

A generalization of Erdős' matching conjecture

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Abstract

Let $\mathcal{H} = (V, \mathcal{E})$ be an r -uniform hypergraph on n vertices and fix a positive integer k such that $1 \leq k \leq r$. A k -matching of \mathcal{H} is a collection of edges $\mathcal{M} \subset \mathcal{E}$ such that every subset of V whose cardinality equals k is contained in at most one element of \mathcal{M} . The k -matching number of \mathcal{H} is the maximum cardinality of a k -matching. A well-known problem, posed by Erdős, asks for the maximum number of edges in an r -uniform hypergraph under constraints on its 1-matching number. In this article we investigate the more general problem of determining the maximum number of edges in an r -uniform hypergraph on n vertices subject to the constraint that its k -matching number is strictly less than a . The problem can also be seen as a generalization of the well-known k -intersection problem. We propose candidate hypergraphs for the solution of this problem, and show that the extremal hypergraph is among this candidate set when $n \geq 4r \binom{r}{k}^2 \cdot a$.

Mathematics Subject Classifications: 05D05, 05C65

1 Prologue, related work and main results

A hypergraph, \mathcal{H} , is a pair (V, \mathcal{E}) where V is a finite set, called the *vertex set*, and \mathcal{E} is a collection of subsets of V . The set \mathcal{E} is called the *edge set* and its elements *edges*. We denote by $\binom{V}{k}$ the family consisting of all subsets of V whose cardinality equals k . A hypergraph is called r -uniform if all of its edges have cardinality r . A hypergraph is called k -intersecting if the intersection of any two of its edges has cardinality at least k . Given a finite set, F , we denote by $|F|$ its cardinality and, given a positive integer m , we

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denote by $[m]$ the set $\{1, \dots, m\}$. A finite set whose cardinality equals m is referred to as an m -set, for short. As an abuse of notation, we sometimes denote by $|\mathcal{H}|$ the number of edges in a hypergraph \mathcal{H} . A *matching* in a hypergraph, \mathcal{H} , is a family of pairwise disjoint edges. The *matching number* of \mathcal{H} , denoted $\nu(\mathcal{H})$, is the maximum cardinality of a matching. The notion of matching is fundamental in combinatorics. Its significance is supported by the fact that several combinatorial problems can be reduced to the problem of determining the matching number of appropriate hypergraphs.

An important problem regarding matchings in uniform hypergraphs was posed by Erdős in 1965, who asked for the determination of the maximum number of edges in an r -uniform hypergraph under constraints on its matching number. More precisely, let \mathcal{H} be an r -uniform hypergraph on n vertices which satisfies $\nu(\mathcal{H}) < a \leq \frac{n}{r}$. What is a sharp upper bound on the number of edges in \mathcal{H} ?

Erdős conjectured that the maximum is attained by two extremal hypergraphs. The first is the hypergraph \mathcal{H}_1 consisting of all r -sets on $ra - 1$ vertices, whose matching number is clearly $a - 1$. The second one is the r -uniform hypergraph, \mathcal{H}_2 , on n vertices whose edge set consists of all r -sets that contain at least one element from a fixed set of $a - 1$ vertices, and whose matching number is $a - 1$ as well.

Conjecture 1 (Erdős' Matching Conjecture, 1965). The number of edges in an r -uniform hypergraph, \mathcal{H} , on n vertices whose matching number satisfies $\nu(\mathcal{H}) < a \leq \frac{n}{r}$ is at most

$$\max\{|\mathcal{H}_1|, |\mathcal{H}_2|\}.$$

When $n \geq (r + 1) \cdot a$ it is not difficult to see that $|\mathcal{H}_2| \geq |\mathcal{H}_1|$ and therefore, in this case, \mathcal{H}_2 is the hypergraph which is conjectured to have the maximum number of edges among all hypergraphs satisfying the assumptions of Conjecture 1. Erdős obtained the following result.

