others in that: 1) The operational probabilistic frameworks have correlations that map to the real numbers which encode probabilistic correlations, while the CNQFT framework has correlations that map to the complex numbers which can incorporate correlations in amplitudes. 2) The operational probabilistic frameworks are for finite dimensional systems, while the CNQFT framework incorporates finite and infinite dimensional systems. 3) Unlike [82, 58] which impose local causality conditions that introduce a time asymmetry, the present framework is time symmetric (Section 3.2.8). 4) The operational probabilistic frameworks make essential use of the Choi operators (or equivalents, such as the operator tensors in [54, 55]) on some preassigned Hilbert spaces, while the CNQFT framework uses the GNS construction, which "creates" Hilbert spaces depending out of the generalized states.

The last point deserves some further comment since the advantages of the GNS construction in infinite dimensions may not be known to many people who study indefinite causal structure, given that previous works are mostly (if not always) conducted within operational probabilistic frameworks. The advantages of GNS can be demonstrated in a comparison to the popular Choi operator representations. Choi operators in infinite dimensions are used, for instance, in the work of Giacomini, Castro-Ruiz, and Brukner as part of the proposal to generalize the process matrices to continuous variable systems [107]. The generalization was not intended to and did not lead to a field theory, and appears more suitable for studying subjects such as quantum optics, but one may ask if a field theory can be found similarly using the Choi operators. Then one runs into some questions. First, the Choi operators are defined to represent CP maps, and there may be some additional technical subtleties for an attempt to adapted them to express the action of the unbounded field operators. Second, even for ordinary CP maps such as the identity channel (which is

[9]There has been other works on quantum causality. See, for example, [106] and references therein.

intuitively speaking quite “bounded”) Choi operator will not be bounded [108, 109], which complicates the matter since one has deal with domain issues. Furthermore, the usual applications of the Choi operators require some pre-established tensor product factorizations for the Hilbert spaces according to subspace structures, but tensor product factorization is a canonical issue in QFT (See for instance discussions about Reeh-Schlieder theorem and the entanglement structure of QFT, e.g., [110], and about Tsirelson’s problem, e.g., [111].). The GNS construction, on the other hand, has the advantage that field operator actions are incorporated in standard ways, ordinary CP map actions can be expressed without domain issues, and a tensor product structure need not be pre-established.

On the other hand the Choi operators and equivalents may have certain advantages, such as the closer connection to tools used in the studies of indefinite causal structure for finite dimensional systems. We should also keep in mind that either the GNS construction or the Choi operators and equivalents is a way to represent the correlation defined more fundamentally as a functional that is not locked to any particular Hilbert space representation. We should keep an open mind for further useful representations other than the ones studied. It remains to be explored how to best exploit the different tools available.

In comparison to all the frameworks mentioned above, a further point to stress is that the algebra factors $\mathfrak{A}_i$ that form the free product algebra in the present framework do not always have to describe observational outcomes of human agents, but could also describe events as realizations of particular possibilities, not necessarily associated to human observations.[10]

The above is a list of works from which the current author draw tools and lessons directly in devising the CNQFT framework. There are also other works on QFT that turn out to be related, and may provide further lessons and tools upon closer inspection on the connections. For this we note especially the works [113, 114, 86, 115, 116, 117, 28, 55, 41, 83, 63, 46].

3.4 Ultraviolet structure of the correlation functions

This section proposes that the spontaneous causal fluctuations of quantum spacetime which become significant in the ultraviolet (UV) offer a UV regularization mechanism.

As an orientation, in Section 3.4.1 we first show that causal fluctuations offer a UV regularization for finite dimensional field detectors. This result is based on the author’s previous Bachelor’s thesis [1] and the paper [5]. Then in Section 3.4.2 we argue that causal fluctuations similarly offer a UV regularization mechanism for the field correlation functions. We propose that the UV regularizing correlation functions

[10]Some ideas sketched by the later Haag [112] are apparently related.

with causal fluctuations characterize physical states, and replace conditions based on the singular structure of the correlation function, such as the Hadamard condition.

3.4.1 Field detectors

That causal fluctuation reduces correlations can be demonstrated quite straightforwardly for field detectors. A field detector model, such as the Unruh-DeWitt model [118, 119], describes the coupling(s) of some finite dimensional field detector(s) to the infinite dimensional quantum field. For instance, the detectors can be atoms containing electrons with their energy level as the degree of freedom for the detector. Only finitely many energy levels are under consideration and the systems are described by density operators on a finite dimensional Hilbert spaces. The detectors are turned on and off in different spacetime regions. The electrons couple to the electromagnetic field, so can become correlated to each other through the field.[11] The transition rates of the electrons to different energy levels can be calculated through the field correlation function [120]. Such detector models have been standardly used to study effects of QFT in curved and flat spacetimes [120], to provide quantitative characterizations of the causal and acausal correlations of the quantum field (see, e.g., [121]), and to discover remarkable effects such as "information transmission without energy exchange" [122].

Quite generally, quantum systems with distinguishable classical outcomes such as the detector systems admit a description based on quantum instruments [50]. Also quite generally, correlations of (finite dimensional) quantum instruments admit a description as process matrices reviewed in Section 2.5 and below. As mentioned above, the correlation in the quantum field can be probed by the correlation of the detectors.[12] Therefore demonstrating the reduction of correlations for field detectors provides a indirect clue correlations for the reduction of correlations for the quantum field itself.

The rest of this subsection is a non-technical summary of results in the author's previous Bachelor's thesis [1] and the paper [5]. Technical details can be found in the original references.

Examples of processes with significant causal fluctuation

In Section 2.5 we touched on the process matrix framework [58], which offers a general way to describe correlations with indefinite causal structure. The basic idea is to take local operations to be described by ordinary quantum theory with definite causal structure, and introduce indefinite causal structure in the global correlations of the

[11] In the literature it is common to study scalar field couplings, partly due to technical simplicity.

[12] In fact one may even argue that all our realistic knowledge about the field correlations on infinite dimensional systems come from finite dimensional detectors, and consequently correlations in detectors are the completely source of the knowledge we have about field correlations.

local operations. Here we briefly review the formalism again, introducing with some shorthand notations of super- and subscripts along the way.

The local laboratories where operations are applied are denoted $A, B, \cdots$. Each party X is associated with an input system with Hilbert space $\mathcal{H}^{x_1}$ where information propagates in, and an output system with Hilbert space $\mathcal{H}^{x_2}$ where information propagates out. The parties can share correlations on these systems. For example, a state ρ shared at A's input and B's input is denoted $\rho^{a_1 b_1}$, and a channel N from A's output to B's input is denoted $N_{a_2}^{b_1}$ (Following [54, 55] we used superscripts and subscripts to distinguish input and output systems.). A process W shared by A and B is denoted $W_{a_2 b_2}^{a_1 b_1}$. It generalizes states and channels to incorporate the correlations among all the input and output systems.

All these objects with inputs and outputs can be represented as operators on Hilbert spaces using the well-known Choi isomorphism [53]. The Choi operator of a object is obtained by inputting a maximally entangled state in a canonical basis on each input (which yields a state described by a density operator). Expressed as a matrix in a canonical basis, the Choi operator of a process W becomes a "process matrix" denoted by the same symbol. The matrix obeys

$$W \geq 0, \tag{3.22}$$

$$\mathrm{Tr}[W] = 1, \tag{3.23}$$

$$L_V(W) = W. \tag{3.24}$$

These follow from some very general physicality conditions on the correlated outcome probabilities (the first from the non-negativity of probabilities, and the last two from the normalization of probabilities[13]). In the last line L_V is the projector given in [59]. Its explicit form of the will be given in Section 4.2.2. For the present section the details of this projector is irrelevant, except that it implies

$$W_{a_2 b_2}^{a_1} = W_{b_2}^{a_1} \otimes W_{a_2}. \tag{3.25}$$

Objects with sub- and superscripts can compose when the output of one object is fed into the input of another. Such a composition is shown with repeated sub- and superscripts, e.g., sequentially composing the channels M and N yields a new channel L: $M_a^b N_b^c = L_a^c$. We use the convention that a discarded system has its label eliminated, e.g., $\rho^a = \mathrm{Tr}_b\, \rho^{ab}$. In addition, when no ambiguity arises we sometimes omit the labels or refer to objects by the relevant parties, e.g., $W_{a_2 b_2}^{a_1 b_1}$ is sometimes referred to as W or W^{AB}.

The following two examples had been conceived to describe indefinite causal structure that arise in quantum spacetime fluctuations. The first example [100, 1] is a mixture of two causal relations $A \to B$ (A causally precedes B) and $A - B$ (A causally

[13] Some other works use for the second condition $\mathrm{Tr}[W] = d_O$, where d_O is the dimension of all the output systems taken together. We use the alternate convention of [1, 2] and absorb the factor d_O into the composition rule. This is merely a choice of convention and does not make a difference for the physics.

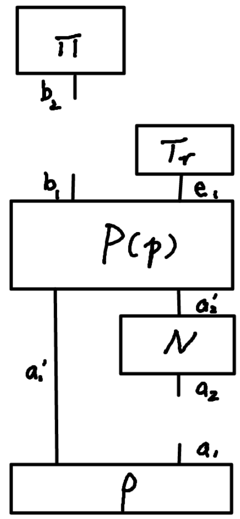

Figure 3.3: Illustration of Equation (3.28)

disconnected with B). Assume $\dim a_1 = \dim a_2 = \dim b_1 = \dim b_2$. The "partial swap" channel $P(p)^{a_2 b_2}_{a_1 b_1}$ $(0 \leq p \leq 1)$ is the channel corresponding to the partial swap unitary:

$$\sqrt{1-p}\ \mathbb{1} + \sqrt{p}\ i\ U_{SW}. \tag{3.26}$$

The identity $\mathbb{1}$ part sends a_1 to a_2 and b_1 to b_2, whereas the swap unitary U_{SW} part sends a_1 to b_2 and b_1 to a_2.

$$W^{a_1 b_1}_{a_2} := \mathrm{Tr}_{e_1} P(p)^{b_1 e_1}_{a'_1 a'_2} \rho^{a_1 a'_1} N^{a'_2}_{a_2}, \tag{3.27}$$

$$W^{AB} = W^{a_1 b_1}_{a_2} \otimes \pi_{b_2}, \tag{3.28}$$

where a'_1 and a'_2 are copies of a_1 and a_2, and π is the maximally mixed density operator. The process W puts A and B into a "coherent superposition" of sharing an acausal state ρ and a causal channel N.

The second example [123, 1, 3] is a mixture of all the three causal relations $A \to B$, $A \leftarrow B$ and $A - B$.

$$\begin{aligned} |w(\alpha)\rangle^{GABE} =& \alpha_1 |1\rangle^g |\Psi(\alpha)\rangle^{a_1 e_2 e_3} |I\rangle^{a_2 b_1} |I\rangle^{b_2 e_1} \\ &+ \alpha_2 |2\rangle^g |\Psi(\alpha)\rangle^{e_1 b_1 e_3} |I\rangle^{b_2 a_1} |I\rangle^{a_2 e_2} \\ &+ \alpha_3 |3\rangle^g |\Psi(\alpha)\rangle^{a_1 b_1 e_3} |I\rangle^{a_2 e_1} |I\rangle^{b_2 e_2}, \end{aligned} \tag{3.29}$$

$$W^{GABE}(\alpha) = |w(\alpha)\rangle\langle w(\alpha)|^{GABE}, \tag{3.30}$$

$$W^{AB}(\alpha) = \mathrm{Tr}_{GE} W^{GABE}(\alpha). \tag{3.31}$$

The party E is the "environment" that collects information not collected by the other parties. $|\Psi(\alpha)\rangle$ is a tripartite state depending on the parameter α, and $|I\rangle\rangle^{xy}$ is vector of the Choi operator $|I\rangle\rangle\langle\langle I|^{xy}$ for the identity channel from x to y. The party G's system g has basis vectors $|0\rangle$, $|1\rangle$ and $|2\rangle$ from which the causal relations $A \to B$, $A \leftarrow B$ and $A - B$ can be read respectively. For example, for the $|1\rangle$ term A causally precedes B through sharing the channel $|I\rangle\rangle^{a_2 b_1}$. g can be thought of as containing quantum gravitational degrees of freedom that induce different causal relation for A and B. $|w(\alpha)\rangle$ puts the relations into a "superposition". The probability amplitudes form a complex 3-vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\|\alpha\|_2 = 1$.

Measures of correlation

In the following S^a denotes the von Neumann entropy of a density operator on system a. A widely used measure of correlation is the mutual information. For a state ρ^{ab}, it is defined as $I^{a:b}(\rho) = S^a + S^b - S^{ab}$, where S^a (S^b) is the von Neumann entropy of ρ^a (ρ^b). The mutual information measures both quantum and classical correlations. To measure only quantum correlations, the coherent information can be used. It is defined as $I^a(\rho^{ab}) = S^a - S^{ab}$ ($I^b(\rho^{ab}) = S^b - S^{ab}$) when the target system is a (b). The coherent information is exactly the negative of the conditional entropy, and differs from the mutual information only by S^b (S^a). It can be positive only if the state is entangled, and attains the maximum value for maximally entangled states. For pure states the coherent information coincides with the entanglement entropy. Although for general states the coherent information is not an entanglement measure since local operations and classical communications (LOCC) may increase it, the LOCC-optimized coherent information $I^a_{\mathrm{LOCC}}(\rho) := \sup_{L \in \mathrm{LOCC}} I^{a'}(L(\rho))$ is. Here the optimization is over the set LOCC of the allowed LOCC operations. Depending on the context, different sets of LOCC operations (e.g., two-way classical communication, one-way classical communication, no classical communication) will be allowed. a' with a prime shows up because L (e.g., by changing system dimensions) may map to a Hilbert space different from the original a.

These "coherent information based measures" for states can be generalized to process matrices [2]. In this letter we use the coherent information $I^B(W^{AB}) := S^B - S^{AB}$ and the LO-optimized coherent information $I^B_{\mathrm{LO}}(W^{AB}) := \sup_{L \in \mathrm{LO}} I^B(L(W^{AB}))$. S^B and S^{AB} are the von Neumann entropies of the process matrix reduced to the party B and of the whole process matrix. The optimization is over the set LO of local operations without classical communication, because in the present context we are interested in indefinite causal structure, and allowing classical communication would make the causal structure become trivially causally connected. No prime needs to be introduced on B because it refers to whatever system the party B obtains after the optimization. The process matrices are treated as density operators for the evaluation. The state represented by the density operator can be operationally obtained from the process by inputting a maximally entangled state to each input system. Hence the above measures are interpreted as the the coherent information of the corresponding states obtained from the processes. The latter optimized measure is an entanglement measure in the generalized sense [2].

Reduction of correlation by causal fluctuation

A and B can have three possible definite causal relations $A \to B$, $A - B$, and $A \leftarrow B$. There are four possible ways to (quantum coherently or classically) mix these relations: three ways to mix two relations, and one way to mix three relations. Among these, the mixture of $A \to B$ with $A \leftarrow B$ is not expected to take place through naturally occurring quantum gravitational fluctuations, because it leaves out the intermediate case $A - B$. In addition, the mixture of $A \to B$ with $A - B$ and that of $A \leftarrow B$ with $A - B$ are of the same type. Therefore we restrict attention to two cases, the mixture of $A \to B$ with $A - B$ and that of all three relations.

The following theorems identify sufficient conditions to reduce coherent information based measures down to zero. By the continuity of the coherent information [124, 125], process matrices close to these characterized by the conditions have measures close to zero. The intuition behind the reduction of correlation is that indefinite causal structure induces leakage of correlation into the environment.

For the indefinite causal structure of $A - B$ with $A \to B$, we have the following theorem.

Theorem 2. Let W^{AB} be a process matrix of the form

$$W^{AB} = W_{a_2}^{a_1 b_1} \otimes W_{b_2}. \tag{3.32}$$

Suppose there is a $|w\rangle\langle w|^{ABE} = W_{a_2}^{a_1 b_1 e_1} \otimes W_{b_2}^{e_2}$ that purifies W^{AB} as a density operator, so that for a subsystem e_0 of e_1,

$$W_{a_2}^{a_1 b_1} = W_{a_2}^{a_1 e_0}. \tag{3.33}$$

Then $I_{LO}^{B}(W^{AB}) = 0$.

