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On extensions of Baer and quasi-Baer modules

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Abstract. Let R be a ring, M_R a module, S a monoid, $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism and $R * S$ a skew monoid ring. Then $M[S] = \{m_1 g_1 + \cdots + m_n g_n \,|\, n \geq 1, m_i \in M \text{ and } g_i \in S \text{ for each } 1 \leq i \leq n\}$ is a module over $R * S$. A module M_R is *Baer* (resp. *quasi-Baer*) if the annihilator of every subset (resp. submodule) of M is generated by an idempotent of R. In this paper we impose S-compatibility assumption on the module M_R and prove: (1) M_R is quasi-Baer if and only if $M[s]_{R*S}$ is quasi-Baer, (2) M_R is Baer (resp. p.p) if and only if $M[S]_{R*S}$ is Baer (resp. p.p), where M_R is S-skew Armendariz, (3) M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[S]_{R*S}$, where M_R is S-skew quasi-Armendariz.

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1 Introduction and preliminaries

Throughout this paper R denotes an associative ring with identity and M_R is a right R-module. According to [16] a ring R is *Baer* if the right annihilator of every nonempty subset of R is generated by an idempotent. Quasi-Baer rings were initially introduced by Clark [10]. A ring R is *quasi-Baer* if the right annihilator of every right ideal of R generated by an idempotent. Another generalization of Baer rings is p.p.-rings. Recall that a ring R is called *right* (resp. *left*) *p.p* if right (left) annihilator of every element of R is generated by an idempotent. Birkenmeier et al. in [7] introduced principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or p.q.-Baer for short) if the right annihilator of a principal right ideal of R is generated by an idempotent.

In [1] Armendariz studied the behaver of a polynomial ring over Baer ring. He proved for a reduced ring R, $R[x]$ is Baer if and only if R is Baer [1, Theorem B]. Also, he provid an example to show that the "Armendariz" condition is not superfluous. Birkenmeier and park [9] extended this result to monoid ring.

We now introduce the definitions and notions used in this paper. If A and B are non-empty subsets of a monoid S, then an element $s_0 \in AB = \{ab : a \in A, b \in B\}$ is said to be a *unique product element* (u.p. element for short) in the product of AB if it is uniquely presented in the form of $s = ab$ where $a \in A$ and $b \in B$.

Recall that a monoid S is called *unique product monoid* (*u.p. monoid* for short) if for any two non-empty finite subsets $A, B \subseteq S$ there exist $a \in A$ and $b \in B$ such that ab is u.p. element in the product of AB. The class of u.p. monoids are quite large. For example this class includes the right or left ordered monoid and torsion free nilpotent groups. Every u.p. monoid S is cancellative [9, Lemma 1.1] and has no non-unit element of finite order.

Assume that R is a ring, S a monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. For each $g \in S$ we denote the image of g by ω_g (i.e., $\omega(g) = \omega_g$). Then all finite formal combinations $\sum_{i=1}^{n} a_i g_i$, with point-wise addition and multiplication induced by $(ag)(bh) = (a\omega_g(b))gh$ form a ring that is called *skew monoid ring* and it is denoted by $R*S$. The construction of skew monoid ring generalizes some classical ring construction such as skew polynomial rings, skew Laurent polynomial rings and monoid rings. Hence any result on skew monoid ring has its counterpart in each of the subclasses.

As a generalization of monoid rings, we introduce the notion of modules over skew monoid rings. For a module M_R, let $M[S] = \{m_1g_1 + \cdots + m_ng_n \mid n \geq 1,\ m_i \in M \text{ and } g_i \in S \text{ for each } 1 \leq i \leq n\}$. Then $M[S]$ is a right module over $R*S$ under the following scaler product operation: for $m(s) = m_1g_1 + \cdots + m_ng_n \in$

$M[S]$ and $f(s) = a_1h_1 + \cdots + a_mh_m \in R * S$, $m(s)f(s) := \sum_{i,j} m_i\omega_{g_i}(a_j)g_ih_j$.

For a nonempty subset X of M_R, let $ann_R(X) = \{r \in R \mid Xr = 0\}$.