Theorem 2 (Erdős [5]). *There exists some constant c_r , which depends only on r , such that among all r -uniform hypergraphs on $n > c_r \cdot a$ vertices that satisfy $\nu(\mathcal{H}) < a$, the hypergraph \mathcal{H}_2 has the maximum number of edges.*

The problem of determining the smallest value of c_r has attracted considerable attention (see [3, 4, 8, 9, 10, 11, 12, 14, 16], among several others). The current best known upper bound on this constant is $c_r \leq 2r + 1$, and is due to Frankl [8]. Let us also remark that Erdős' matching conjecture, if true, has implications in game theory (see [6]), distributed storage allocation (see [3, Section 5]) as well as in probability theory (see [3, 17]).

In this article we shall be interested in a generalization of Erdős' conjecture. Our work is motivated by the following notion of matchings in hypergraphs.

Definition 3 (k -matching). Let $\mathcal{H} = (V, \mathcal{E})$ be an r -uniform hypergraph on n vertices and fix a positive integer k such that $1 \leq k \leq r$. A k -matching of \mathcal{H} is a collection of edges $E_1, \dots, E_j \in \mathcal{E}$ such that every $T \in \binom{V}{k}$ is contained in at most one $E_i, i \in \{1, \dots, j\}$. The maximum cardinality of a k -matching in a hypergraph, \mathcal{H} , is its k -matching number and is denoted by $\nu_k(\mathcal{H})$.

Equivalently, a k -matching of $\mathcal{H} = (V, \mathcal{E})$ is a subset $\mathcal{M} \subset \mathcal{E}$ such that $|E_i \cap E_j| \leq k-1$, for all $E_i \neq E_j$ from \mathcal{M} . Let us mention that the notion of k -matching arose in the study of certain search games on hypergraphs (see [18, Appendix C]) as well as in the study of certain generalisations of Tuza's conjecture (see [1]).

Notice that a 1-matching of a hypergraph coincides with a matching. Notice also that when $\nu_k(\mathcal{H}) = 1$, then any two edges, say E_1, E_2 , in \mathcal{H} satisfy $|E_1 \cap E_2| \geq k$ and therefore the problem of maximizing the number of edges in a r -uniform hypergraph whose k -matching number equals 1 is equivalent to the problem of maximizing the number of edges in an r -uniform k -intersecting hypergraph, which we refer in this paper as the *k -intersection problem*. This problem, having been open for several decades, was proven to be of great importance in the development of extremal set theory (see [13, 15]) and was finally resolved by Ahlswede and Khachatrian (see [2]). In particular, the following holds true.

Theorem 4 (see [2, 19]). *Let $1 \leq k < r$. For every positive integer i such that $0 \leq i \leq \frac{n-k}{2}$, let \mathcal{B}_i be the family consisting of all $E \in \binom{[n]}{r}$ such that $|E \cap [k+2i]| \geq k+i$. The number of edges in an r -uniform k -intersecting hypergraph on n vertices is at most*

$$\max_{0 \leq i \leq \frac{n-k}{2}} |\mathcal{B}_i|.$$

Moreover, when $n \geq (k+1)(r-k-1)$ the number of edges in an r -uniform k -intersecting hypergraph on n vertices is at most $\binom{n-k}{r-k}$.

The families \mathcal{B}_i , defined in Theorem 4, are referred to as *Frankl families* (see [13]). In this work we address the problem of determining the number of edges in an r -uniform hypergraph on n vertices under constraints on its k -matching number. More precisely, we examine the following.

Problem 5. Let $\mathcal{H} = (V, \mathcal{E})$ be an r -uniform hypergraph on n vertices. Fix a positive integer k such that $1 \leq k \leq r$ and assume further that $\nu_k(\mathcal{H}) < a$. What is a sharp upper bound on the number of edges in \mathcal{H} ?

Later, after constructing suitable candidates for the extremal graph of Problem 5, we will formulate a generalization of Erdős' conjecture which is the main target of this note. Then we verify the conjecture for large values of n .

Note that Problem 5 is trivial when $k = r$: the maximum number of edges in \mathcal{H} is equal to $\nu_r(\mathcal{H})$. Hence, from now on, we assume that $k < r$. Notice also that in Erdős' matching conjecture, the extremal hypergraph \mathcal{H}_2 is obtained by fixing $a-1$ vertices, say v_1, \dots, v_{a-1} , and then taking all r -sets of the vertex set, V , that contain at least one of the vertices $v_i, i = 1, \dots, a-1$. In the same way, candidates for the extremal hypergraph in Problem 5 can be obtained as follows.