In Equation (3.28) the most significant indefinite causal structure comes with $p = 1/2$, which has equal probability amplitudes $A \to B$ and $A - B$. By taking e_0 to be e_1 itself, the above conditions are met. In Equation (3.31) purified by (3.29), the most significant indefinite causal structure for mixing $A - B$ with $A \to B$ comes with $\alpha_1 = \alpha_3 = 1/\sqrt{2}$ and $\alpha_2 = 0$. The above conditions are met under the relabelling $(e_1 = g e_0 e_3)$:

$$\begin{aligned} |w(\alpha)\rangle^{GABE} =& \alpha_1 |1\rangle^g |\Psi(\alpha)\rangle^{a_1 e_0 e_3} |I\rangle^{a_2 b_1} |I\rangle^{b_2 e_2} \\ &+ \alpha_3 |3\rangle^g |\Psi(\alpha)\rangle^{a_1 b_1 e_3} |I\rangle^{a_2 e_0} |I\rangle^{b_2 e_2}. \end{aligned} \tag{3.34}$$

In general, (3.32) says that there is no A to B signal term [58], which holds when $A \to B$ mixes with $A - B$. Condition (3.33) reflects significant indefinite causal structure. A has equal chance to be causally prior to B (so e_0 is causally disconnected with A) and causally disconnected with B (so e_0 is causally prior to A), such that B and the environment e_0 share correlations with A in the same way.

Next we consider indefinite causal structure of all three relations.

Theorem 3. Let W^{AB} be a process matrix, whose density operator is purified by $W^{a_1b_1e}_{a_2b_2}$. Suppose $W^{a_1b_1}_{a_2b_2}$ is symmetric in A and B. Suppose further that e has a subsystem e_1 such that $S^{b_1} \leq S^{e_1}$ and $S^{a_1b_2} \geq S^{b_1e_1}$. Then $I^B(W^{AB}) \leq 0$.

An example of a process matrix satisfying the conditions is (3.31) with $\alpha_1 = \alpha_2 = \alpha_3 = 1/\sqrt{3}$ and $|\Psi(\alpha)\rangle^{xyz} = |\Phi_+\rangle^{xy}$ (a maximally entangled state). The probability amplitudes of the three causal relations are equal, so the causal structure is significantly indefinite. In general, W^{AB} is symmetric in A and B if indefinite causal structure washes out any asymmetry that comes from an initial definite causal structure. The other condition can be heuristically interpreted as saying that an environmental subsystem e_1 is correlated with b_1 no less strongly than a_1 is correlated with b_2 ($S^{a_1b_2} \geq S^{b_1e_1}$), while obeying the technical condition $S^{e_1} \geq S^{b_1}$.

Whether the LO-optimization can increase the measure to a some positive number is an open question. Finally, the mutual information $I^{a:b}$ is related with the coherent information by $I^{a:b} := S^a + S^b - S^{ab} = I^a + S^b$. For each coherent information based measure there is corresponding mutual information based measure (taking into account classical correlation) defined by substituting $I^{a:b}$ for I^a. The reduction of the coherent information based measures implies the reduction of the mutual information based measures.

3.4.2 Field correlation functions

In comparison to the above results about finite dimensional field detectors, the following ideas about field correlation functions of infinite dimensional systems still need significant developments. However, the above results on causal fluctuations reducing field detectors correlations already offer strong support that the field correlation functions also gets reduced by the same effects, since ultimately the detector correlation draws from the infinite dimensional field correlation. The question is how the regularization is realized in the specific context of the field correlation functions, which has structures not present for the detectors. This forms the topic for the present subsection.

A simple observation

The idea that causal fluctuation may regularize field correlation functions in the UV draws support from the following simple observation. Consider the ordinary QFT two-point correlation for a real scalar field in the UV limit,

$$\langle \phi(x)\phi(x+\Delta x)\rangle \xrightarrow{\Delta x \to 0} \frac{C}{\Delta x_0^2 - \Delta \vec{x}^2}, \tag{3.35}$$

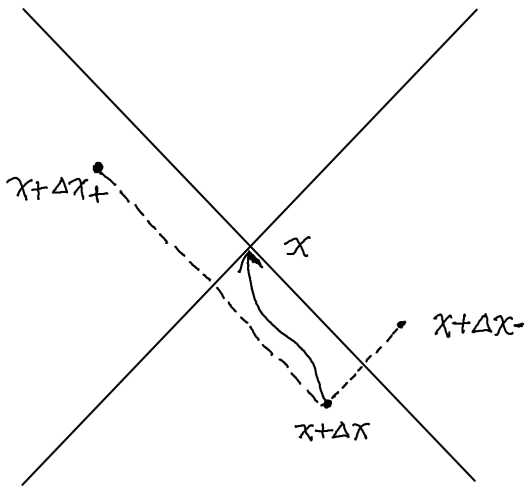

Figure 3.4: Scalar field correlation function in the UV.

where C is a constant. As expected, this diverges in the limit the two arguments x and $x + \Delta x$ coincide, whence the denominator on the right hand side goes to zero. Furthermore, the divergence appears also in the limit of lightlike separation for the two arguments.

The key observation is that the correlation takes on opposite values for spacelike separations timelike separations in a "symmetric way". Pick an arbitrary point with $\Delta x = (\Delta x_0, \Delta x_1, 0, 0)$ for, say a timelike separation (we chose a coordinate system so that $\Delta x_2 = \Delta x_3 = 0$). As illustrated in Figure 3.4, It has two dual spacelike separation points $\Delta x'_{\pm} = (\pm\Delta x_1, \pm\Delta x_0, 0, 0)$. The RHS of Equation (3.35) are exactly opposite for Δx and $\Delta x'_{\pm}$ for either plus or minus. The points of $\Delta x'_{\pm}$ are the mirror images of the point of Δx with respect to the lightcone boundaries as the reflection axis. Hence the above observation can be summarized as saying that if we sum up two points symmetrically with respect to either lightcone boundary, the field correlations cancel.

The observation suggests that if somehow causal fluctuations induce a "quantum mixture" of spacelike and timelike separations, there could be a cancellation for the correlation.[14] Just for purely illustration purposes, suppose in some model the quantum mixture in the end results in a correlation that can be effectively calculated using a uniformly weighted average of correlations that take values over a circular region around the point Δx. As the center point Δx moves towards any point on the lightcone from far away, the average will first increase since shorter distances are being approached, but the average must stop increasing and decrease at some point, since the circle will cover correlations form the other causal relation, which contribute oppositely to the average. Also when the point on the lightcone is reached the average

[14] I was informed that [30] could be relevant to this idea.

will drop to zero be the above "symmetry" argument. Conceptually, this implies that in this "uniform circular average" setup, as the invariant distance zero is approached from a greater distance, the correlation will not increase all the way and will drop down to zero in the end.

More generally, the exact circular shape of the region and the exactly uniform weight are not essential to see the same qualitative behavior. A generic average with a weighting function that is not too singular will likely result in the same qualitative behavior. Note that the average value and the value at the center point differ significantly only near invariant distance zero (near the lightcone). Here the center point correlation value tend to be either very large or very small, and the average value is a sum over large magnitudes with different signs that tend to cancel. Away from the lightcone the sum is over magnitudes with similar values and the same sign so the average value differs less from the center point correlation. Thus the UV modification is significant only at small absolute invariant distances.

The missing piece of evidence is to show that "quantum mixtures" can actually produce correlations that are effectively described by a sum like the above. This will be provided by a model that generalizes the quantum switch in the next part.

The above sum may appear to some to resemble a smearing, which is a standard procedure of QFT to be performed in treating field correlations rigorously as operator-valued distributions. The smearing in ordinary QFT does not cure the UV divergence. For example, a delta function may appear as the UV leading part of the integral kernel of the correlation function. In the evaluation of Feynman diagrams for an interacting theory products of the correlation functions (propagators) appear and there will be products of delta functions at the same point. While smearing against a delta function cures its divergence, smearing against a product of delta functions does not cure the UV divergence. In contrast, for the field correlation UV structure proposed here a delta function or any divergent part should never appear. Therefore even for an interacting theory a divergence shall not arise.

As a comment on Lorentz invariance, first we note that it is a classical spacetime notion, and it is unclear what it means for quantum spacetime. Nevertheless, suppose we can speak of it in a meaningful way regarding "quantum spacetime distances" which are akin to classical spacetime distances. In particular, suppose there is a distinction between spacelike/timelike distance and invariant distance. Then there are at least two possibilities for the relationship between spacetime distance and the expected strength of the causal fluctuation. The UV effects may become significant according to a small spacelike/timelike distance, or a small invariant distance. To the extent that the notion of "Lorentz invariance" makes sense, the former is not Lorentz invariant, while the latter is. We do not want to commit to either possibility but simply note that the proposed regularization mechanism for quantum spacetime can be adapted to either case. There is much freedom in picking generalized states and in associating them with different types of effective distances.

A toy model: the quantum fuzz

Here we present an explicit generalized state that realizes the idea of inducing a correlation that is described effectively by an average. This is only a toy model, because the intention is to provide an explicit and clean example that exhibits the proposed characteristic feature of a UV regularization induced by causal fluctuations. The question of determining nature's actual generalized state is discussed in Section 3.5.

The toy model is a generalization of the quantum switch and will be called the "quantum fuzz".[15] The quantum switch of Section 3.2 quantum coherently implements a "superposition" of causal order. In order to achieve a regularization a switch of causal order is not necessary. It suffices to quantum coherently "superpose" different configurations.

Recall the quantum switch transition amplitude of (3.3),

$$\begin{aligned}&\mathcal{A}(|\phi\rangle, x, y, u, v)\\&= \langle\phi|\, v\Big(\frac{1}{\sqrt{2}}\,|0\rangle\langle 0| \otimes U_{v,x}xU_{x,y}yU_{y,u} + \frac{1}{\sqrt{2}}\,|1\rangle\langle 1| \otimes U'_{v,y}yU'_{y,x}xU'_{x,u}\Big)u\,|\psi\rangle\end{aligned} \tag{3.36}$$

for four original factors with elements of the form x, y, u and v. We can fix some particular u and v (e.g., to be the identity) and view this amplitude as a "two-point" transition amplitude for the two points of x and y. This is viewed as two individual correlation configurations in a "superposition". We can turn it into an integral "superposition" of multiple correlation configurations of the form

$$\begin{aligned}&\mathcal{A}(|\phi\rangle, x, y, u, v)\\&= \langle\phi|\, v\Big(\int_\alpha d\alpha\, c_\alpha\, |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)}_{v,x}xU^{(\alpha)}_{x,y}yU^{(\alpha)}_{y,u} + \int_\beta d\beta\, c_\beta\, |\beta\rangle\langle\beta| \otimes U^{(\beta)}_{v,y}yU^{(\beta)}_{y,x}xU^{(\beta)}_{x,u}\Big)u\,|\psi\rangle\,.\end{aligned} \tag{3.37}$$

Here α and β are variables in some abstract parameter space, which can be continuous or discrete. The variable label the different correlation configurations to be put in a "superposition". c_α and c_β are complex numbers that encode the "weights" of the correlation configurations. The vectors $|\alpha\rangle$ and $|\beta\rangle$ are understood to describe quantum gravitational degrees of freedom and are not accessible to the operators x and y of the two "points". In this simple model the all the different vectors are mutually orthogonal.

For $a, b \in \mathfrak{A} = \star_{i=1,2,3,4}\mathfrak{A}_i$, define the generalized state ω by

$$\omega(b,a) = \sum_{|\phi\rangle} \overline{\mathcal{A}}(|\phi\rangle, x_b\cdots, y_b\cdots, u_b\cdots, v_b\cdots) \tag{3.38}$$

$$\times\, \mathcal{A}(|\phi\rangle, x_a\cdots, y_a\cdots, u_a\cdots, v_a\cdots), \tag{3.39}$$

[15] A secret reason for the name is that we suspect that, when applied to black holes, a quantum fuzz may grow hair for them.

where we grouped elements of a and b according their original factors. For example, factor $(x_a \cdots)$ collects all the elements of a coming from $\mathfrak{A}_1$ in the order of their appearance, ..., the factor $(v_a \cdots)$ collects all the elements of a coming from $\mathfrak{A}_4$ in the order of their appearance, and similarly for b.

The kind of weighted average over different correlation configurations mentioned above can be induced by setting u and v to be the identity. Through the summation over $|\psi\rangle$, the $|\alpha\rangle\!\langle\alpha|$ and $|\beta\rangle\!\langle\beta|$ from $\overline{\mathcal{A}}$ act on those from $\mathcal{A}$, which effectively induces a loss of coherence for the "superposition". The result is effectively equivalent to a classical mixture of the correlation configurations parametrized by α and β with weights $|c_\alpha|^2$ and $|c_\beta|^2$.

Discussion

The previous considerations suggest a field correlation function UV structure different from those in other contexts, for example, from that specified by the well-known Hadamard condition for quasi-free QFT in curved spacetime [126, 92]. This is not surprising since QFT in curved spacetime does not incorporate quantum gravitational effects, which give rise to the structure we considered. The implications of the differences remain to be investigated.

Having the renormalizaion group flow in mind, one may wonder how the proposed UV structure with strong causal fluctuations is reached when one zooms in from a low energy scale. This may perhaps be modelled by a parametrized family of generalized states, ω_λ, where he parameter λ plays the role of an energy scale. As higher energy scales are reached causal fluctuation becomes more significant. AQFT studies of the "scaling limit" [127] seem to provide some relevant guidance on how to approach this question of scale change.

The UV structure considered here could have a "fixed point" corresponding to a configuration with maximal causal fluctuation. How does this relate to the asymptotic safety program of quantum gravity [128]? Can the UV structure proposed here provide a particular mechanism to establish the basic hypothesis of the asymptotic safety program that the physical renormalization group trajectories lie in the UV critical surface?

The question of how to determine the exact form of the causal fluctuation modified correlation function will be discussed next.

3.5 Crossroads: dynamical constraint; quantum gravity

In this section we discuss two crossroads the new framework presents to us. These are deep foundational questions whose ultimate answers will perhaps be found through trial and error. At present we can only state the options without offering any suggestion which road to take.

Dynamical constraint?

The first crossroad is about deriving the form of the quantum gravitational modification of the correlation function. Will it be derived from a dynamical constraint or will it be derived from something else?

Possibility 1: A quantum gravitational dynamical constraint fixes the form of the modified correlation function.

In ordinary QFT the form of the correlation function is determined by the equation of motion/Lagrangian/Hamiltonian. For example, the correlation function can be obtained by finding the Green's functions for the differential equation of motion. The same possibility is pursued by some approaches to quantum gravity such as canonical quantization. The hope is to solve the dynamical constraint of the Wheeler-DeWitt equation satisfactorily to conquer quantum gravity.

Possibility 2: The paradigm of dynamical constraint breaks down for quantum spacetime. The correlation function is not found by solving some differential equation (including quantized versions) type dynamical constraint.

One reason to expect this possibility is that differential equations of motion are usually written with respect to a continuum spacetime manifold, which can be absent for quantum spacetime. Another reason is that indefinite causal structure may take away the definite causal order that a dynamical "evolution" follows. Neither reason is conclusive, though. For instance, causal dynamical triangulation is a theory of discrete spacetime, but implements a dynamical constraint based on an action [129]. Loop quantum gravity may contain indefinite causal structure encoded in the spin networks on the boundaries, but Wheeler-DeWitt equation type dynamical constraints are implemented for the correlation [28].

If one pursues the second possibility then the question is how the relaxation of the dynamical constraint is related to quantum spacetime effects such as discreteness and indefinite causal structure. One example of a relaxation of the equation of motion seems to be offered by the causal set inspired QFT summarized in [86]. The equation of motion for the field may be said to hold only approximately for usual models within the framework.

Quantum gravity?

The next crossroad is about the nature of the framework of CNQFT. Do we understand it as a a framework to formulate effective theories that incorporate quantum gravitational effects, or a framework to formulate a complete theory of quantum gravity?

Possibility 1: The CNQFT framework allows us to study quantum gravitational effects for quantum fields in an effective way, but the ultimate quantum theory of gravity needs to be based on a different framework.

Under this possibility, we can hope to obtain results resembling, e.g., [130, 131, 132, 133, 134], which incorporate effective quantum gravitational in the modified correlation functions.

Possibility 2: A complete theory of quantum gravity can be expressed in the present framework of CNQFT.

An example of a framework of QFT supporting a complete theory of quantum gravity is Rovelli's background independent QFT. The general boundary formulation of QFT [115] is used to express LQG states as part of the boundary configuration.