The notion of reduced, Armendariz, Baer, p.p and quasi-Baer module introduced in [18] by Lee and Zhou. A module M_R is called *reduced* if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A module M_R is called *Baer* if, for any nonempty subset X of M, $ann_R(X) = eR$ where $e^2 = e \in R$. A module M_R is called *p.p* if for any element $m \in M$, $ann_R(m) = eR$ where $e^2 = e \in R$. A module M_R is called *quasi-Baer* if, for any right R-submodule X of M, $ann_R(X) = eR$ where $e^2 = e \in R$. Clearly, R is reduced (resp. Baer, right p.p, quasi-Baer) if and only if R_R is reduced (resp. Baer, right p.p, quasi-Baer). Lee and Zhou [18] proved that M_R is reduced if and only if $M[x]_{R[x]}$ is reduced. Various results of reduced rings were extended to modules in [18, 2].

Recall that from [6] an idempotent $e \in R$ is *left (resp. right) semicentral* in R if $exe = xe$ (resp. $exe = ex$) for all $x \in R$. Equivalently, $e = e^2 \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right R-module is an ideal, then the right annihilator of a right R-module is generated by a left semicemtral idempotent in a quasi-Baer module. We denote the set of all left (resp. right) semiccentral idempotents of R with $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$).

A module M_R is called *principally quasi-Baer* (or p.q.-Baer for short) if, for any $m \in M, ann_R(mR) = eR$ where $e^2 = e \in R$. Clearly R is a right p.q.-Baer if and only if R_R is p.q.-Baer module.

In this paper we introduce and study the concept of S-skew Armendariz modules as a generalization of S-Armendariz rings [19]. For a u.p. monoid S and monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$ we show that reduced module M_R is S-skew Armendariz. We investigate the quasi-Baer and related conditions on right $R * S$-module $M[S]$ for a u.p. monoid S and monoid homomorphism $\omega : S \longrightarrow \mathrm{Aut}(R)$. We impose S-compatibility assumption on the module M_R and prove: (1) M_R is quasi-Baer if and only if $M[s]_{R*S}$ is quasi-Baer, (2) M_R is Baer (resp. p.p) if and only if $M[S]_{R*S}$ is Baer (resp. p.p), when M_R is S-skew Armendariz, (3) M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[S]_{R*S}$, when M_R is S-skew quasi-Armendariz. Our results extend Armendariz [1, Theorem B], Groenewald [11, Theorem 2], Birkenmeier, Kim and Park [8, Theorem 1.2], Birkenmeier and Park [9, Theorem 1.2, Corollary 1.3].

2 S-skew Armendariz modules

Let R be a ring with an endomorphism σ. According to [4] for a module M_R and an endomorphism $\sigma : R \longrightarrow R$, we say that M_R is σ-*compatible* if for each $m \in M$ and $r \in R$, we have $mr = 0$ if and only if $m\sigma(r) = 0$. For more details on σ-compatible rings refer to [13, 14].

Definition 1 *Let* R *be a ring,* S *a monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. We say that a module* M_R *is* S*-compatible if* M_R *is* ω_g*-compatible for each* $g \in S$.

Notic that R is S-compatible if and only if R_R is S-compatible. Now we give some examples of S-compatible modules.

Example 1 [4, Example 4.4] *Let* R_0 *be a domain of characteristic zero, and* $R := R_0[t]$. *Define* $\sigma|_{R_0} = \mathrm{id}_{R_0}$ *and* $\sigma(t) = -t$. *Let* $M_R := R_0 \oplus R_0 \oplus R_0 \oplus \cdots$, *where* $t \in R$ *acts on* M_R *as follows: for* $(m_0, m_1, m_2, \ldots) \in M$, *we set* $(m_0, m_1, m_2, \ldots).t := (0, m_0k_0, m_1k_1, m_2k_2, \ldots)$ *where the* $k_i (i \in \mathbb{N})$ *are fixed nonzero integers. We show that* M *is* σ*-compatible. For this, it suffices to show that* $\mathrm{ann}(m) = 0$ *whenever* $0 \neq m \in M$. *Suppose that* $(a_0, a_1, a_2, \cdots)(b_rt^r + b_{r+1}t^{r+1} +$ *"higher terms"*$) = 0$, *where* $a_i, b_i \in R_0$ *for every* $i \in \mathbb{N}$ *and* $b_r \neq 0$. *First applying* t^r *to* $(a_0, a_1, a_2, \ldots)$ *gives*

$$(0, 0, \cdots, 0, a_0k_0k_1 \cdots k_{r-1}, a_1k_1k_2 \cdots k_r, \ldots)(b_r + b_{r+1}t + \text{" higher terms"}) = 0.$$