Fix a set V consisting of n vertices. For every positive integer i such that $0 \leq i \leq \frac{\lfloor \frac{n}{a-1} \rfloor - k}{2}$ and every family $\mathcal{T} = \{T_1, \dots, T_{a-1}\} \subset \binom{V}{k+2i}$ let $\mathcal{H}(\mathcal{T})$ denote the r -uniform hypergraph on the vertex set V whose edge set consists of all $E \in \binom{V}{r}$ such that $|E \cap T_j| \geq$

$k + i$, for some $j \in [a - 1]$. Notice that the $(k + 2i)$ -sets T_j need not be disjoint. In fact, there are several ways to choose the sets T_1, \dots, T_{a-1} . When the sets $T_1, \dots, T_{a-1} \in \binom{V}{k+2i}$ are *pairwise disjoint*, we will refer to the hypergraph $\mathcal{H}(\mathcal{T})$ as a (n, r, k, a, i) -Frankl family. Notice that $(n, r, 1, 2, i)$ -Frankl families are precisely the Frankl families from Theorem 4. Observe also that a $(n, r, 1, a, 0)$ -Frankl family is precisely the hypergraph \mathcal{H}_2 in Erdős' matching conjecture. The following result implies that the number of edges in $\mathcal{H}(\mathcal{T})$ is maximized when the sets are disjoint.

Theorem 6. *Let $V = [n]$ be a set of vertices and fix positive integers r, k, a such that $n \geq ra$ and $r > k \geq 1$. For every positive integer i such that $0 \leq i \leq \frac{\lfloor \frac{n}{a-1} \rfloor - k}{2}$ and every $\mathcal{T} = \{T_1, \dots, T_{a-1}\} \subset \binom{V}{k+2i}$ let $\mathcal{H}(\mathcal{T})$ be the r -uniform hypergraph on the vertex set V whose edge set consists of all $E \in \binom{V}{r}$ such that $|E \cap T_j| \geq k + i$, for some $j \in [a - 1]$. Then the number of edges in $\mathcal{H}(\mathcal{T})$ is less than or equal to the number of edges in a (n, r, k, a, i) -Frankl family.*

In other words, Theorem 6 suggests that candidates for the hypergraph that maximizes the number of edges in Problem 5 can be found among (n, r, k, a, i) -Frankl families.

We now proceed to find a hypergraph whose k -matching number equals $a - 1$ and corresponds to the hypergraph \mathcal{H}_1 in Erdős' conjecture. Notice that the hypergraph \mathcal{H}_1 is a complete r -uniform hypergraph on $ra - 1 = r(a - 1) + (r - 1)$ vertices that has the following property: one can find $a - 1$ edges $E_1, \dots, E_{a-1} \in \mathcal{H}_1$ such that for every $T \in \binom{[ra-1]}{r}$ there exists $i \in \{1, \dots, a - 1\}$ such that $|T \cap E_i| \geq 1$. Similarly, we are looking for an r -uniform hypergraph having the property that there exist $a - 1$ edges E_1, \dots, E_{a-1} such that for every $T \in \binom{[ra-1]}{r}$ there exists $i \in [a - 1]$ such that $|T \cap E_i| \geq k$. Suppose that $r \geq (a - 1)(k - 1) + 1$ and let $n_0 = r(a - 1) + r - (a - 1)(k - 1) - 1$. Consider the hypergraph, \mathcal{H}_0 , on the vertex set $[n_0]$ whose edge set is $\binom{[n_0]}{r}$. It is not difficult to see that \mathcal{H}_0 has the required property. Notice that, when $k = 1$, the hypergraph \mathcal{H}_0 is the same as the hypergraph \mathcal{H}_1 . The discussion thus far leads us in the formulation of the following.

Conjecture 7. Let $\mathcal{H} = (V, \mathcal{E})$ be an r -uniform hypergraph on n vertices. Fix a positive integer $k < r$ and assume further that $\nu_k(\mathcal{H}) < a$, for some $a \geq 2$, as well as that $n \geq r \cdot a$. Set $n_0 = ra - (a - 1)(k - 1) - 1$. Then the number of edges in \mathcal{H} is at most

$$\max \left\{ \binom{n_0}{r}, |\mathcal{F}_i|; 0 \leq i \leq \frac{\lfloor \frac{n}{a-1} \rfloor - k}{2} \right\},$$

where \mathcal{F}_i is a (n, r, k, a, i) -Frankl family.