3.6 Outlooks

We have not yet discussed the spectrum condition. The widely adopted spectrum condition based on microlocal analysis does not seem to apply to quantum spacetime, because of the strong UV-regularizing causal fluctuations proposed in Section 3.4. It is an open question what the spectrum condition axiom generalizes to in quantum spacetime. Related to this, the causally fluctuating correlation function proposed in Section 3.4 is not compatible with the Hadamard condition. There are arguments in favour of the Hadamard condition to hold for QFT in curved classical spacetime (e.g., [135]). Can the arguments be extended to quantum spacetime? What is to be learned from the differences between the UV regularizing causally fluctuating correlation functions and the singular Hadamard correlation functions?

An important departure of CNQFT from QFT in classical spacetime is the causal fluctuations which are expected to render the imaginary part of the correlation functions (which substitutes the "commutator" in the new framework (Section 3.2.7)) generically non-vanishing. This effect of causal fluctuation is expected to be generically present, but the strength is expected to be weak away from the UV scale.[16] An open question is to find out patterns that govern the strength of the generic causal fluctuations. An analogy may be drawn with the acausal correlation in ordinary QFT. On the one hand, the Reeh-Schlieder Theorem [137] says that correlation is generically present for regions however far separated. On the other hand, the Cluster Decomposition Property [91] indicates that the correlation drops off quickly as the separation is increased. What one can aim for is an analogous result bounding the strength of the correlations arising from causal fluctuations.

If one wants to make contact with practical QFT and consider interactions, there is the question about time ordering. Time ordering shows up in Dyson's formula and is omnipresent in perturbative QFT to study interactions. How to study interacting QFT in the new framework when there is no definite time order? Does one still want to use time ordering? If so, how is it carried out?

[16] The question of when physical predictions can be made in the presence of indefinite causal structure is discussed in [18, 19, 136, 46].

In Section 4.1 of the next chapter, we study quantitative measures of causal strength, which may provide clues to the above two questions. In particular, regarding time ordering an obvious suggestion is to compare the causal strength (as quantified by some suitable causality measure) from ϕ_1 to ϕ_2 and the causal strength from ϕ_2 to ϕ_1 and order them according to which strength is stronger.

In the new framework indefinite causal structure is introduced across the original algebras used to form the free product. Can there be indefinite causal structure within individual factors? If so, how to describe it? Section 4.2 to Section 4.4 in the next chapter may provide some clues to this question.

Finally, moving the causal structure from the algebra to the generalized states is a fundamental structural change of ordinary QFT. We expect some repercussions that change our understanding of concepts previously established on the traditional structure of QFT. As an example, we present a new understanding of field entanglement in Chapter 5.

Chapter 4

Fine-grained causality

For quantum gravity, previous studies of causal structure need to be fine-grained for "causal strength" and for "fine structures".

Consider classical mechanics that supports the notion of a point particle. The location of point particle is definite. For each possible location, we can ask if the particle is there, and the answer is either yes or no, and no additional information is left to be given. Now consider quantum mechanics that supports the notion of a particle or an approximation notion thereof. For each possible location we again ask if the particle is there. The simple answer of yes or no will not yield all the information that can be given. For example, it does not tell us about the probability amplitudes, which give more quantitative information regarding the location of the particle.

Now consider the causal structure of physical variables/physical events. At the coarse-grained level, the causal relation between two variables/events A and B is A causes B, B causes A, or A and B are causally independent. Does this offer all the information about causal structure? More quantitative information can be given, such as how strongly one variable/event depends on another, i.e., the causal structure.

For quantum spacetime with indefinite causal structure, causal strength is especially important because of the generic causal fluctuations. Quantum causal fluctuations are expected to induce generic causal connections for physical events. Many of the induced causal connections are expected to have extremely low strength. Causal strength should be used to distinguish causal connections with significant strength and those with extremely low strength.

In Section 4.1, we study quantitative measures of causal strength.

For the rest of this chapter we study the "fine structure" of causal structure. One way to study causal structure is through "causal diagrams" [138]. For example, this can be a directed acyclic graph with the nodes representing variables, and the arrows representing causal connections. For the study of classical spacetime, the nodes can represent spacetime regions, or variables in the spacetime regions, and the arrows again representing causal connections. In the study of causal structure for spacetime [31] we can break one region into finer regions and probe the finer causal structure.

For a graph this intuitively corresponds to breaking a node into a new “subgraph” and redraw the arrows among the new and old nodes.

Traditional causal modelling was studied for classical variables [138]. There have been studies to generalize causal models to quantum systems with definite causal structure, e.g., [139, 140], and studies of quantum models with indefinite causal structure (e.g., see [106] and references therein). To our knowledge the fine structure (e.g., decomposing systems into finer subsystems) has not been investigated for quantum theory with indefinite causal structure. Studying quantum theory with indefinite causal structure for more general system structures forms the topic of Section 4.2 to Section 4.4. The motivations will be presented at the beginning of Section 4.2.

Section 4.1 is based on the [3]. The rest of the chapter contains original work.

4.1 Quantitative measures of causal strength

The precise definitions of the the causality measures, especially quantum causality measures, refer to some prior knowledge of one-shot quantum communication theory, which can be complicated to the uninitiated. For those who just want a broad understanding of causality measures, here is the idea in a nutshell.

A causality measure is a function that maps a correlation to its causal strength (a real number). There are some basic requirements a causality measure should obey. Performing local operations on the correlation should not increase its causal strength. The minimum value of a measure is conventionally set to zero. A correlation that cannot be used to send any signal should not be assigned a positive causal strength. A canonical example of causality measure is communication capacities.

In the context of quantum theory, in the study of acausal correlation strength it is useful to distinguish classical correlations and quantum correlations. For example, the latter can be characterized through entanglement measures. For causal correlation strength it is also useful to distinguish classical correlations and quantum correlations. This leads to the notion of quantum causal strength. One possible way to define quantum causal strength is to use the analogy with entanglement and require that, on top of being a causality measure, a quantum causality measure only assigns positive values to correlations that cannot be written as a convex combination of product correlations. Another possible way is to update the signalling condition (positive measure only if can signal) of causality measures to a “quantum signalling” condition (positive measure only if can “quantum signal”). What counts as “quantum signal”? Having positive quantum communication capacity (which does not tolerate error and requires that the achievable message matches exactly the target message) is one option, but one wants to have a weaker notion, and we picked out having positive one-shot quantum communication capacity (which tolerates error and allows the achievable message to differ from the target message with certain thresholds) as the requirement.

The following substitutes in the details for the above ideas, and can be skipped if all one wants is a broad understanding.

4.1.1 Causality measures

In this subsection we list the axioms for causality measures. There are many different frameworks for operational probabilistic theories with definite causal structure and with indefinite causal structure (e.g., [18, 48, 47, 141, 52, 142, 143, 144, 54, 55, 82, 58, 59, 60, 83, 63, 116, 145, 100, 146, 139, 140]). The following definition of causality measures applies to a wide range of frameworks, not restricted to quantum theory. The only preliminary concepts are local operations with different choices, and their correlations (such as a channel) that mediate the causal influence.

A **causality measure** $\mu^{A\to B}(G)$ on parties A and B sharing the correlation G is a real-valued function obeying the following axioms:

1. $\mu^{A\to B}(G)$ is non-increasing under local operations within A and B.
2. $\mu^{A\to B}(G) \geq 0$.
3. $\mu^{A\to B}(G) > 0$ only if A can signal to B using G.

Here "A can signal to B using G" means that by exploiting the correlation G, A can change the measurement outcome probabilities of B by choosing different operations. A **normalized causality measure** further obeys $\sup_G \mu^{A\to B}(G) = 1$ so that $0 \leq \mu^{A\to B}(G) \leq 1$ for all G. The causality measure $\mu^{A\leftarrow B}(G)$ in the opposite direction is defined similarly except that it obeys Axiom 3 with A and B swapped.

Axiom 1 is the main axiom for causality measures. It captures the intuition that the local operations cannot generate causal correlations. An arbitrary G can be mapped to any correlation G' that can be prepared by local operations alone (such as product states). The parties simply discard G and prepare G'. Axiom 1 implies that $\alpha = \mu^{A\to B}(G')$ is the minimum value $\mu^{A\to B}$ can reach for all G, because starting from any G the parties can apply local operations to prepare G'. Axiom 1 also implies that any two different G' must share the same value of α for $\mu^{A\to B}$, because each can be prepared from the other. Axiom 2 sets this minimum value α to zero.

Axioms 1 and 2 resemble the axioms for entanglement measures [147], which was originally defined for states and recently generalized to general quantum correlations including those with indefinite causal structure [2]. The defining axioms of entanglement measures are that the measures do not increase under the LOCC (local operations and classical communications) operations, and that the measures are non-negative. More precisely, the first axiom for entanglement measures says that they should not increase under LOCC operations allowed by the LOCC setting that one is considering (monotonicity). Here an LOCC setting dictates what LOCC operations are allowed. For example, in some LOCC settings only one-way classical communication is allowed, and in some others no classical communication is allowed. The only difference between the entanglement measure axioms and Axioms 1 and 2 above is that entanglement measures must also be monotonic in the presence of classical communications if the LOCC setting allows them. In LOCC settings where all local operations are allowed (which is the case for most LOCC settings of interest), the monotonicity axiom of entanglement is stronger than Axiom 1 for causality measures. Therefore in a framework[1] where they are defined the entanglement measures obey Axioms 1 and 2. However, a correlation that contains entanglement certainly does not necessarily contain causal correlation. Therefore Axiom 3 is needed to make sure that causality measures indeed measure causality. Incidentally, in a model where entanglement and causality measures are defined, if the LOCC setting only allows local operations, then causality measures obey the entanglement measure axioms. One could view causality measures as special cases of entanglement measures which obey Axiom 3 in the LOCC setting without classical communication.

We believe the remarks above justify the three axioms as necessary to define causality measures. There remains the question of whether more axioms are needed. One obvious option is to strengthen Axiom 3 by also requiring that $\mu^{A\to B}(G) > 0$ if A can signal to B using G. We do not to make this requirement because it exclude some useful information transmission capacities as causality measures. For example, there are channels that can signal but have zero quantum channel capacity.

Another option is to require $\sum_i p_i \mu^{A\to B}(G_i) \geq \mu^{A\to B}(\sum_i p_i G_i)$ for probability vectors p_i. We do not to make this convexity requirement because again it would exclude quantum channel capacity as a causality measure [148]. This choice echoes the choice in entanglement theory not to require entanglement measures to be convex (some useful measures such as distillable entanglement are not know to be convex) [147].

There are potentially other conditions one may want to impose on causality measures, just like there are conditions one may want to impose on entanglement measures in addition to the basic monotonicity and non-negativity axioms. For entanglement theory, the common view is that the two axioms above are the only ones necessary in defining entanglement measures, and other conditions may be imposed depending on particular contexts [147]. It seems the case is the same for causality measures and we regard axioms 1 to 3 as sufficient to define causality measures at the basic level. Other conditions may be imposed to suit particular interests. For example, in Section 4.1.3 we study the the additional condition based on "quantum signalling" to define quantum causality measures.

In addition to general causality measures, we also defined normalized causality measures $\mu^{A\to B}$ for which $\sup_G \mu^{A\to B}(G) = 1$. Normalized measures are useful when one compares correlations for systems with different dimensions. For example, the qubit identity channel and the qutrit identity channel are both channels with no noise

[1] Although as stated in [2] entanglement measures are defined specifically for quantum theory, they can easily be generalized to apply to a broad family of probabilistic theories which supports the notion of LOCC operations.

and with the maximum causal strength on their respective systems. Yet the quantum channel capacity as a standard causality measure assigns a larger value to the qutrit channel. This assignment is reasonable from the perspective that the qutrit channel is capable of transmitting more information per use. Nevertheless, in other contexts where one quantifies causal strength according to how much noise there is in the correlation, a normalized measure that assigns the value one to both channels would be preferable.

Examples

- The zero measure.

$$\mu_{\text{zero}}^{A\to B}(G) = 0 \quad \text{for all } G. \tag{4.1}$$

  This function trivially obeys all the three axioms and also the axioms for entanglement measures. It is of no practical value but shows that some function is both a causality measure and an entanglement measure.

- The signalling measure.

$$\mu_{\text{sg}}^{A\to B}(G) = \begin{cases} 1, & \text{A can signal to B} \\ 0, & \text{A cannot signal to B.} \end{cases} \tag{4.2}$$

  This function clearly obeys Axioms 1 to 3 and is a causality measure. It is also a normalized causality measure.

  Better than, for example, quantum channel capacity, it meets the condition that $\mu^{A\to B}(G) > 0$ if A can signal to B using G. Yet it is not convex. Let G_1 and G_2 be channels that can and cannot signal. Then

$$\frac{1}{2}\mu_{\text{sg}}^{A\to B}(G_1) + \frac{1}{2}\mu_{\text{sg}}^{A\to B}(G_2) = \frac{1}{2} < 1 = \mu_{\text{sg}}^{A\to B}(\frac{1}{2}G_1 + \frac{1}{2}G_2). \tag{4.3}$$

  The biggest drawback is that the signalling measure does not really *quantify* causal strength.

- For the special case that G is a quantum or classical channel, the various channel capacities are causality measures, as one can easily check that they obey all the three axioms.

  The channel capacities quantify how many qubits the channel can transmit per use and are not normalized measures in general. One can easily normalize them by dividing the maximum capacity a channel on the same input and output systems can reach. Precisely, given the channel capacity $C(N)$ on channels N as a causality measure, we normalize it by

$$C_{\text{norm}}(N) := \frac{C(N)}{\sup_{N' \in \mathfrak{C}(N)} C(N')}, \tag{4.4}$$

where $\mathfrak{C}(N)$ is the set of channels on the same input and output systems of N. This suits the need mentioned at the end of the last subsection of finding a measure that assigns the same value one to all noiseless channels.

- The primary examples of causality measures for general correlations not restricted to channels are the one-shot entanglement transmission capacities $Q^{\rightarrow}_{\mathrm{ent}}(G^{AB};\epsilon)$ and the one-shot subspace transmission capacities $Q^{\rightarrow}_{\mathrm{sub}}(G^{AB};\epsilon)$. In the next subsection we define these quantities correlations without definite causal structure.

- Given any causality measure $\mu^{A\to B}(G)$, a standard way to define a normalized measure is

$$\mu^{A\to B}_{\mathrm{norm}}(G) := \frac{\mu^{A\to B}(G)}{\sup_{\mathfrak{C}(G')}\mu^{A\to B}(G')}, \tag{4.5}$$

  where the supremum is taken over a set of correlations $\mathfrak{C}(G')$ that depends on G'. The normalization (4.4) for channel capacities is a special case of this general procedure.

4.1.2 One-shot quantum capacities

The channel capacities provide fairly natural quantitative measures of the causal strength. Yet the traditional definitions of capacities only apply to channels, which are correlations with definite causal structure. The various definitions of capacities can be generalized to apply to correlations with possibly indefinite causal structure. These generalized capacities can then be used to quantify causality for e.g., quantum gravity, where indefinite causal structure is important.

In this section we focus on two canonical causality measures based on one-shot quantum communication capacities. In communication theory, the asymptotic capacities are usually regarded as the canonical capacities [149]. Yet there is more than one reason to consider the one-shot capacities as the truly canonical ones. The conceptual reason is that practically all correlations for communication comes with noise, so the copies of correlations cannot be strictly identical. In addition, the copies may correlate with each other. Moreover, there is no supply of infinitely many copies of the correlation. The asymptotic capacities do not account for these practical limitations but the one-shot capacities do. The technical reason for preferring one-shot capacities over asymptotic ones is that the asymptotic capacities can be viewed as special cases of the one-shot capacities when the correlation used for communication is a tensor product of n identical correlations and in the limits $n \to \infty$ and $\epsilon \to 0$ where ϵ is the error tolerance. An added reason from indefinite causal structure is that in its presence the asymptotic capacities cannot be defined in the most straightforward way [61]. At present it is still an open question what the best way to define asymptotic capacity is in the presence of indefinite causal structure, but the one-shot capacities do not suffer the same issue.

The one-shot communication tasks of entanglement transmission and subspace transmission and their capacities are originally defined for channels in [150]. We generalize the previous definitions to incorporate communication resources with indefinite causal structure. In the next section we solve for the values of the capacities for some simple but important models of indefinite causal structure.

As mentioned, the notion of causality measures applies to frameworks more general than quantum ones and relies only on the concepts of correlations that mediate the causal influence, and local operations that can change the correlations in order to exert causal influence. The following tasks require in addition that the states to be transmitted live on complex Hilbert spaces, and that the allowed local operations contain preparations of maximally entangled states for the entanglement transmission task, and preparations of arbitrary pure states on subspaces for the subspace transmission tasks.