Upon computing this expression, we deduce that $a_0k_0k_1 \ldots k_{r-1}b_r = 0$. *Since the characteristic is zero,* R *is a domain, and* $k_0k_1 \ldots k_{r-1}b_r \neq 0$, *we deduce that* $a_0 = 0$. *Now, we may proceed inductively to show that all* $a_i = 0$. *From this calculation, we deduce that* M_R *is* σ*-compatible.*

Example 2 [14, Example 1.1] *Let* R_1 *be a ring,* D *a domain and* $R = T_n(R_1) \oplus D[y]$, *where* $T_n(R_1)$ *is upper* $n \times n$ *triangular matrix ring over* R_1. *Let* $\alpha : D[y] \longrightarrow D[y]$ *be a monomorphism which is not surjective. We define an endomorphism* $\overline{\alpha} : R \longrightarrow R$ *of* R *by* $\overline{\alpha}(A \oplus f(y)) = A \oplus \alpha(f(y))$ *for each* $A \in T_n(R_1)$ *and* $f(y) \in D[y]$. *In* [14, Example 1.1] *it is shown that* R *is an* $\overline{\alpha}$*-compatible.*

Example 3 *Let* R *be a ring and* σ_i *an endomorphism of* R *such that* R *be a* σ_i*-compatible for each* $1 \leq i \leq n$. *Let* S *be a monoid generated by* $\{x_1, x_2, \ldots, x_n\}$ *and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism such that* $\omega_{x_i^j} = \sigma_i^j$. *One can show that* R *is* S*-compatible and* $R * S \cong R[x_1, x_2, \ldots, x_n; \sigma_1, \sigma_2, \ldots, \sigma_n]$.

According to Lee and Zhou [18] a module M_R is *Armendariz* if, for elements $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$, $m(x)f(x) = 0$ implies $m_ia_j = 0$ for each $1 \leq i \leq n$, $1 \leq j \leq m$. In [21] Zhang and Chen, introduced the concept of a σ-skew Armendariz module and studied its properties. A module M_R is called σ-*skew Armendariz module*, if, whenever $m(x)f(x) = 0$ where $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x; \sigma]$, we have $m_i\sigma^i(b_j) = 0$ for each $0 \leq i \leq n$, $0 \leq j \leq m$. In [19], Liu introduced the concept of a S-Armendariz ring and studied its properties. In the following we introduce the concept of S-skew Armendariz module as a generalization of S-Armendariz rings.

Definition 2 *Let* R *be a ring,* S *a monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. We say that a module* M_R *is* S*-skew Armendariz module if, for elements* $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ *and* $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$, $m(s)f(s) = 0$ *implies* $m_i\omega_{g_i}(a_j) = 0$ *for each* $1 \leq i \leq n, 1 \leq j \leq t$. *In the case of* ω *is identity homomorphism, we say* M_R *is* S*-Armendariz module.*

Notice that for a ring R and monid S with monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$, R is S-skew Armendariz (resp. S-Armendariz) if and only if R_R is S-skew Armendariz (resp. S-Armendariz).

Theorem 1 *Let* R *be a ring,* S *a monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. Then* M_R *is* S*-skew Armendariz if and only if for every elements* $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ *and* $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$, $m(s)f(s) = 0$ *implies* $m_{i_1}\omega_{g_{i_1}}(a_j) = 0$ *for each* $1 \leq j \leq t$ *and some* $1 \leq i_1 \leq t$.

Proof. The forward direction is clear. For the converse, suppose that $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ with $m(s)f(s) = 0$. Then there exists $1 \leq i_1 \leq n$ such that $m_{i_1}\omega_{g_{i_1}}(a_j) = 0$ for each $1 \leq j \leq t$. Without loss of generality we can assume that $i_1 = 1$. Thus $0 = m(s)f(s) = (m_2g_2 + \cdots + m_ng_n)f(s)$. Then by induction on n we can conclude that $m_i\omega_{g_i}(a_j) = 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq t$. Hence M_R is S-skew Armendariz. □

If S is a monoid generated by $\{x\}$ and $\omega : S \longrightarrow \mathrm{End}(R)$ such that $\omega_{x^i} = \sigma^i$ for an endomorphism σ of R, then the skew monoid ring $R * S$ is isomorphic to skew polynomial ring $R[x; \sigma]$ and $M[S]$ is isomorphic to $M[x]$. Thus we have the following equivalent condition for a module to be σ-skew Armendariz.