Notice that when $k = 1$ the previous conjecture reduces to Conjecture 1. Notice also that when $\nu_k(\mathcal{H}) = 1$ Conjecture 7 reduces to the k -intersection problem. Hence Conjecture 7 can be seen as a generalization of both Erdős' matching conjecture and the k -intersection problem. In this note we verify the validity of this conjecture for large values of n . The proof is by induction on $\nu_k(\mathcal{H})$, where the k -intersection problem is the base case. In particular, we obtain the following result.

Theorem 8. Let $\mathcal{H} = (V, \mathcal{E})$ be a r -uniform hypergraph on n vertices. Assume further that $\nu_k(\mathcal{H}) < a$, where $1 \leq k < r$, and that $n \geq 4r \binom{r}{k}^2 \cdot a$. Let \mathcal{F}_0 be a $(n, r, k, a, 0)$ -Frankl family. Then

$$|\mathcal{E}| \leq |\mathcal{F}_0|.$$

The remaining part of this note is organized as follows. In Section 2 we prove Theorem 6. The proof is probabilistic and is based on a coupling argument. In Section 3 we prove Theorem 8 by adapting Erdős' proof of Theorem 2 to our setting. Section 4 includes some concluding remarks.

2 Proof of Theorem 6

In this section we prove Theorem 6. The proof is divided into several lemmata and requires some extra piece of notation and definitions.

Clearly, we may assume that $k \geq 2$ and $a \geq 3$; otherwise there is nothing to prove. Let $\mathcal{H} = (V, \mathcal{E})$ be an r -uniform hypergraph. Let \mathbb{I} be a subset of V whose cardinality equals r which is chosen uniformly at random from the family $\binom{V}{r}$. Given a set \mathcal{T} consisting of $a - 1$ elements $T_1, \dots, T_{a-1} \in \binom{V}{k+2i}$, we say that \mathbb{I} captures \mathcal{T} if there exists $j \in [a - 1]$ such that $|\mathbb{I} \cap T_j| \geq k + i$.

Now let $\mathcal{T} = \{T_1, \dots, T_{a-1}\} \subset \binom{V}{k+2i}$ and suppose that the sets $T_j, j = 1, \dots, a - 1$ are not disjoint. This means that we can find two $(k + 2i)$ -sets, say T_1 and T_2 , such that $T_1 \cap T_2 \neq \emptyset$. Let $S = T_1 \cap T_2$ and set $s = |S|$. Now choose s vertices $v_1, \dots, v_s \in V \setminus \bigcup_{i=1}^{a-1} T_i$ (recall that $n \geq ra$) and set $R = \{v_1, \dots, v_s\}$. Now define the family $\mathcal{T}^* = \{T_1^*, T_2, \dots, T_{a-1}\}$, where $T_1^* = (T_1 \setminus S) \cup R$ and note that $T_1^* \in \binom{V}{k}$. Finally, fix a bijection $\phi : S \rightarrow R$, from S onto R .

We claim that the number of edges in $\mathcal{H}(\mathcal{T})$ is less than or equal to the number of edges in $\mathcal{H}(\mathcal{T}^*)$. To prove this claim, it is enough to show that the probability that \mathbb{I} captures \mathcal{T} is less than or equal to the probability that \mathbb{I} captures \mathcal{T}^* .

Now let A_1 be the event " \mathbb{I} captures \mathcal{T} and does not capture \mathcal{T}^* " and let A_2 be the event " \mathbb{I} captures \mathcal{T}^* and does not capture \mathcal{T} ".

Lemma 9. We have $\mathbb{P}[A_1] \leq \mathbb{P}[A_2]$.