Entanglement transmission capacity

The goal of the entanglement transmission task is to transmit locally prepared entanglement through the correlation into shared entanglement. Suppose A and B share a correlation G^{AB} that allows A to send quantum states on Hilbert spaces of at most dimension $\tilde{m}$. In the A to B one-shot entanglement transmission task for the correlation G^{AB}, for each given dimension $m \leq \tilde{m}$, A first prepares a maximally entangled state $\Phi^{MM'} \in L(\mathcal{H}^M \otimes \mathcal{H}^{M'})$ ($L(\mathcal{H}^x)$ denotes bounded linear operators on Hilbert space $\mathcal{H}^x$) with $\mathcal{H}^M \subset \mathcal{H}^A$, where $\mathcal{H}^A$ is the largest system A can prepare states on, $\dim \mathcal{H}^M = m$, and M' is a copy of the system M. Then A keeps the M part of the state intact to herself and send the M' part of the state to B using G^{AB} such that they share a state $\Psi^{MM'}(E, D)$ with part M held by A and part M' held by B. In this transmission A applies some encoding local operation E (which must keep the M part of the original state $\Phi^{MM'}$ intact) and B applies some decoding local operation D. The goal is for $\Psi^{MM'}(E, D)$ to be as close to $\Phi^{MM'}$ as possible.

Here we make a distinction between *active* entanglement transmission and *passive* entanglement transmission. The task of passive entanglement transmission is just as stated above, for which A must keep the system M intact after the original maximally entangled state $\Phi^{MM'}$ is prepared. The task of active entanglement transmission on the other hand allows A to apply some local operation E' on the M part of $\Psi^{MM'}(E, D)$ to obtain $\Psi^{MM'}(E, E', D)$ before it is compared with the target state $\Phi^{MM'}$.[2] In the literature the distinction between the active and passive entanglement

[2]We require that A must finally share entanglement with B by keeping the M part of the originally prepared state in order not to confuse the task of entanglement transmission with entanglement generation. Otherwise the parties can use an entangled state G^{AB} that does not allow signalling to set up shared entanglement by simply discarding the originally prepared $\Phi^{MM'}$ and keeping G^{AB}.

transmission tasks is often not stated explicitly, with some articles adopting the former (e.g., [150]) and some others (e.g., [151]) adopting the latter as the "entanglement transmission task". It is not clear to us whether the two tasks are equivalent (having the same capacity), so we prefer to state them as different tasks.

For any positive integer $m \leq \tilde{m}$, define the A to B **entanglement transmission fidelity** for G^{AB} as:

$$F_{\text{ent}}(G^{AB}; m) := \max_{\substack{\mathcal{H}^M \subset \mathcal{H}^A \\ \dim \mathcal{H}^M = m}} \max_{E,D} \left\langle \Phi^{MM'} \middle| \Psi^{MM'}(E, D) \middle| \Phi^{MM'} \right\rangle \quad \text{for the passive task,} \tag{4.6}$$

$$F_{\text{ent}}(G^{AB}; m) := \max_{\substack{\mathcal{H}^M \subset \mathcal{H}^A \\ \dim \mathcal{H}^M = m}} \max_{E,E',D} \left\langle \Phi^{MM'} \middle| \Psi^{MM'}(E, E', D) \middle| \Phi^{MM'} \right\rangle \quad \text{for the active task,} \tag{4.7}$$

where $\left|\Phi^{A\bar{A}}\right\rangle \in \mathcal{H}^A \otimes \mathcal{H}^{\bar{A}}$ is the pure state corresponding to $\Phi^{A\bar{A}}$. In the first maximization the parties try over all the encodings and decodings. In the second maximization A tries over all the subspaces. Let $0 \leq \epsilon \leq 1$ be a real number. $R = \log m$ is an ϵ**-achievable rate** if

$$F_{\text{ent}}(G^{AB}; m) \geq 1 - \epsilon. \tag{4.8}$$

The A to B **one-shot entanglement transmission capacities** of G^{AB} are defined as

$$Q^{\rightarrow}_{\text{ent}}(G^{AB}; \epsilon) := \max\{R : R \text{ is } \epsilon\text{-achievable}\}. \tag{4.9}$$

The tasks of B to A transmission with capacity $Q^{\leftarrow}_{\text{ent}}(G^{AB}; \epsilon)$ can be defined analogously.

Subspace transmission capacity

The definitions for subspace transmission are analogous. The goal is to transmit any state in some subspace with high fidelity. Suppose A and B share a correlation G^{AB} that allows A to send quantum states on Hilbert spaces of at most dimension $\tilde{m}$. In the A to B transmission task, for each $m \leq \tilde{m}$, A picks a subspace $\mathcal{H}^M \subset \mathcal{H}^A$ where $\mathcal{H}^A$ is the largest system A can prepare states on and $\dim \mathcal{H}^M = m$. Arbitrary pure states $|\psi\rangle \in \mathcal{H}^M$ are sent through G^{AB} from A to B such that in the end B gets a state with density operator $\Psi(E, D) \in L(\mathcal{H}^M)$. In the transmission A applies some encoding local operation E and B applies some decoding local operation D. The goal is for $\Psi(E, D)$ to be as close to $|\psi\rangle\langle\psi|$ as possible.

For any positive integer $m \leq \tilde{m}$, define the **minimum output fidelity** for G^{AB} as:

$$F_{\min}(G^{AB}; m) := \max_{\substack{\mathcal{H}^M \subset \mathcal{H}^A \\ \dim \mathcal{H}^M = m}} \max_{E,D} \min_{|\psi\rangle \in \mathcal{H}^M} \langle\psi| \Psi(E, D) |\psi\rangle . \tag{4.10}$$

In the first maximization the parties try over all the encodings and decodings. In the second maximization A tries over all the subspaces. Let $0 \leq \epsilon \leq 1$ be a real number. $R = \log m$ is an **ϵ-achievable rate** if

$$F_{\min}(G^{AB}; m) \geq 1 - \epsilon. \tag{4.11}$$

The **one-shot subspace transmission capacities** of G^{AB} are defined as

$$Q_{\text{sub}}^{\rightarrow}(G^{AB}; \epsilon) := \max\{R : R \text{ is } \epsilon\text{-achievable}\}. \tag{4.12}$$

The task for B to A transmission with capacity $Q_{\text{sub}}^{\leftarrow}(G^{AB}; \epsilon)$ can be defined analogously.

4.1.3 Quantum causality measures

In operational probabilistic theories, signalling is commonly used as *the* criterion for causality. Yet for quantum models (complex Hilbert space quantum operational probabilistic theories, for which ordinary quantum theory and quantum theories with indefinite causal structure are special cases) there are motivations to introduce another criterion for causality. Indeed, quantum and classical information are different types of information, and we know from communication theory that one may be able to transmit classical information without being able to transmit quantum information. The signalling criterion is defined with respect to influencing classical measurement outcomes, so it may be regarded as a causality criterion based on classical information. Is there a causality criterion based on quantum information?

We propose a quantum causality criterion based on the one-shot quantum transmission tasks defined in Section 4.1.2. Roughly speaking the criterion says that if a correlation performs any of the one-shot quantum transmission task better than all non-signalling correlations for any error tolerance ϵ, then the correlation can be used to "quantum signal". The traditional signalling criterion is weak in the sense that any influence of the measurement outcome probabilities qualifies as signalling. Similarly, the quantum signalling criterion is weak in the sense that a better-than-non-signalling-resource performance for any one-shot quantum transmission task qualifies as a correlation to quantum-signal.

One use of the quantum causality criterion is to distinguish natural models of quantum spacetime which support indefinite causal structure from unnatural ones. The models of quantum spacetime that only support indefinite causal structure according to the signalling criterion are unnatural. When the medium of causal influence is some material such as a telephone line, it is conceivable that the material only allows the transmission of classical but not quantum information. However, for quantum spacetime itself as the medium, it would be very unnatural for two causally connected parties to share correlations that can only send classical but not quantum information. A natural model of quantum spacetime should have indefinite causal structure according to both the signalling and the quantum signalling criteria.

Another use of the quantum signalling criterion is to update the axioms of causality measures to define quantum causality measures that quantify quantum causal strengths (Subsection 4.1.3). Quantum causality measures have applications, for instance, in quantifying the causal strength of quantum spacetime correlations.

Quantum signalling

Suppose G is the quantum correlation the two parties A and B share. We say that A can **quantum signal** to B if there exists an error tolerance ϵ for which they can perform any one-shot quantum transmission task better than the non-signalling resources in the traditional sense. In other words, we say that A can quantum signal to B if there exists $\epsilon > 0$ for which $Q^{A\to B}(G;\epsilon) > \sup_{H\in\mathcal{N}} Q^{A\to B}(H;\epsilon)$, where $\mathcal{N}$ is the set of non-signalling resources defined on the same systems, and Q is any of the one-shot quantum transmission capacities including the active and passive entanglement transmission capacities and the subspace transmission capacity. We call this the "quantum signalling criterion". To distinguish quantum signalling from the traditional notion of signalling, we call the latter "classical signalling", because it is defined based on classical observational outcomes.

Quantum signalling is stronger than than classical signalling, because by definition in order to quantum signal the parties must beat all classically non-signalling resources, which implies that they share a resource that is can classically signal.

Classical causal correlations do not allow quantum signalling. Classical correlations break entanglement and coherence. For the entanglement transmission task they can only set up shared separable states (otherwise entanglement may be created by LOCC) but not entangled states. Yet separable states will not have more entanglement fidelity than product states. Let $\sum_i p_i\rho_i^M \otimes \sigma_i^{M'}$ be an arbitrary separable state. Then

$$F = \left\langle \Phi^{MM'} \middle| \sum_i p_i\rho_i^M \otimes \sigma_i^{M'} \middle| \Phi^{MM'} \right\rangle = \left\langle \Phi^{MM'} \middle| \sum_i p_i\rho_i^M \otimes \sigma_i^{M'} \middle| \Phi^{MM'} \right\rangle \tag{4.13}$$

$$= \frac{1}{|M|} \operatorname{Tr}_M \left[\sum_i p_i \rho_i^M \sigma_i^M \right] \tag{4.14}$$

$$\leq \max_i \frac{1}{|M|} \operatorname{Tr}_M \left[\rho_i^M \sigma_i^M \right] \tag{4.15}$$

$$= \max_i \frac{1}{|M|} \left\langle \Phi^{MM'} \middle| \rho_i^M \otimes \sigma_i^{M'} \middle| \Phi^{MM'} \right\rangle . \tag{4.16}$$

Therefore the separable state does not have greater entanglement fidelity than the product state $\rho_i^M \otimes \sigma_i^{M'}$ for some i. Because any product state can be created without signalling, classical correlations do not perform better than the classically non-signalling resources for entanglement transmission. For the subspace transmission task, note that even the most effective classical causal correlation, the classical identity channel, cannot achieve a greater minimum output fidelity than classically non-signalling correlations. Suppose the classical identity channel projects onto the $\{|i\rangle\}_{i=1}^d$ basis. The most effective encodings and decodings are unitaries. Without loss of generality assume they are the quantum identity channels. The worst case scenario for the minimum output fidelity is with the input state $|\psi\rangle = \sum_{i=1}^d \frac{1}{\sqrt{d}} |i\rangle$. Then $\Psi = \frac{1}{d}\mathbb{1}$ and $F = \langle\psi| \Psi |\psi\rangle = 1/d$. This minimum output fidelity can be matched if A and B share a classically non-signalling correlation and for the transmission B traces out whatever he receives and outputs the maximally mixed state. Therefore classical correlations do not perform better than the classically non-signalling resources for subspace transmission either and hence they cannot quantum signal.

We note that entanglement *generation* capacities do not count as *transmission* capacities, because as mentioned in Subsection 4.1.2 a correlation (e.g., an entangled state) that does not allow the transmission of quantum information may have a positive entanglement generation capacity. This situation contrasts that with the asymptotic capacities for quantum channels, for which the three capacities of entanglement transmission, subspace transmission, and entanglement generation agree. One reason for the difference is that restricted to channels nothing can generate entanglement without being able to transmit entanglement. On the other hand, correlations with indefinite causal structure such as process matrices contain entangled states as special cases, but these can generate entanglement without being able to transmit entanglement. The inclusion of such correlations break the "degeneracy" of the three capacities.

Another reason for the difference is that quantum signalling is defined using one-shot capacities rather than asymptotic ones. For quantum channels there is an inequality that relates the entanglement transmission and subspace transmission capacities [150]:

$$Q_{\text{ent}}(N;\epsilon) - 1 \leq Q_{\text{sub}}(N;2\epsilon) \leq Q_{\text{ent}}(N;4\epsilon), \tag{4.17}$$

which shows that the two capacities are closely related. However, it does not set up an equivalence of the two capacities. Neither can it be used to pick one out of the two capacities to define quantum signalling to yield a weaker quantum signalling criterion than with the other one. The incomparability of the one-shot quantum capacities leaves us with the need to check each type of capacity to qualify for quantum signalling.

Axioms for quantum causality measures

A **quantum causality measure** $\mu^{A\to B}(G)$ on local parties A and B sharing correlation G is a real-valued function obeying the following axioms:

1. $\mu^{A\to B}(G)$ is non-increasing under local operations within A and B.

2. $\mu^{A\to B}(G) \geq 0$.

3. $\mu^{A\to B}(G) > 0$ only if A can quantum signal to B using G.

A **normalized quantum causality measure** further obeys $\sup_R \mu^{A\to B}(G) = 1$ so that $0 \leq \mu^{A\to B}(G) \leq 1$ for all G. The quantum causality measure $\mu^{A\leftarrow B}(G)$ in the opposite direction is defined similarly except that it obeys Axiom 3 with A and B swapped.

In comparison to causality measure axioms, the only difference is that in axiom 3 "quantum signal" is used in place of "signal".

Examples

- The zero measure.

$$\mu_{\text{zero}}^{A\to B}(G) = 0 \quad \text{for all } G. \tag{4.18}$$

  This function trivially obey all the three axioms. It is a causality measure, a quantum causality measure, and an entanglement measure.

- The quantum signalling measure.

$$\mu_{\text{qsg}}^{A\to B}(G) = \begin{cases} 1, & \text{A can quantum signal to B} \\ 0, & \text{A cannot quantum signal to B.} \end{cases} \tag{4.19}$$

  This function clearly obeys axioms 1 to 3 and is a quantum causality measure. It is also a normalized measure.

- For quantum channels the quantum channel capacities are quantum causality measures, as one can easily check. Their normalization as in (4.4) are normalized quantum causality measures that assign the value one to noiseless channels.

- For arbitrary correlations that may or may not contain indefinite causal structure, the one-shot entanglement transmission and subspace transmission capacities defined and studied in previous sections are quantum causality measures. Axioms 1 to 3 hold for these capacities directly by their definitions.

  Definitions similar to (4.5) yield normalized capacities that assign the value one to the maximally causal correlations such as the identity channel:

$$Q_{\text{norm}}(G;\epsilon) := \frac{Q(G;\epsilon)}{\sup_{G'\in\mathfrak{C}(G)} Q(G';\epsilon)}, \tag{4.20}$$

  where $\mathfrak{C}(G)$ is the set of correlations on the same systems of G, and Q stands for any of the one-shot quantum capacities.

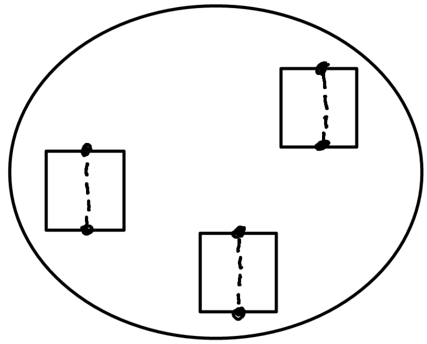

Figure 4.1: The picture of the process matrix framework. Three local laboratories are shown as boxes. Each allows correlations through arbitrary local operations shown as a dashed line across the input and output systems shown as dots.

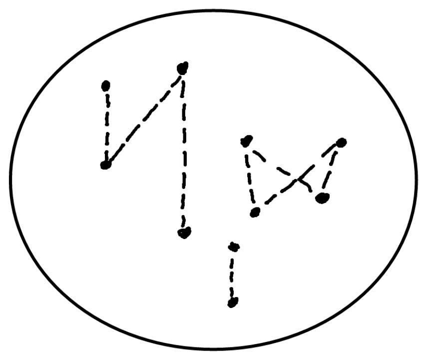

Figure 4.2: The picture of the special framework of correlation networks. Local laboratories and boxes are not necessary.

4.2 Correlation networks: a special framework

The idea of developing the framework of correlation networks grew out of working with the process matrices (Section 2.5) as a framework to study correlations with indefinite causal structure for finite dimensional systems. In this section, we present the special framework of correlation networks as a generalization of the process matrix framework [58, 59, 60].

One motivation for the generalization is to develop a more adaptable framework for studying indefinite causal structure. The main idea of the generalization is to weaken the role "local laboratories" plays in the study of correlations with indefinite causal structure, and to allow for more general setups for operations. The original process matrix framework adopts the picture that there are many local laboratories that exchange information with the outside world only once (Figure 4.1). For a local laboratory labelled by X, some influence of the outside world passes into it once through an input system x_1. A party applies some operation on the input system.

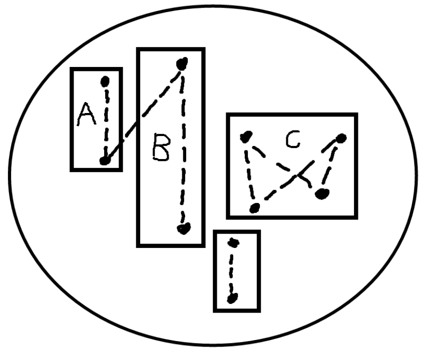

Figure 4.3: Local laboratory introduced into Figure 4.2. It is allowed to have correlations across the local laboratories A and B, and restrictions on the local operations within C through introducing the two sub-input systems and two sub-output systems.

Then the influence after the operation is let out of the local laboratory through an output system x_2. The input and output systems are associated with Hilbert spaces $\mathcal{H}^{x_1}$ and $\mathcal{H}^{x_2}$. It is assumed that inside each local laboratory ordinary quantum theory with definite causal structure holds, and the parties can perform any arbitrary operation across the input and output systems compatible by ordinary quantum theory with definite causal structure. Precisely, this means that any quantum instrument across the input and output Hilbert spaces $\mathcal{H}^{x_1}$ and $\mathcal{H}^{x_2}$ are allowed.

For correlation networks, the notion of "local laboratory" is not used in setting up the framework (Figure 4.2). There is a set of systems that can be correlated with operations. A system may be correlate to many other system through operations.

Although local laboratories is not a necessary ingredient for the new framework, in applications to some particular physical situations it can be useful to consider local laboratories. When one reintroduces local laboratories one can now do the following:

- Describe restrictions on the local operations within local laboratories.
- Describe influences across local laboratories.
- Allow indefinite causal structure within a local laboratory.

The first two points are illustrated in (Figure 4.3). The third point and further applications will be mentioned in Section 4.3. In Section 4.4 we outline an even more general framework suitable for some other applications to be discussed in the same section.

Although we stated the generalization in particular with respect to the process matrix framework, the general idea of allowing one system to be correlated with multiple other systems by different operations has further applicabilities. It is possible to apply the idea to generalize other frameworks for quantum theory with or without indefinite causal structure, such as [52, 82, 152, 83, 63].

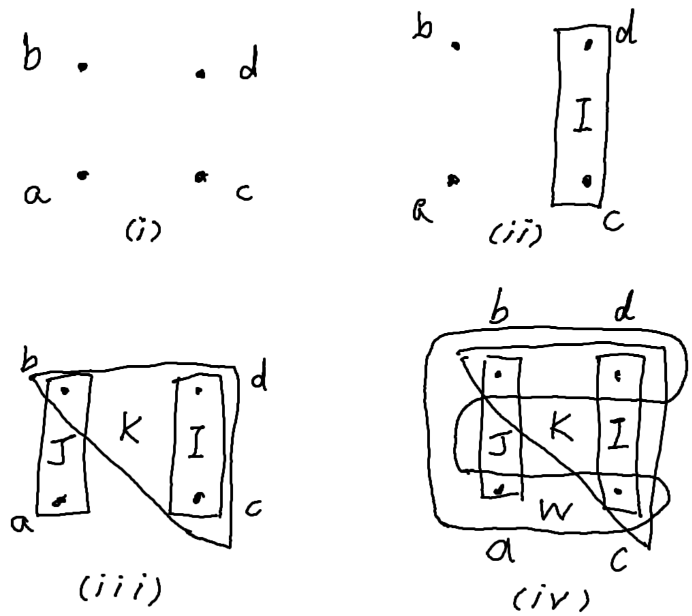

Figure 4.4: (i) A set of systems; (ii) A set of systems and a connection I (which comprises a setting); (iii) A more complicated setting with three connections; (iv) A network with the setting of (iii) and the correlation W.

4.2.1 Basic concepts

Let there be a set of *systems* $\mathcal{S} = \{a, b, \cdots, c\}$. Each system is either an *input system* or an *output system*. A *connection* (e.g., $I = \{a, b, c\}$) is a subset of systems. Two connections are *compatible* if their intersection is empty.

An *operation* on a set of systems consists of an action and an observation. Depending on the observational outcome, different correlations are established among the systems. We denote an operation using symbols of the form $\{A_{ab\cdots}^{cd\cdots}[i]\}_i$, $\{B_{ef\cdots}^{gh\cdots}[j]\}_j$ etc. The letters $A, B, \cdots$ distinguish different operations. The subscripts are input systems, the superscripts are output systems, and the letters $i, j, \cdots$ are observational outcomes. An operation $\{A_{ab\cdots}^{cd\cdots}[i]\}_i$ can alternatively be denoted $\{A_I[i]\}_i$ when I is the connection that consists of all the systems of the operation. A *setting* is a set of systems with a set of connections, each associated with a set of allowed operations.

The notions are illustrated in Figure 4.4. Connections are drawn with sharp corners, and correlations (to be introduced below) with round corners. The physical picture is that there is a list of quantum systems relevant for the study. In labelling the set of systems we implicitly assumed that the systems can be distinguished. A connection specifies that some operations may be performed to correlate the systems belonging to the connection. A connection with only input systems allows the operations of measurements, and a connection with only output systems allows operations of state preparations. If two connections overlap, their associated operations may not be performed in a same experiment because they beg access to the same systems. A set of mutually compatible connection are associated with operations that can be applied in the same experiment. A setting specifies a configuration of which operations correlating which systems can be applied.

A setting has a hypergraph structure, with the systems as the nodes, and the connections as the hyperedges. A hypergraph is more suitable than a graph as a pictorial tool since a connection does not always contain two systems.

From the operation-correlation perspective of Chapter 2, we want to consider probabilistic maps that correlate the possible operational outcomes. Given a setting, *a full set of compatible connections* $\{I, J, \cdots, K\}$ is a subset of all the connections of the setting such that all systems of the setting are contained in one and only one connection. Given a full set of compatible connections $\{I, J, \cdots, K\}$, we can pick an allowed operation for each connection to form *a full set of compatible operations* $\{A_I, B_J, \cdots, C_K\}$. Given a full set of compatible operations $\{A_I, B_J, \cdots, C_K\}$, we can pick an outcome for each operation to form *a full set of compatible outcomes* $\{A_I[i], B_J[j], \cdots, C_K[k]\}$. A *probabilistic correlation* (or *correlation* in short when no ambiguity arises), W, is a map from all the full sets of compatible outcomes to the reals, with the physical interpretation that it yields the probability for the joint outcome $(i, j, \cdots, k)$ to happen conditioned on performing the operations $(A_I, B_J, \cdots, C_K)$:

$$W : \{A_I[i], B_J[j], \cdots, C_K[k]\} \mapsto p(i, j, \cdots, k|A_I, B_J, \cdots, C_K). \tag{4.21}$$

A correlation is required to obey the basic probability rules that the conditional probabilities are non-negative, i.e., $p(i, j, \cdots, k|A_I, B_J, \cdots, C_K) \geq 0$, and that all full sets of compatible outcomes for a full set of compatible operations sum up to one, i.e., $\sum_{i,j,\cdots,k} p(i, j, \cdots, k|A_I, B_J, \cdots, C_K) = 1$.

For a given setting there is a family of correlation maps compatible with the probability rules. A setting with one fixed probabilistic correlation is called a *network*. A network is understood to describe one physical setup that comes with a well-defined probability calculus for the outcomes of operations allowed by the setting of the network.

As a remark on the physical interpretation, the systems and the operations are not necessarily defined according to spacetime notions. The necessary requirement for considering different systems or operations is that the systems or operations are distinguishable. The location in spacetime is only one out of many ways to distinguish systems or operations. They can also be distinguished, for instance, according to their locations in Hardy's operational space [41, 46], which is coordinatized by physical fields rather than the virtual spacetime coordinates.

4.2.2 Quantum theory

The basic notions introduced in the last subsection are applicable in a general operational probabilistic framework. In this subsection we restrict attention to finite dimensional quantum theory.

Each system is associated with a finite dimensional complex Hilbert space. On a set of systems $\mathcal{S}$, a *quantum operation*, or *operation* in short when no ambiguity arises, is a set of completely positive (CP) trace non-increasing maps from the input systems of $\mathcal{S}$ to the output systems of $\mathcal{S}$, (When there is no input or output system in $\mathcal{S}$, take the input or output system to be the trivial system, and the operation becomes a preparation or a measurement.) and they sum up to a CP trace-preserving map. Each map represents a different observational outcome.

It is convenient to represent the physical correlations as Hilbert space operators. This can be done (assuming the correlation map is multilinear in the operational outcomes), for example, through the Choi operators as in the process matrix framework [58]. Use hats to denote the Choi operators of the operations and operational outcomes, i.e., $\hat{A}_I$ for A_I, and $\hat{A}_I[i]$ for $A_I[i]$. Then a correlation W obtains the representation $\hat{W}$:

$$\begin{aligned} W :&\{A_I[i], B_J[j], \cdots, C_K[k]\} \mapsto \\ &p(i,j,\cdots,k|A_I,B_J,\cdots,C_K) = \mathrm{Tr}\Big[\hat{W}\hat{A}_I[i]\otimes\hat{B}_J[j]\otimes\cdots\otimes\hat{C}_K[k]\Big]. \end{aligned} \tag{4.22}$$

We call $\hat{W}$ a *generalized process matrix*, or *process matrix* in short. When no ambiguity arises we sometimes omit the hats in referring to the Choi operators for the operational outcomes and the process matrices for simplicity.

Characterization of process matrices

What is the most general set of physical correlations compatible with the probability rules for a given setting? The process matrices gives a simple characterization provided we make the following assumptions.

Assumption 1 (Singleton inclusion)**.** A setting always includes all the singleton sets of systems.

Assumption 2 (Operation inclusion)**.** For each given connection, all mathematically allowed operations are allowed in the theory.

Assumption 3 (Ancilla inclusion)**.** For each individual system an isomorphic ancilla system can be introduced for operations to be performed on the two-system connection.

These assumptions may be relaxed at a more general level of study, but imposing these assumptions yields a simple framework to start with.

The conditions of probabilities being non-negative and normalized imply the following constraints on the process matrices W.

$$W \geq 0, \tag{4.23}$$

$$\mathrm{Tr}\, W = d_O, \tag{4.24}$$

$$L_V(W) = W, \tag{4.25}$$

where d_O is the product of the dimensions of all the output systems, and L_V is a projection onto a subspace to be specified below. This projection is a generalization of the one in [59] to account for the more general settings.

The condition of non-negative probabilities implies that the process operators are positive semidefinite, i.e., $W \geq 0$, by considering operations of inputting maximally entangled states on single systems and their ancillas. The condition of normalized probabilities implies $\operatorname{Tr} W = d_O$ and $L_V(W) = W$.

Before stating the explicit form of $L_V(W)$, which can appear complicated at first sight, we spell out the idea behind it, which is simple. Let a set of compatible connections be given. This set of connections holds a set of jointly applicable operations. After "tracing out" all the systems not contained in these connections (physically corresponding to inputting a maximally mixed state to every input system and do a trivial measurement on every output system), the correlation, or the process matrix, should yield a probability for each joint outcome of the operations. The normalization of probability imposes a constraint on the process matrices, which turns out to be a projection L onto a subspace in the space where the process matrices are defined. Had we started with a different set of compatible connections, a different projection L' would be given. All such projections turn out to commute with each other. The overall normalization constraint then turns out to be the product L_V of all such constraints.

With the above understanding in mind we come to the technical specification of L_V. Given a set of systems X, define ${}_XW := \frac{\mathbb{1}_X}{d_X} \otimes \operatorname{Tr}_X W$. We refer to the trivial system as 1, and introduce the notation

$$
{}_{[\sum_X \alpha_X X]}W = \sum_X {}_{\alpha_X X}W, \tag{4.26}
$$

where α_X are numbers indexed by X. For simplicity we also all "multiplications of systems" inside of a square bracket. For instance, ${}_{[1-(1-a)(1-b)]}W = {}_{[1-(1-a-b+ab)]} W = {}_{[1-1+a+b-ab]} W = W - W +_a W +_b W -_{ab} W =_a W +_b W -_{ab} W$. Given a connection I, define I_O to be the set of output systems of I.

Given a set of compatible connections $\{I, J, \cdots, K\}$, we define $\mathcal{C} = I \cup J \cup \cdots \cup K$, and $\overline{\mathcal{C}} := \mathcal{S} \backslash \mathcal{C}$. Given a set of compatible connections $\{I, J, \cdots, K\}$, we also define

$$
L_{IJ\cdots K}W :=_{[1-(1-I_O)(1-J_O)\cdots(1-K_O)\overline{\mathcal{C}}]} W. \tag{4.27}
$$

For instance, suppose $\mathcal{S} = I \cup J \cup K$, where I, J and K are compatible. Then $L_{IJ}W =_{[1-(1-I_O)(1-J_O)K]} W$. In general, one can check that all the L's are a projections onto subspaces, and they all commute with each other.

As mentioned above, it turns out that each set of compatible connections $\{I, J, \cdots, K\}$ imposes a normalization condition $L_{IJ\cdots K}(W) = W$. The overall normalization condition takes the form $L_V(W) = W$, where L_V is the product of all such L's associated compatible connections. This gives an explicit definition of the projector L_V of Equation (2.12) and Equation (3.24).

As an illustrative example, consider a simple "Alice-Bob" setup. There are four systems $\mathcal{S} = \{a_1, a_2, b_1, b_2\}$, with a_1 and a_2 as Alice's input and output systems, and b_1 and b_2 as Bob's input and output systems. The two connections are $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, allowing Alice to perform operations that correlate her input and output systems a_1 and a_2 belonging to A, and Bob to perform operations that correlate his input and output systems b_1 and b_2 belonging to B (Note that here A and B denote connections rather than operations.). There are three non-trivial sets of compatible connections, $\{A\}, \{B\}$, and $\{A, B\}$. They are associated with three projectors that act on W according to Equation (4.27),

$$L_A W =_{[1-(1-a_2)B]} W = W -_{b_1 b_2} W +_{a_2 b_1 b_2} W, \tag{4.28}$$
$$L_B W =_{[1-(1-b_2)A]} W = W -_{a_1 a_2} W +_{b_2 a_1 a_2} W, \tag{4.29}$$
$$L_{AB} W =_{[1-(1-a_2)(1-b_2)]} W =_{a_2} W +_{b_2} W -_{a_2 b_2} W. \tag{4.30}$$

The overall projector $L_V W = L_A L_B L_{AB} W$ acts on W as the product of these three projectors. It is required that $L_V W = W$.

The above constraints are stated using the language of [59], and the claims can be proved straightforwardly using the techniques of the same paper. Equivalently one can state and prove the constraints in the language of [60] or [152].

4.2.3 Compositions

In the framework set up so far, a given a network contains one and only one correlation. It could be that the correlation's process matrix W factors into a tensor product $W = X \otimes Y \otimes \cdots Z$ with the individual matrices $X, Y, \cdots, Z$ acting on different systems. In this case we regard the individual $X, Y, \cdots, Z$ as independent process matrices that specify independent correlation maps.