Proof. Notice that the event A_1 happens if and only if $|\mathbb{I} \cap T_1| \geq k + i$, $|\mathbb{I} \cap T_1^*| < k + i$ and $|\mathbb{I} \cap T_j| < k + i$, for all $j \in \{2, \dots, a - 1\}$. Similarly, the event A_2 happens if and only if $|\mathbb{I} \cap T_1^*| \geq k + i$ and $|\mathbb{I} \cap T_j| < k + i$, for all $j \in \{1, 2, \dots, a - 1\}$. Now let \mathbb{I} be an outcome for which the event A_1 occurs. Set $\mathbb{I}_R = \mathbb{I} \cap R$, $\mathbb{I}_S = \mathbb{I} \cap S$ and define the set

$$J_{\mathbb{I}} = (\mathbb{I} \setminus (\mathbb{I}_S \cup \mathbb{I}_R)) \cup (\phi(\mathbb{I}_S) \cup \phi^{-1}(\mathbb{I}_R)).$$

Since $\mathbb{I}_R \cap \phi(\mathbb{I}_S) = \phi^{-1}(\mathbb{I}_R) \cap \mathbb{I}_S$, it follows that $J_{\mathbb{I}}$ preserves the size of \mathbb{I} . Notice also that $J_{\mathbb{I}}$ is an outcome for which the event A_2 occurs and that $\mathbb{I}_1 \neq \mathbb{I}_2$ implies $J_{\mathbb{I}_1} \neq J_{\mathbb{I}_2}$. This shows that for every outcome for which A_1 occurs we can associate, in an injective way, an outcome for which A_2 occurs. Since the all r -sets have the same probability of occurring, the result follows. \square

The previous lemma yields the following.

Lemma 10. *We have $\mathbb{P}[\mathbb{I} \text{ captures } \mathcal{T}] \leq \mathbb{P}[\mathbb{I} \text{ captures } \mathcal{T}^*]$.*

Proof. Notice that Lemma 9 yields

$$\begin{aligned} \mathbb{P}[\mathbb{I} \text{ captures } \mathcal{T}] &= \mathbb{P}[\mathbb{I} \text{ captures } \mathcal{T} \text{ and } \mathcal{T}^*] + \mathbb{P}[A_1] \\ &\leq \mathbb{P}[\mathbb{I} \text{ captures } \mathcal{T} \text{ and } \mathcal{T}^*] + \mathbb{P}[A_2] \\ &= \mathbb{P}[\mathbb{I} \text{ captures } \mathcal{T}^*]. \end{aligned} \quad \square$$

Hence Theorem 6 follows upon iterating the previous two lemmata until we get a family \mathcal{T} whose elements are pairwise disjoint.

In the next section we prove that $(n, r, k, a, 0)$ -Frankl families are the extremal hypergraphs of Problem 5 for large values of n . For the sake of completeness, let us also count the maximum number of edges in such a family.

Lemma 11. *Let the set V and parameters n, k, a be as in Theorem 6 and let $\mathcal{T} = \{T_1, \dots, T_{a-1}\}$ be a family consisting of $a - 1$ pairwise disjoint k -sets from V . Then the number of $E \in \binom{V}{r}$ that contain at least one element from \mathcal{T} is equal to*

$$g(n, r, k, a) := \sum_{j=1}^{\min\{a-1, \lfloor r/k \rfloor\}} (-1)^{j-1} \binom{a-1}{j} \binom{n-jk}{r-jk}.$$

Proof. For $j = 1, \dots, a-1$ let \mathcal{D}_j be the family consisting of all $E \in \binom{V}{r}$ such that $T_j \subset E$. The inclusion-exclusion principle yields that $g(n, r, k, a)$ is equal to

$$\left| \bigcup_{i=1}^{a-1} \mathcal{D}_i \right| = \sum_{\emptyset \neq J \subset [a-1]} (-1)^{|J|-1} \left| \bigcap_{j \in J} \mathcal{D}_j \right|.$$

Now notice that for $\emptyset \neq J \subset [a-1]$ we have

$$\left| \bigcap_{j \in J} \mathcal{D}_j \right| = \binom{n - |J| \cdot k}{r - |J| \cdot k}$$

and the result follows upon observing that $\left| \bigcap_{j \in J} \mathcal{D}_j \right| = 0$, when $|J| > \min\{a-1, \lfloor r/k \rfloor\}$. \square

3 Proof of Theorem 8

We imitate Erdős' proof of Theorem 2. Notice that it is enough to show that if $\mathcal{H} = (V, \mathcal{E})$ is an r -uniform hypergraph on $n > 4r \binom{r}{k}^2 \cdot a$ vertices such that $|\mathcal{E}| \geq 1 + g(n, r, k, a)$, where $g(n, r, k, a)$ is as in Lemma 11, then we have $\nu_k(\mathcal{H}) \geq a$. We prove this statement

by induction on a . When $a = 2$, then the result follows from the second statement of Theorem 4. Assuming it holds true for $a - 1 > 1$, we prove it for a .