Operations and processes can compose into operations (Figure 4.5). Suppose a given network has a list of independent processes $U, V, \cdots, W$ and a list of compatible connections $\{I, J, \cdots, K\}$ with the associated operations $A_I, B_J, \cdots, C_K$. Let $\mathcal{S}_c$ be the set of all the systems of these processes, and $\mathcal{S}_o$ be the set of all systems of these connections. If $\mathcal{S}_c \subset \mathcal{S}_o$ (each system of a correlation is occupied by one and only one operation), we can compose the correlations and the operations into a new operation D_E on the connection $E := \mathcal{S}_o \backslash \mathcal{S}_c$. The operational outcomes are given by the partial trace

$$D_E[i, j, \cdots, k] = \mathrm{Tr}_{\mathcal{S}_c}[(U \otimes V \otimes \cdots W \otimes \mathbb{1}_E)(A_I[i] \otimes B_J[j] \otimes \cdots \otimes C_K[k])]. \tag{4.31}$$

Note that this "into operation" kind of composition may not give rise to all the mathematically allowed operations, due to restrictions coming from the correlations or the configuration of the operations. In this case, we say that the composition restricts the operations, and the systems of the new operation is regarded as a *generalized connection*, which does not obey Assumption 2. Suppose we start with a

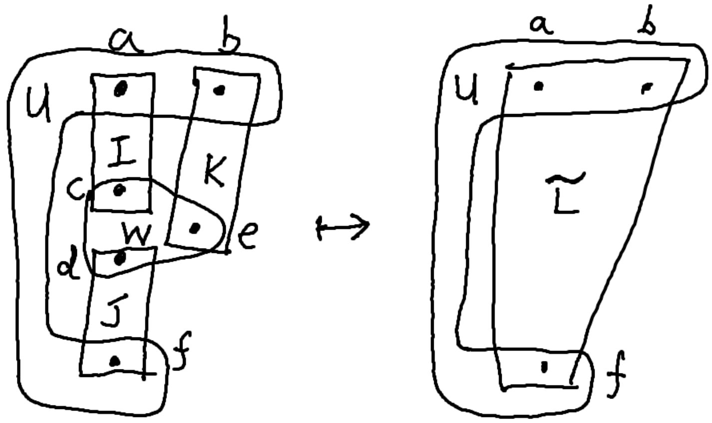

Figure 4.5: I, J, K and W composes into a generalized connection $\tilde{L}$ (tilde for it is generalized). The systems c, d, e are annihilated from the old network.

network and conduct an "into operation" composition. The result is either a new network, when there is no restriction coming from the composition, or a *generalized network*, containing generalized connections that break Assumption 2. In both cases the resulting network or generalized network has the systems in $\mathcal{S}_c$ annihilated in comparison to the original network.

Operations and processes can also compose into correlations (Figure 4.6). This happens in the opposite situation with $\mathcal{S}_o \subset \mathcal{S}_c$ (each system of a operation is occupied by one and only one correlation). The composition yields a new correlation Q on the systems $F := \mathcal{S}_c \backslash \mathcal{S}_o$ by

$$Q = \sum_{i,j,\cdots,k} \mathrm{Tr}_{IJ\cdots K}[(U \otimes V \otimes \cdots W)(A_I[i] \otimes B_J[j] \otimes \cdots \otimes C_K[k] \otimes \mathbb{1}_F)], \tag{4.32}$$

where the sum is over all outcomes of the operations.

After an "into correlation" composition, a new network is generated out of an old network with the systems in either $\mathcal{S}_o$ annihilated. The networks are "closed" under "into correlation" compositions.

In other frameworks of operational probabilistic theories, operations can usually "overlap" on some systems, and be sequentially composed on the overlapping systems (e.g., as in Figure 4.7). The present framework has a different setup such that compatible operations do not have overlapping systems. Operations can be connected only through correlations. Composition of operations can be reproduced, though, by introducing trivial correlations (e.g., as in Figure 4.8).

Incidentally, one can ask what happens if one system is "overlapped" by three or more operations. This kind of overlapping is usually forbidden in frameworks of operational probabilistic theories, but can be made legal, for example, by introducing another operation attached to three or more systems (e.g., as in Figure 4.9). In the

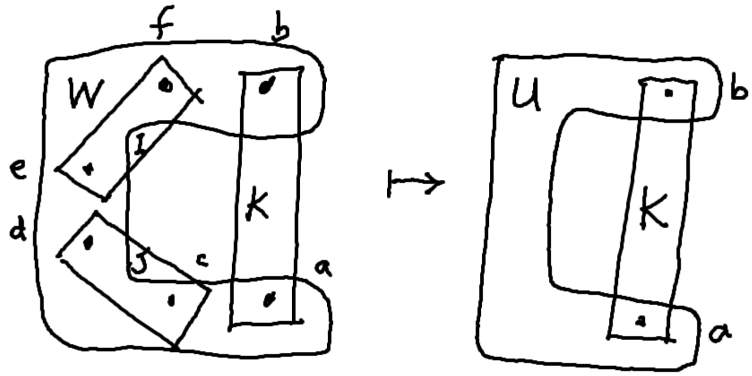

Figure 4.6: Some operation on I, some operation on J and W composes into a correlation U. The systems c, d, e, f are annihilated from the old network.

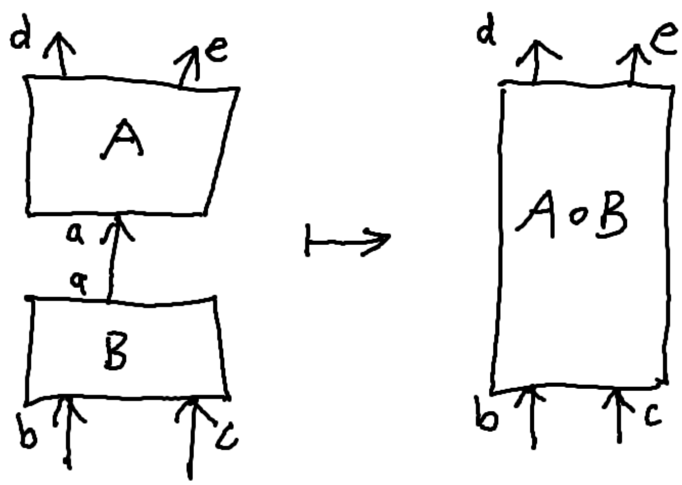

Figure 4.7: An example of sequential composition in other formulations

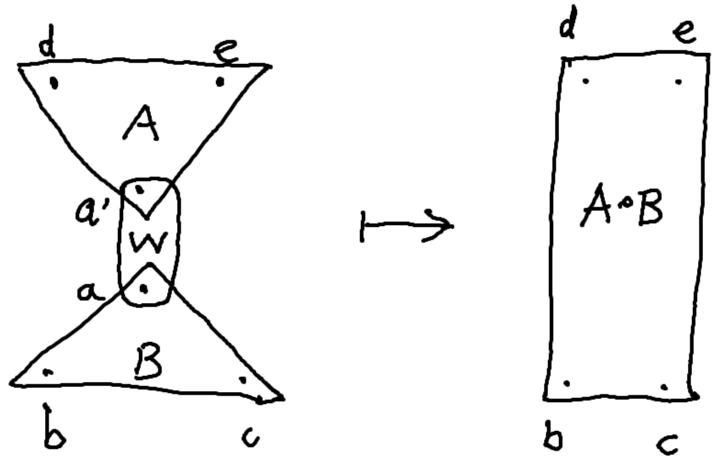

Figure 4.8: An example of sequential composition in the present frameworks. a and a' are isomorphic systems, and W is a trivial correlation that realizes an isomorphism.

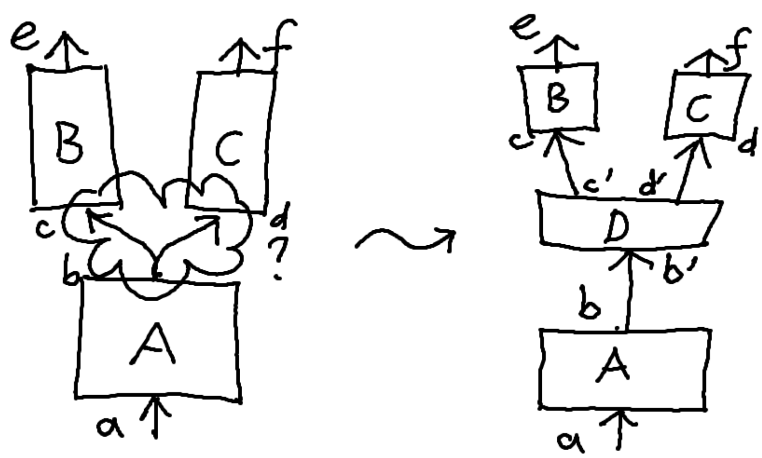

Figure 4.9: System overlapped by three or more operations? Made legal by introducing an additional operation D in other formulations.

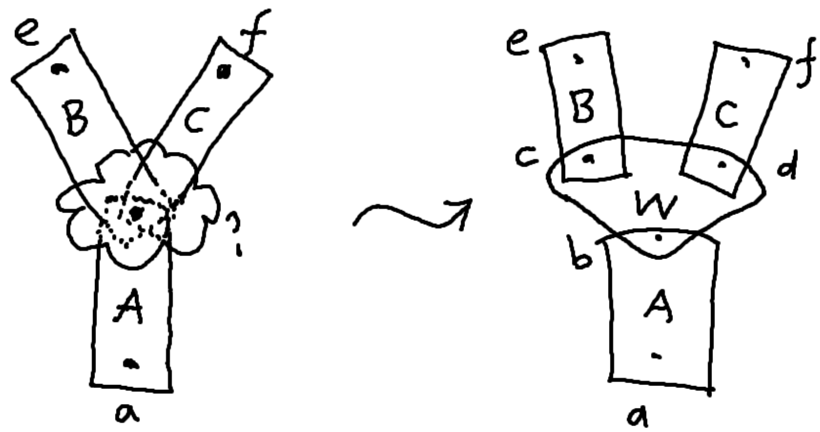

Figure 4.10: System overlapped by three or more operations? Made legal by introducing an additional correlation W in the present formulation.

present framework, this can be made legal by introducing a correlation (e.g., as in Figure 4.10). Since a general correlation may causally correlate different operations in an indefinite way, indefinite causal correlation can arise in this kind of "multiple operation overlapping" situation. One may even amuse oneself with the idea that this kind of indefinite causal structure replaces spacetime singularities in a theory of quantum gravity.

Compositions "into operation" and "into correlation" either require $\mathcal{S}_o \subset \mathcal{S}_c$ or $\mathcal{S}_c \subset \mathcal{S}_o$. When neither holds one can conceive "compositions into hybrid", since the resulting object would interact with operations on some systems and correlations on the rest. We exclude this exotic case from the special framework, which we intend to keep simple, and discuss it in the general framework of Section 4.4.

In other frameworks of operational probabilistic theories there are usually two kinds of compositions: sequential composition and parallel composition. What we presented above corresponds to sequential composition. As for parallel composition, in quantum theory this usually corresponds to forming tensor products of the Hilbert spaces of systems, and then of operations and correlations on the systems. In the present framework for a given network there is no need to parallel compose because all the operations/correlations on joint systems are assumed to be given already. One reason one may want to consider parallel composition in the present framework is to join networks with different systems. Then a non-trivial parallel composition would involve introducing new connections across systems from different networks and taking tensor product of process matrices from different networks. For correlations with indefinite causal structure, one must be careful since naively forming tensor products can lead to unphysical correlations that violate the probability normalization condition [61]. Different ideas on dealing with this issue had been conceived. For example, one is to restrict the new connections that can be introduced. Another is to modify the process matrices after taking tensor products. Yet another option is to not allowing parallel composition of networks at all. In our view, which option to choose depends on what physical situation the networks are supposed to describe, and cannot be decided on the mathematical model-building level.

In the following, we count the "into operation" composition and the "into correlation" compositions, but not any form of composition, such as parallel composition or "into hybrid" composition, as standard forms of compositions of the special framework.

An important notion for the applications to be presented next is "fine-graining", which relates different networks. Let $\mathcal{N}$ and $\mathcal{N}'$ be two networks. If by some standard compositions we can generate $\mathcal{N}$ out of $\mathcal{N}'$, then we say that $\mathcal{N}'$ is a *fine-graining* of $\mathcal{N}$ (or $\mathcal{N}'$ *fine-grains* $\mathcal{N}$), and write $\mathcal{N} \preceq \mathcal{N}'$.

4.3 Applications of the special framework

Here we sketch some applications of the special framework of correlation networks.

4.3.1 Extensions

It was pointed out in [61] that the indefinite causal structure causes some subtleties in forming tensor products of correlations. The simple matrix tensor product of two process matrices does not necessarily lead to a valid process matrices. The reason is that the probability normalization condition on the original processes induce constraints that take care of only the operations attached to the original process matrix, but not the operations attached jointly to the two process matrices.

In the original process matrix framework it is tricky to deal with this issue, since if any operation is allowed across some input and output systems then all operations are allowed across them. In the present framework we can impose restrictions on the operations through subsystems, which means that non-trivial tensor products of process matrices are incorporated.

Figure 4.11, Figure 4.12, and Figure 4.13 shows an example to obtain a non-trivial tensor product of process matrices $W_1 \otimes W_2$. We think of such a procedure as an extension of networks. Starting from the network of W_1 or W_2, we introduce some new systems, connections, and correlations to obtain a new network that fine-grains the original one.

The present framework specifies a language to express some of the ideas (in the example above, the idea is to not allow arbitrary operations across the two process matrices) conceived to clarify the issue of tensor product forming/parallel composition of correlations with indefinite causal structure. However, it in no way resolves the issue itself. An important question that begs a separate analysis is regarding the

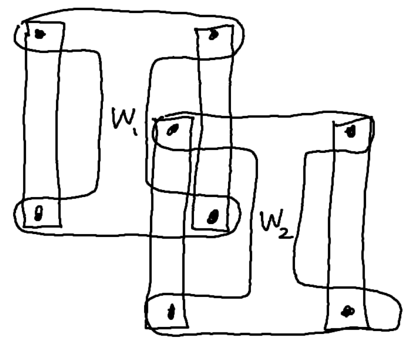

Figure 4.11: Two independent networks. W_1 and W_2 are classical mixtures of channels in opposite directions.

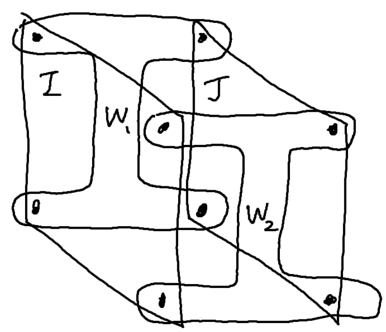

Figure 4.12: An illegal parallel composition by allowing arbitrary operations on the left and on the right.

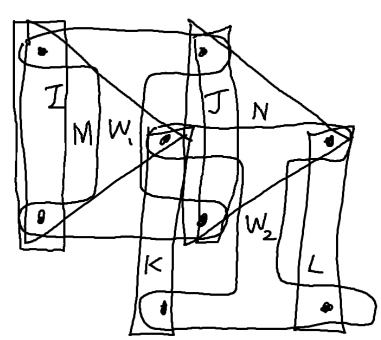

Figure 4.13: A legal parallel composition by allowing only information flow from W_1 to W_2.

experimental realizations of the quantum switch (and other correlations with indefinite causal structure in general). Suppose two quantum switches can be realized very adjacent to each other, what would be the correct description of the physics? Is it a tensor product of the quantum switches with restricted operations? Is it a correlation that differs from a tensor product of the quantum switches with unrestricted operations? Or is it something else? In the present framework, a starting point of one attempt to address this question is to decompose a system into many subsystems, and to find the description of the physical correlation and operations at this fine-grained level.

4.3.2 Decomposition

Not only composition, but also decomposition is important for operational quantum theory and indefinite causal structure. For example, one may apply operational/information theory tools to investigate renormalization and renormalization group flows, and study the role indefinite causal structure plays for renormalization and renormalization group flows.

Decomposition can be modelled using the notion of fine-graining introduced at the end of Section 4.2.

In the original paper [58] the process matrix formalism makes an assumption of "local quantum mechanics". Part of this assumption says that the operations an agent performs inside a local laboratory are described by ordinary quantum mechanics with definite causal order. Technically, this means that the operations are described by quantum instruments obeying the causality condition. This leaves it unspecified whether through fine-graining the local laboratory, or technically the operations associated to the local laboratory, it is allowed to see correlations with indefinite causal structure. Our opinion is that it should be allowed to see correlations with indefinite causal structure. For instance, a quantum switch on a fine scale can appear as a unitary on a broader scale to other agents [153]. A unitary loses no information, so can be used to reproduce arbitrary operations on an even larger scale of a local laboratory that obeys the "local quantum mechanics" assumption.

4.3.3 Counterfactual systems

Does a generalized process matrix always describe correlations in the same universe? The answer is no.

Usually in operational quantum theory an agent's choice is to apply different quantum instruments on the same pair of input and output systems. The systems are assumed to be there to begin with, and is not subject to the choice of the agent. What if the agent has a freedom to bring one or another system into existence? As an example, consider a highly dense and slowly expanding gas that has almost enough energy density to form a black hole. Suppose an agent has a choice of either injecting

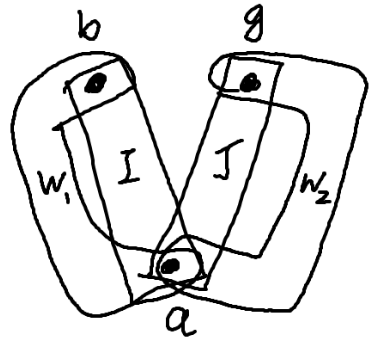

Figure 4.14: An agent can choose to I to bring a black hole system b into existence, or J that interacts with a gas system g. Two different correlations W_1 and W_2 are associated with these counterfactual situations.

more energy into the gas to make it collapse into a black hole, or accelerating the expansion of the gas to avoid a black hole formation. Namely, the agent can choose to bring a black hole system into existence or not.

This shows that an agent's choice of operation may be associated with systems that exist in different counterfactual situations. In a correlation network, it is possible to have systems that belong to different counterfactual situations.

In describing counterfactual situations, it is convenient to generalize the notion of network a little bit to allow the same system to be associated with multiple correlations corresponding to the counterfactual situations. For example, in Figure 4.14 system a is associated with two correlations W_1 and W_2.

4.4 Correlation networks: general frameworks

Here we briefly summarize the motivations for more general frameworks and outline the basic elements of one generalization.

4.4.1 Motivations

The above are theoretical motivations for more general frameworks based on inspecting the mathematical model.

Recall Assumption 2 "operation inclusion", which says that for each given connection, all mathematically allowed operations are allowed in the theory. We noted already in considering the "into operation" composition that this assumption cannot be retained. In a more general theory Assumption 2 may be relaxed.

An operation can be viewed as a set of correlations each labelled by an observational outcome. From this view all operators are regarded correlations. Some come in sets and are labelled by outcomes. Some come as individuals.

In the special framework all these sets of operational correlations are described by ordinary quantum theory with definite causal structure. In Section 4.3.2 we showed that it is possible such correlations labelled by outcomes with definite causal structure arise out of coarse graining of correlations with indefinite causal structure. What about moving one step forward and allowing correlations labelled by outcomes to be correlations with indefinite causal structure.

Now that one could relax the assumption that operations obey definite causal structure, it is natural to think of removing the distinction between input and output systems and treat all systems on an equal footing.

4.4.2 Basic elements

The different possibilities of generalizations suggested above may be conducted independently. This is why the title of this section speaks about "general frameworks" in the plural form, since there could be a hierarchy of generalized frameworks of correlation networks.

Here we briefly sketch the basic elements of a general framework of our choice. In comparison to the special framework, the distinction between input and output systems is removed, the operational correlations are allowed to carry indefinite causal structure, and each connection is now associated with a possibly restricted family of operations.

The basic setup of the framework is the same as in Section 4.2.1, except that there is no longer a distinction between input and output systems.

When it comes down to quantum theory, the elements of an operation are directly associated with positive semidefinite operators. There is no causality requirement that they sum up to the Choi operator of a CP trace-preserving map – they too can carry indefinite causal structure. Naturally, we relax Assumption 2 so that there can be restrictions on which operations are possible. Similar to in the special framework, the correlations can be represented as operators. An additional consideration must be made for the conditions that probabilities are non-negative and normalized. There may not be any correlation that obeys these conditions for an arbitrary setting, since now there is no causality constraint on the operations.

There is the technical question of characterizing the possible correlations in the absence of the causality constraint for the operations and in the presence of restrictions on the possible operations. There does not seem to be a straightforward answer to this question and further investigations are needed.

In terms of composition, now that operational correlations and correlation are now both viewed as correlations, we can encompass more general compositions under the type of "into correlation", which result in operational correlations that are labelled by outcomes, or individual correlation. The composition rule is still the partial trace rule, as in the special framework.

4.4.3 Discussion

The framework outlined above is very close to the Oreshkov-Cerf theories [63, 83]. The Oreshkov-Cerf theories relax the assumption of only allowing pre-selection for operations, and allows post-selection as well. We have not made this step of generalization yet. Perhaps the general framework above may be viewed as a specialization of the Oreshkov-Cerf theories?

Chapter 5

Entanglement in quantum spacetime

Entanglement is deemed an important concept to studies of quantum gravity. For example, quantum field entanglement is believed to contribute to black hole entropy [154, 155], and its scaling is believed to characterize Einstein's equation [156]. There is the question of whether entanglement can be studied in a causally neutral way in quantum spacetime. This is the topic of this chapter.

In Section 5.1, we present a way to study entanglement causally neutrally based on the CNQFT framework of Chapter 3. In Section 5.2, we briefly present some preliminary ideas on finding causally neutral analogues of Einstein's equation in quantum spacetime. Apart from the work reviewed in Section 5.1.2, the content of this chapter has not been published by the author elsewhere.

5.1 Entanglement in quantum spacetime

In this section, we present a generalized notion of entanglement in accordance with the principle of causal neutrality Section 1.5.5. For finite dimensional systems the work was done in the author's previous thesis [1] and paper [2]. This section contains the original work of carrying out the same task for quantum field theory.

5.1.1 Clarifying some misunderstandings

The following are some common misunderstandings about the concept of entanglement one can occasionally encounter among the discussions of physicists, especially in the context of entanglement in spacetime.

1. Entanglement can only be considered for *pure* states.

2. Entanglement entropy is the only entanglement measure to be studied.

3. Entanglement can only be considered for acausal correlations/quantum *states*.

4. Entanglement must vanish for causal correlations.

1 and 2 are very simple to refute. There is a whole subject of studies on mixed state entanglement [147]. Entanglement entropy, defined as $S(\rho^a)$ (the von Neumann entropy for the reduced state $\rho^a := \mathrm{Tr}_b\, \rho^{ab}$) for a bipartite state ρ^{ab}, is really only a meaningful entanglement measure for pure states. For mixed states, there are families of entanglement measures, such as distillable entanglement, entanglement of formation, relative entropy of entanglement etc., suitable for different contexts.

There are some positive reasons to incorporate mixed states into the study of entanglement in spacetime. It often happens in the context of studying entanglement for systems in spacetime (e.g., entanglement of black holes) that one only specifies one system, and implicitly takes the second system to be the rest of the global system. A difficulty for quantum field entanglement entropy in this setting is that it diverges. One option is to regularize the divergence in order to derive a finite entanglement entropy. However, if the regularization is assumed to arise from a physical mechanism rather than from a mathematical trick, it is unclear that the physical regularization mechanism keeps the state on the two systems pure, and the use of entanglement entropy as a measure of entanglement becomes questionable. There are other issues attached to common regularization schemes such as the breaking of symmetries. Another option is to not let the systems touch each other and leave some safe corridor between the regions [157]. Apart from the prospect of obtaining a finite entanglement, this can be motivated by the fact that when the system touch each other the Hilbert space may not have a tensor product structure for the systems of the regions, which makes entanglement an ill-defined concept. By the "split property" [87, 56], leaving a corridor implies a tensor product Hilbert space structure. This option needs to have the corridor system "traced out", and makes it necessary to study entanglement for mixed states. For either option, it is more general and safer to allow for both pure and mixed states in a study of field entanglement.

Points 3 and 4 may appear reasonable, as traditionally entanglement is only defined for quantum states on tensor product spaces. Since the tensor product structure traditionally implies acausal separation, the concept of entanglement is undefined for causal correlations.

Point 3 can be refuted by explicitly demonstrating that entanglement can be considered for correlations such as quantum channels which are not acausal correlations/quantum states. The principle of causal neutrality (Section 1.5.5) offers a strong motivation for doing so. Causal fluctuations will turn acausal correlations into correlations with indefinite causal structure. Entanglement should better be defined for such correlations, especially given the wide expectation that the notion of entanglement is crucial to quantum spacetime [154, 155, 158, 159]. As we shall see, such extensions of the concept of entanglement and entanglement measures automatically refute 4.

Entanglement for causal correlations had been studied by several different groups of authors, e.g., [160, 161, 146, 162, 163, 164, 165]. The step of extending the study to correlations of arbitrary (including indefinite) causal structure and having a causally neutral notion of entanglement was made by the author in [1, 2]. We give a brief review of the causally neutral entanglement next.

5.1.2 Causally neutral entanglement

In this subsection we briefly review the causally neutral notions of entanglement and entanglement measures. We focus on presenting the basic idea, and the technical details can be found in [2].

We are guided by the **causally neutral view on entanglement**:

> Entanglement, as a property of correlations, can be carried by acausal, causal, and causally indefinite correlations.

Once this view is set, a generalization of traditional entanglement to causally neutral entanglement is simple. One simply applies the old definition of entanglement and entanglement measures for states (which are acausal correlations) to general correlations that can be acausal, causal, and causally indefinite, such as the process matrices.

Given any correlation of operations, an *entanglement measure* is a function from the correlation to the reals such that the value of the function is non-increasing under the action of the allowed local operations and classical communications (LOCC). This is called the *monotonicity axiom* for entanglement measures. Any such entanglement measure has a minimum value that is reached for correlations that can be created through LOCC alone. As a convention, we usually subtract this minimal value from the measure such that the minimum value of entanglement measures are zero. A correlation that is positive for some entanglement measure is said to be contain *entanglement*. Depending on different contexts, different LOCC operations are allowed. Each prescription of allowed LOCC operation is called an *LOCC setting*.

Apart from the fundamental consideration of causal neutrality, a practical advantage of using the generalized notion of entanglement is that it offers a unification of previously independent concepts. For example, the entanglement distillation capacity of a state, the quantum communication capacity of a channel, and the entanglement generation capacity of a network or a process become different manifestations of one and the same entanglement measure – the entanglement generation capacity for general correlations. The capacity theorem for all these different cases can be proved in a unified way [2], as opposed to separately, which is what happened historically, and which appears redundant in hindsight. In particular, under the causally neutral view the quantum communication capacity of a channel is viewed as a causally neutral entanglement measure specialized to the causal correlation of quantum channels. This demonstrates the usefulness of assigning positive values of entanglement to causal correlations.

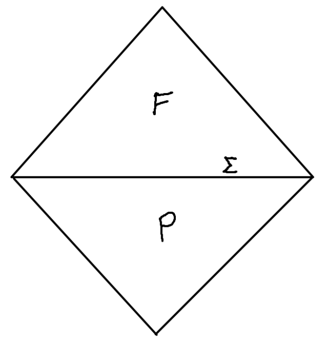

Figure 5.1: A causal diamond is partitioned by the Cauchy surface Σ into a future region F and a past region P, both of which are Cauchy regions.

5.1.3 Field entanglement in quantum spacetime

Hidden behind the above causally neutral generalizations of entanglement and entanglement measures is that a causally neutral description of correlations should be available. In other words, a unified description of acausal, causal, and causally indefinite correlations should be available. For finite dimensional systems this is provided, for example, by the process matrices. What about for infinite dimensional systems such as those in QFT?

A causally neutral definition of entanglement and entanglement measures is not possible based on ordinary AQFT, since the framework is "causally biased" as opposed to causally neutral. A generalization of entanglement based on the causally biased ordinary AQFT leads to Sorkin's spacetime entanglement [166]. Here only the pure state entanglement measure of the entanglement entropy S is studied. For a spacetime region R with the causal domain of dependence Σ (a spacelike surface), it holds that $S(R) = S(\Sigma)$. This implies that in a situation of Figure 5.1, there is no entanglement between the future region F and the past region P, because the domain of dependence for either F or P is the Cauchy surface Σ, which contains the global pure state and has entropy zero [167].

Is this reasonable? It is reasonable from the perspective of ordinary AQFT with the time-slice/primitive causality axiom [56]. In this case an equation of motion or a dynamical constraint reduces the algebra $\mathfrak{A}(R)$ of the region R to the algebra $\mathfrak{A}(\Sigma)$ of its domain of dependence. In a sense, the causal correlation is encoded through the equation of motion/dynamical constraint in the algebra, and only the acausal correlation is encoded in the state. Therefore when the two regions F and P have no acausal quantum correlation there is zero entanglement as accounted for by the state.

This is not reasonable, though, from the perspective of the causal neutrality principle. Because the causal correlation and the acausal correlation must be strictly distinguished and separately encoded in the algebra and the state, ordinary AQFT is not a causally neutral framework. A causally neutral framework for quantum fields is provided by the causally neutral QFT framework (CNQFT) of Chapter 3. The generalized states provide a unified description of acausal, causal, and causally indefinite correlations. This is what one needs to define causally neutral entanglement and entanglement measures.

In this case the entanglement between F and P (to the extent that it is meaningfully defined when supplied with a tensor product space structure) is expected to be positive, just like a quantum channel's entanglement between the input and the output is expected to be positive (given that the channel contains quantum causal correlation and is not, e.g., a classical channel). In general, causal correlations in the field theory are expected to generically have positive entanglement.

Causally neutral entanglement for quantum fields

The entanglement of quantum fields have been studied by many since the seminal works of [154, 155]. Issues such as the UV divergence and the lack of a tensor product space are usually glossed over. A more mathematically rigorous study of entanglement and entanglement measures is carried out by Hollands and Sanders in [157] in the ordinary AQFT framework. We obtain a causally neutral definition of entanglement and entanglement measures by following the same strategy of Section 5.1.2. Namely, we take the definitions of [157] and substitute in causally neutral correlations for the causally biased correlations. In a nutshell, this gives us the following (see [157] for the definitions of the technical terms).

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two commuting von Neumann algebras, and let $\mathcal{H}$ be a Hilbert space of generalized states that the von Neumann algebras act on. Denote the von Neumann algebra generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (the smallest von Neuman algebra containing both) by $\mathfrak{A}_1 \vee \mathfrak{A}_2$, and their tensor product algebra by $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, which acts on $\mathcal{H} \otimes \mathcal{H}$. Suppose $\mathfrak{A}_1 \vee \mathfrak{A}_2 \simeq \mathfrak{A}_1 \otimes \mathfrak{A}_2$, i.e., they are unitarily equivalent.

A normal state[1] ω on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is *separable* if it can be written as $\sum_j \phi_j \otimes \psi_j$, where the sum is norm convergent and ϕ_j, ψ_j are positive normal functionals on $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively. A normal state that is not separable is said to be "entangled". A property that an entanglement measure $E(\omega)$ should obey is the monotonicity under separable operation condition: Let $\{\mathcal{F}_j = \mathcal{F}_{1,j} \otimes \mathcal{F}_{2,j}\}_j$ be a family of normal completely positive maps on the algebra $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ so that $\sum \mathcal{F}_j(1) = 1$, and define $p_j := \omega(\mathcal{F}_j(1))$. Then we require that E satisfy $\sum_j p_j E(\frac{\mathcal{F}_j^*(\omega)}{p_j}) \leq E(\omega)$. An example of an entanglement measure is the relative entropy of entanglement based on Araki's relative entropy for von Neumann algebras, suitably (and straightforwardly) generalized to CNQFT.

Entanglement and entanglement measures in the CNQFT framework differ from the traditional notions at least at three levels. First, in ordinary AQFT, von Neumann algebras for causally related spacetime regions generically do not commute, and do

[1] A normal state can be characterized in several ways. One way is to define it as an ultraweakly continuous state.

not have the tensor product structure, so entanglement cannot be studied for them. In CNQFT the algebras generically commute (Section 3.2.7). The tensor product structure may be obtained, for instance by considering situations where a generalized split property holds.[2] It is meaningful to speak of "timelike entanglement" in the CNQFT framework.

Second, in ordinary AQFT, von Neumann algebras that allow a meaningful study of entanglement must be associated with non-overlapping spacetime regions. This is not necessary for CNQFT. By a reason similar to the above timelike separation case, the tensor product structure can exist even for overlapping spacetime regions.

Third, in ordinary AQFT, the von Neumann algebras must be associated to regions of spacetime. This is not necessary for CNQFT. In principle, the framework can abstain from associating the algebras to spacetime regions while still allows a meaningful study of entanglement.

The generalized notions of entanglement will be applied in Section 5.2 to search for analogues of the Einstein's equation in quantum spacetime.

5.1.4 Aside: Possible lessons for causal set QFT and its entanglement

It is not our intention to study causal set QFT in this thesis, so this subsection is an "aside". I include this discussion here because what we presented above seems to clarify some issues regarding causal set QFT and its entanglement studied in the literature [167, 168].[3]

1. The version of QFT that forms an analogue of causal set QFT [86] is not ordinary AQFT, but CNQFT of Chapter 3. In AQFT, the dynamics is encoded in the algebra. In both causal set QFT and CNQFT, the "dynamics" is encoded in the correlation functions.

2. For causal set QFT, one should broaden the horizon to study mixed state entanglement in addition to pure state entanglement. The entropy of a single region of a causal set spacetime may not be an entanglement measure, since the global state may not be analogous to an ordinary pure state in AQFT. Mixed state entanglement measures such as the relative entropy of entanglement [157] has a better chance of being a meaningful entanglement measure for causal set QFT.