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph on n vertices which satisfies $|\mathcal{E}| \geq 1 + g(n, r, k, a)$. For every $T \in \binom{V}{k}$ denote by $d(T)$ the number of edges $E \in \mathcal{E}$ such that $T \subset E$ and choose a k -set, say T_1 , for which $d(T_1)$ is maximum. We distinguish two cases.

Assume first that $d(T_1) < \frac{1+g(n,r,k,a)}{(a-1)\binom{r}{k}}$. Let E_1, \dots, E_l be a maximal k -matching of \mathcal{H} . Notice that this implies that for any $E \in \mathcal{E}$, there exists $j \in [l]$ such that $|E \cap E_j| \geq k$. We claim that $l \geq a$. To see this, notice that if $l < a$ then the edges E_1, \dots, E_l would contain at most $(a-1)\binom{r}{k}$ k -sets and therefore the total number of edges in \mathcal{H} satisfies $|\mathcal{E}| < 1 + g(n, r, k, a)$. Hence \mathcal{H} contains an edge E_{l+1} which satisfies $|E_{l+1} \cap E_j| \leq k-1$, for all $j \in [l]$ and contradicts the maximality of E_1, \dots, E_l . Therefore the claim follows and so does Theorem 8.

Assume now that $d(T_1) \geq \frac{1+g(n,r,k,a)}{(a-1)\binom{r}{k}}$. Let $\mathcal{H}(T_1)$ be the hypergraph whose vertex set is V and whose edge set, $\mathcal{E}(T_1)$, consists of all $E \in \mathcal{E}$ such that $T_1 \not\subset E$. Clearly, we have $|\mathcal{E}(T_1)| \geq 1 + g(n, r, k, a) - \binom{n-k}{r-k}$. Now notice that

$$g(n, r, k, a) - \binom{n-k}{r-k} = g(n, r, k, a-1)$$

and therefore $|\mathcal{E}(T_1)| \geq 1 + g(n, r, k, a-1)$. The induction hypothesis implies that there exist at least $a-1$ edges E_1, \dots, E_{a-1} in $\mathcal{H}(T_1)$ such that $|E_i \cap E_j| \leq k-1$, for all $i \neq j$. Now notice that the proof will follow once we show that there exists $E \in \mathcal{E}$ such that $T_1 \subset E$ which does not contain any of the $(a-1)\binom{r}{k}$ k -sets that are contained in E_1, \dots, E_{a-1} . Let T be a k -set which is contained in some $E_i, i = 1, \dots, a-1$. Notice that the number of r -sets which contain T_1 and T is at most $\binom{n-|T \cup T_1|}{r-|T \cup T_1|}$. Since $|T \cup T_1| \geq k+1$ it follows that the number of r -sets that contain T_1 and any of the k -sets contained in E_1, \dots, E_{a-1} is at most

$$(a-1) \binom{r}{k} \binom{n-k-1}{r-k-1}.$$

We now claim that

$$d(T_1) > (a-1) \binom{r}{k} \binom{n-k-1}{r-k-1}, \text{ when } n \geq 4r \binom{r}{k}^2 \cdot a,$$

which in turn implies that there exists $E \in \mathcal{E}$ such that $T_1 \subset E$ and for which $|E \cap E_i| \leq k-1$, for all $i \in [a-1]$. Hence $\nu_k(\mathcal{H}) \geq a$ and Theorem 8 follows. To prove the claim, note that the estimate

$$g(n, r, k, a) \geq (a-1) \binom{n-(a-1)k}{r-k},$$

combined with the assumption $d(T_1) \geq \frac{1+g(n,r,k,a)}{(a-1)\binom{r}{k}}$, implies that it is enough to show

$$\binom{n-(a-1)k}{r-k} \geq (a-1) \binom{r}{k}^2 \binom{n-k-1}{r-k-1}, \text{ for } n \geq 4r \binom{r}{k}^2 \cdot a.$$