[2] To be sure, one needs to write down a "generalized split property" explicitly and show that a tensor product structure can be derived before one can draw any definitive conclusion. We postpone this since we suspect the UV regularization mechanism proposed in Section 3.4 may be used to derive a tensor product structure directly.

[3] I call these "possible" lessons because my knowledge of causal set theory is limited. Someone more familiar with the theory should check if these "lessons" actually make sense.

3. For a causal set QFT, the entropy of a "Cauchy region" [167] should not be expected to vanish, and $S(R) = S(\Sigma)$ [166] (R is a spacetime region, and Σ is its domain of dependence) should not be expected to hold in general. $S(R) = S(\Sigma)$ holds for ordinary AQFT, because the dynamics is encoded in the algebra, whence $\mathfrak{A}(\Sigma)$ generates $\mathfrak{A}(R)$. There is no analogous ground to expect the same equation to hold for a causal set QFT. As mentioned above in point 1, the analogue of causal set QFT is not ordinary AQFT, but CNQFT. For CNQFT there is nothing wrong if the entropy of a "Cauchy region" does not vanish.

   In fact in the case of classical spacetime we expect that the entanglement of "Cauchy regions" not only does not vanish, but also has a generalized area scaling for suitable measures. For example, for F and P separated by Σ in Figure 5.1 (they should share a noiseless channel), we expect that entanglement measures such as optimized coherent information scales as the number of degrees of freedom on Σ [2]. If the number of degrees of freedom on Σ scales as the area of Σ, then the entanglement is expected to be proportional to the area of Σ. Note that in 4D spacetime Σ is 3D, so the "area" of the entangling surface Σ is its 3D-volume.

5.2 Einstein constraints?

One task of quantum gravity is to find an analogue of Einstein's equation in quantum spacetime. Causal neutrality again poses a challenge. A classical spacetime manifold, carrying a definite causal structure, cannot be retained, so Task 1 is to formulate a theory with an analogue of Einstein's equation without reference to classical spacetime manifolds. Achieving Task 1 is not necessarily enough – one also needs to incorporate indefinite causal structure and formulate fundamental notions in accordance with the causal neutrality principle. Hence Task 2 is to formulate a theory with an analogue of Einstein's equation so that indefinite causal structure is incorporated and the causal neutrality principle is obeyed.

There have been many attempts on Task 1, such as the well-known approaches of quantum gravity [169]. The topic of this section is to make progress on Task 2, which is less visited.

5.2.1 The reconstruction paradigm

While there will perhaps be different routes to achieve the tasks, we focus on what we call the "reconstruction paradigm", pioneered by Jacobson [156].[4] In the reconstruc-

[4] Other names such as "emergent gravity", "entropic gravity", and "thermodynamic gravity" had been attached to this or similar programs. We find the word "reconstruction" more appropriate in the current context, since we are not embracing the view that Einstein's equation is an effective

tion paradigm one finds some expression not directly referring to the gravitational metric and then show that Einstein's equation can be derived from this expression. In other words, one "reconstructs" Einstein's equation starting from some other expression. A primary example is Jacobson's work [156] that initiated the paradigm. The Clausius relation $dE = TdS$ is used to reconstruct the Einstein equation. Here dE is the energy difference, T is the Unruh temperature, and dS is the horizon entanglement entropy difference. In this perspective, the key content of Einstein's equation is that S is area-scaling, as different scalings of S would yield modified gravitational field equations [170].

In the context of this section, the salient feature of a reconstruction like this is that it yields an Einstein's equation analogue that may still make sense in the absence of classical spacetime manifolds, or even in the absence of definite causal structures.

This is almost the case for Jacobson's reconstruction, but not quite, since the entanglement entropy refers to causal horizons, whose meaning is unclear without classical spacetime manifolds or definite causal structures. We present some preliminary thoughts on upgrading horizon entanglement in accord with quantum spacetime and the causal neutrality principle next.

5.2.2 Horizon entanglement without horizon

A notion of horizon defined through the global definite spacetime causal structure [31] (such as the event horizon) or through the local properties of geodesic congruence [171] (such as the trapped surfaces, the isolated horizon, and the dynamical horizon) is not expected to be meaningful in a quantum spacetime with causal fluctuations.

The task now is to reinterpret T, S and E in the formula $dE = TdS$ in quantum spacetime. In Jacobson's work T is the Unruh temperature of constantly accelerating trajectories. One route to replacing it with a notion meaningful in quantum spacetime is summarized as

> constant acceleration $\Longleftrightarrow$ constant proper distance $\Longleftrightarrow$ causal fluctuation equilibrium.

The first correspondence between (timelike) trajectories with constant acceleration and (timelike) trajectories with constant proper distance from the origin is elementary in flat classical spacetime (in Jacobson's derivation only local properties near the origin is relevant, which justifies the consideration of a flat spacetime).

The second correspondence allows us to move from classical to quantum spacetime. First we need to explain what we mean by "causal fluctuation equilibrium". Then we need to argue for its connection with "constant proper distance".

The idea of causal fluctuation equilibrium is that the "magnitude" of causal fluctuation stays the same along some flow for the state. In the language of Chapter 3, the idea can be captured as follows. Consider a free product algebra $\mathfrak{A} = \star_i \mathfrak{A}_i$ formed

description on top of some further microscopic degrees of freedom.

out of many algebras $\mathfrak{A}_i$. Here i can be a discrete or a continuous parameter, and there can be infinitely many $\mathfrak{A}_i$. The set of all i forms an "abstract parameter space" I. Take ω to be the generalized state that describe the physics of interest. Without loss of generality let $\mathfrak{A}_0$ be associated to the central physical event of interest (the physical event whose causal horizon Jacobson's derivation focuses on). We are concerned with "two-point" correlations, with one "point" fixed to be $\mathfrak{A}_0$, and the other "point" i is allowed to vary. For each $i \neq 0$, we can obtain the reduced generalized state ω_{0i} induced from ω by supplying the identity elements e for all the other factors.

We can consider "flows" in the abstract parameter space I defined by functions of the form $\lambda : \mathbb{R} \to I \backslash \{0\}$. Each flow in the abstract parameter space induces a flow of "two-point" generalized states with ω_{0i} as the images. Different ω_{0i}'s describe causal fluctuations between $\mathfrak{A}_0$ and $\mathfrak{A}_i$ with different "magnitudes". Some describe large causal fluctuations, while some describe small causal fluctuations. A "causal fluctuation equilibrium flow" maps to ω_{0i}'s describing causal fluctuations with the same magnitude or approximately the same magnitudes.

In setting up the causal fluctuation equilibrium flow we have only used quantum spacetime notions. How does this relate to constant proper distance trajectories in classical spacetime? The connection is based on the expectation that the magnitude of the causal fluctuation relates to the distance in classical spacetime in a direct "Lorentz invariant" way. Suppose that there is a classical correspondence from the algebras $\mathfrak{A}_i$ describing quantum spacetime to classical spacetime regions (this point will be discussed further below). Then one can ask about the distance between the classical regions corresponding to $\mathfrak{A}_0$ and $\mathfrak{A}_i$ for different i in the same causal fluctuation equilibrium flow. Because of Lorentz symmetry, if the magnitude of the causal fluctuations is related to spacetime distance in any direct way, the most natural expectation is that the *proper distance* is the same or approximately the same for the different i's in the same flow. Under the assumption that the expectation holds, there is a correspondence between causal fluctuation equilibrium and constant proper distance.

The above argument for second correspondence admittedly has some gaps to be filled. First, we need a more mathematically precise characterization of the magnitude of causal fluctuation used in setting up the "causal fluctuation equilibrium" flow. Some conditions on the generalized states similarly succinct as the KMS conditions on ordinary QFT states would be ideal. Second, we need a concrete way to setup the "classical correspondence" from the algebras to classical spacetime. At the quantum level description of the algebras and the generalized states, we know only probabilities for realizing different classical spacetime relations. One potential way to arrive at a classical correspondence is to pick the classical configurations with high probabilities.

Suppose these gaps can be filled. Then we can use the the "causal fluctuation equilibrium flow" to replace the constant acceleration trajectory. In ordinary QFT the Unruh temperature can be obtained by calculating the excitation rates of a detector along the constant acceleration trajectory [118, 120]. The input to this calculation is essentially the the two-point function along the trajectory. In quantum spacetime we can replace this with ω_{0i} along the causal fluctuation equilibrium flow to obtain the temperature T.

The next task is to reinterpret S. In Jacobson's original derivation this is the entanglement entropy associated with the causal horizon. Section 5.1 offers a way to consider entanglement causally neutrally in quantum spacetime. Regarding the use of classical spacetime horizon one can avoid it and focus on the entanglement of $\mathfrak{A}_0$ and $\mathfrak{A}_i$ in the causal fluctuation equilibrium flow. In classical spacetime, the area scaling of the entanglement encodes the idea that only highly local contributions near the entangling surface contribute significantly to the entanglement, and it does not make an essential difference if we ignore contributions from any other regions. Translated to quantum spacetime[5], this allows us to choose suitable $\mathfrak{A}_i$ with a suitable causal fluctuation equilibrium flow that encompass the significant contributions to the entanglement. This is the entanglement we use for S.

Incidentally, in Jacobson's derivation the area scaling of entanglement hinges on the assumption of a UV cutoff, whose particular origin is not specified. A salient feature of the present consideration incorporating indefinite causal structure is that causal fluctuations supplies a particular mechanism for the UV regularization (Section 3.4). When this regularization is "Lorentz invariant" it is consistent with the correspondence between causal fluctuation equilibrium and constant proper distance mentioned above.

As of the energy perturbation dE, the difficulty is that traditionally it is defined with respect to the timelike symmetry of a classical spacetime. In quantum spacetime we propose to consider energy operationally. For example, we can pick the energy perturbation to be a transfer of energy into a battery, which results in a transition of the battery to a new state characterizing a different energy level. This offer an at least approximate characterization of energy perturbation in quantum spacetime.

We have reinterpreted all the ingredients of $dE = TdS$ in order to make it meaningful in quantum spacetime. This can be viewed as a restatement of the Einstein's constraint in quantum spacetime. Indefinite causal structure played an active role twice in the reformulation, once in setting up an alternative to constant acceleration trajectories through the causal fluctuation equilibrium flow, and once in offering the UV regularization for the entanglement.

5.2.3 Discussion

We found a way to make sense of $dE = TdS$ in quantum spacetime with indefinite causal structure in a causally neutral way. There is a long way ahead to address many questions that naturally arise. As discussed above, which exact way to charac-

[5] The area scaling of entanglement holds in 3+1 dimensions, but not for, e.g., 1+1 dimensions. Here we consider a quantum spacetime model that yields 3+1 dimensions under the classical correspondence.

terize the magnitudes of causal fluctuation do we choose? How exactly to obtain the classical correspondence? How to show that the UV regularization actually gives rise to an approximate area scaling entanglement in a suitable limit? Is there anything interesting to say about the fluctuations around the approximate area scaling? How to use the analogue of Einstein's equation as an "Einstein's constraint" for physical states (This is related to questions discussed in Section 3.5), as Einstein's equation is used to solve for physical states in classical spacetime?

Chapter 6

Outlook

We conclude with a brief outlook into some important directions for further research.

- Follow the principle of causal neutrality to upgrade concepts important to quantum gravity and formulate theories accordingly.

  For example, can we characterize black holes through their information processing properties instead of causal boundaries or geodesic congruence?

  As another example, what is the quantum spacetime analogue of the gravitational metric itself? In classical spacetime the metric is equivalently characterized by the causal structure plus the conformal factor [31, 32, 172]. Causal structure seems to be a causally neutral notion, but what about the conformal factor? For this we note a very interesting result showing that the classical gravitational metric can be recovered from the quantum field correlation function [30]. Perhaps there is an analogue recovery in CNQFT?

- Further investigate the correlation functions with the UV structure of strong causal fluctuation proposed in Section 3.4.

  For example, does it say anything about singularities?

- Look for analogues of Einstein's equation in quantum spacetime in accordance with the principle of causal neutrality.

  For example, follow the suggestions of Section 5.2.

- Design observational schemes for causal fluctuations and spacetime fluctuations in general.

  This belongs to the field of quantum gravity phenomenology [173]. Although much work has been done on general spacetime fluctuations, causal fluctuation may provide some new experimental windows. As an example of a question to study, do causal fluctuations introduce anything new to the speed of light? As another example, what does causal fluctuations say about early universe cosmology, and in particular, about the "horizon problem"?

Appendix A

Communication capacities for general correlations

In this appendix we present a theorem on the communication capacities for general correlations. Here a general correlation is general in the sense that it could be a correlation with definite causal structure (such as a quantum channel) or indefinite causal structure (such as a process matrix). The result is technically trivial to derive, but is worth to be included here for future reference.

The result says that for state transmission tasks of communication, the capacity of a general correlation can be reduced to the capacity of channels. In the following we explain what we mean by "state transmission tasks", and why the reduction holds.

The basic setting for a state transmission task is illustrated in Figure A.1. Party A shares a preexisting state ρ with party C and a communication resource R with party B. The goal is for A to "transmit" her share of the preexisting state to B so that in the end B and C share a state that meets certain criteria. For example, one way to define the quantum channel capacity for a channel is to let R be multiple copies of a quantum channel and ρ be a maximally entangled state. The goal is for B and C to share as much maximally entangled state as possible in the end [149]. Similarly, the standard channel capacities of classical and private classical communication can be defined as transmitting classical maximally correlated states [149].

The reason that for transmission tasks the capacity of a general correlation reduces to the capacity of a channel is very simple: to transmit a state using a general correlation, the general correlation must always effectively be turned into a channel. This shall be clear from observing the setup in Figure A.1. The most general protocol to reach the goal that B shares a state with C consists of A and B applying operations M and N to generate a channel as the composition of M, N and R, as illustrated in Figure A.2.

Therefore we have the following result.

Theorem. For a state transmission communication task, the capacity of a general correlation R^{AB}, $C(R^{AB})$, can be expressed as an optimization over the capacities for

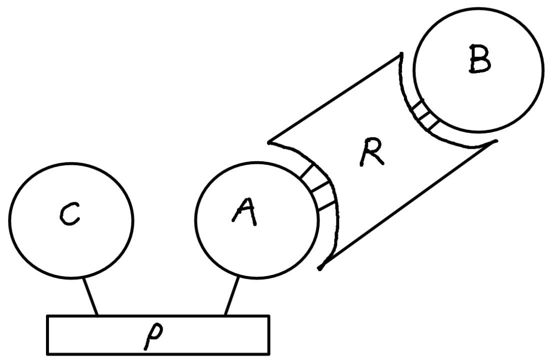

Figure A.1: Basic setting for state transmission. Here R has multiple wires connected to A and B, because a general correlation may interact with the parties through more than one system. For example, a process matrix interacts with a party twice, once through an input system and once through an output system.

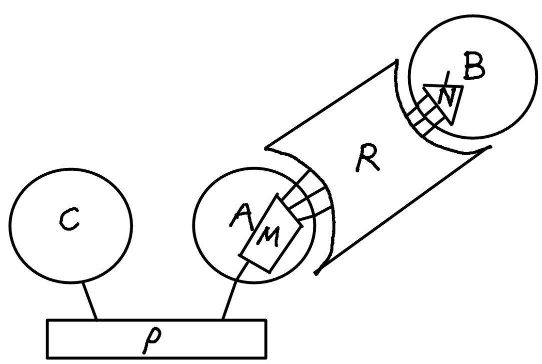

Figure A.2: State transmission protocol.

the channels it generates. Namely,

$$C(R^{AB}) = \sup_{\mathcal{O}} C(N^{AB}(R^{AB}, \mathcal{O})). \tag{A.1}$$

Here $N^{AB}(R^{AB}, \mathcal{O})$ denotes the channel N^{AB} generated from the general correlation R^{AB} from applying the operation $\mathcal{O}$ conducted by A and B. The optimization is over all operations $\mathcal{O}$ allowed by the communication task. $C(N^{AB})$ is the capacity of the same task for the channel N^{AB}.

The proof is elementary. The LHS is no less than the RHS, because one can apply the protocol to generate $N^{AB}(R^{AB}, \mathcal{O})$ from R^{AB} first and then use this channel to achieve the task at the capacity of the RHS. The LHS is no greater than RHS, because as illustrated above to transmit a state, R^{AB} always has to be turned into a channel first. Hence the RHS encompasses all possible protocols and upper-bounds the LHS.

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