The last inequality can be equivalently written as

$$(n-r) \prod_{i=1}^{(a-2)k-1} \left(1 - \frac{r-k}{n-k-i}\right) \geq (a-1)(r-k) \binom{r}{k}^2.$$

Now observe that, since $r > k$, we can estimate

$$(n-r) \prod_{i=1}^{(a-2)k-1} \left(1 - \frac{r-k}{n-k-i}\right) \geq (n-r) \left(1 - \frac{r}{n-ar}\right)^{ar}.$$

Note that the right hand side is an increasing function of n , for fixed r, k , which upon substituting $n = 4r \binom{r}{k}^2 \cdot a$ becomes

$$\left(4r \binom{r}{k}^2 a - r\right) \left(1 - \frac{1}{4 \binom{r}{k}^2 a - a}\right)^{ar} \geq \left(4r \binom{r}{k}^2 a - r\right) \cdot \left(1 - \frac{1}{4 \binom{r}{k}^2 a - a}\right)^{4 \binom{r}{k}^2 a - a},$$

since $4 \binom{r}{k}^2 a - a \geq ar$. Since the sequence $(1 - \frac{1}{m})^m$ is increasing and $4 \binom{r}{k}^2 a - a \geq 2$, we conclude

$$\left(4r \binom{r}{k}^2 a - r\right) \cdot \left(1 - \frac{1}{4 \binom{r}{k}^2 a - a}\right)^{4 \binom{r}{k}^2 a - a} \geq \left(4r \binom{r}{k}^2 a - r\right) \cdot \frac{1}{4}.$$

Now it is not difficult to verify that

$$\left(4r \binom{r}{k}^2 a - r\right) \cdot \frac{1}{4} \geq (a-1)(r-k) \binom{r}{k}^2$$

and the claim follows.

4 Concluding remarks

A well known technique that has been proven to be very fruitful in extremal set theory involves the notion of *shifting* (see [7, 13]).

Let $\mathcal{H} = (V, \mathcal{E})$ be an r -uniform hypergraph whose vertex set is indexed by the positive integers, i.e., $V = [n]$ for some positive integer n , and fix $1 \leq i < j \leq n$. The (i, j) -shift of \mathcal{H} , denoted $S_{i,j}(\mathcal{H})$, is the hypergraph with vertex set $[n]$ and with edges

$$S_{i,j}(E) = \begin{cases} E \setminus \{i\} \cup \{j\}, & \text{if } i \in E, j \notin E, E \setminus \{i\} \cup \{j\} \notin \mathcal{E} \\ E, & \text{otherwise.} \end{cases}$$

Clearly, for every $1 \leq i < j \leq n$, the hypergraph $S_{i,j}(\mathcal{H})$ is r -uniform and it contains the same number of edges as \mathcal{H} . Moreover, it can be shown that the k -matching number of \mathcal{H}

does not increase under an (i, j) -shift. It is known (see [7]) that if we keep on shifting a hypergraph then after a finite number of steps we end up with a *stable* hypergraph, that is, a hypergraph \mathcal{H} for which $S_{i,j}(\mathcal{H}) = \mathcal{H}$, for all $1 \leq i < j \leq n$.

The shifting technique is usually combined with induction on the number of vertices in a stable hypergraph and allows to obtain sharp estimates on the cardinality of the families $\mathcal{H}[n] = \{E \setminus \{n\} : n \in E \in \mathcal{E}\}$ and $\mathcal{H}(n) = \{E \in \mathcal{E} : n \notin E\}$ (see [7]). However, as can already be seen in the proof of Theorem 8, Problem 5 appears to depend on estimates on the cardinality of the families $\mathcal{H}[T] = \{E \setminus T : T \subset E \in \mathcal{E}\}$ and $\mathcal{H}(T) = \{E \in \mathcal{E} : T \not\subseteq E\}$, where $T \in \binom{V}{k}$, and we were not able to adapt standard shifting arguments in this setting. We believe that suitable generalizations of the notion of shifting may provide improvements upon the constant $4r \binom{r}{k}^2$ in Theorem 8 and we hope that we will be able to report on that matter in the future.

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