Corollary 1 *Let* M_R *be a module and* σ *an endomorphism of* R. *Then* M_R *is* σ*-skew Armendariz if and only if for every polynomials* $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ *and* $f(x) = a_0 + a_1x + \cdots + a_tx^t \in R[x; \sigma]$, $m(x)f(x) = 0$ *implies* $m_{i_1}\sigma^{i_1}(a_j) = 0$ *for each* $0 \leq j \leq t$ *and some* $0 \leq i_1 \leq n$.

Corollary 2 *Let R be a ring and σ an endomorphism of R. Then R is σ-skew Armendariz if and only if for every polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma]$, $f(x)g(x) = 0$ implies $a_{i_0}\sigma^{i_0}(b_j) = 0$ for each $0 \le j \le m$ and some $0 \le i_0 \le n$.*

Recall that a module M_R is reduced if, for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$.

Lemma 1 *The following are equivalent for a module M_R.*

(i) *M_R is reduced and S-compatible.*

(ii) *The following conditions hold for any $m \in M, a \in R$ and $g \in S$,*

 (a) *$ma = 0$ implies $mRa = 0$.*

 (b) *$ma = 0$ if and only if $m\omega_g(a) = 0$.*

 (c) *$ma^2 = 0$ implies $ma = 0$.*

Proof. The proof is straightforward. □

For an element $f(s) = a_1g_1 + \cdots + a_ng_n \in R * S$ with $a_i \ne 0$ for each i, we say that length $(f(s)) = n$ and denote it by $\ell(f(s))$. Similarly, we can define $\ell(m(s)) = t$ for an element $m(s) = m_1h_1 + \cdots + m_th_t \in M[S]$.

Proposition 1 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Then S-compatible reduced module M_R is S-skew Armendariz.*

Proof. Assume that $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ with $m(s)f(s) = 0$. We proceed by induction on $\ell(m(s)) + \ell(f(s)) = n + t$. If $\ell(m(s)) = 1$ or $\ell(f(s)) = 1$, then the result is clear Since u.p. monoids are cancellative by [6, Lemma 1.1]. From $m(s)f(s) = 0$ there exist $1 \le i \le n, 1 \le j \le t$ such that g_ih_j is u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_t\}$ of S. Without loss of generality we can assume that $i = j = 1$. Thus $m_1\omega_{g_1}(a_1) = 0$ and so $m_1a_1 = 0$ since M_R is S-compatible. Therefore $0 = m(s)f(s)a_1 = (m_1g_1 + \cdots + m_ng_n)(a_1\omega_{h_1}(a_1)h_1 + \cdots + a_t\omega_{h_t}(a_1)h_t)$. By using of Lemma 1, from $m_1a_1 = 0$ we have $m_1\omega_{g_1}(a_j\omega_{h_j}(a_1)) = 0$ for each $1 \le j \le t$ since M_R is reduced and S-Compatible. Thus $0 = m(s)f(s)a_1 = (m_2g_2 + \cdots + m_ng_n)f(s)a_1 = m'(s)(f(s)a_1)$. Since $\ell(m'(s)) + \ell(f(s)a_1) < n + t$ satisfying $m'(s)f(s)a_1 = 0$, by induction hypothesise $m_i\omega_{g_i}(a_j\omega_{h_j}(a_1)) = 0$ which implies that $m_ia_ja_1 = 0$ for each $2 \le i \le n, 1 \le j \le t$, since M_R is S-compatible. Thus $m_ia_1^2 = 0$

and so $m_i a_1 = 0$ for each $2 \leq i \leq n$, by Lemma 1. Hence $0 = m(s)f(s) = m(s)(a_2h_2 + \cdots + a_th_t)$. Then by induction $m_i\omega_{g_i}(a_j) = 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq t$. Therefore M_R is S-skew Armendariz. □

If ω is identity homomorphism (i.e. $\omega_g = id_R$ the identity homomorphism of R for each $g \in S$) we deduce the following corollary.

Corollary 3 *Let M_R be a reduced and S a u.p. monoid. Then M_R is S-Armendariz.*

Corollary 4 [2, Theorem 2.19] *Every reduced module is Armendariz.*

Corollary 5 *Let R be a reduced ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Then R is S-skew Armendariz.*

Proposition 2 *Let S be a monoid and M_R a S-skew Armendariz module. If $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f_i(s) = a_1^ih_1^i + \cdots + a_{t_i}^ih_{t_i}^i \in R * S$ for $1 \leq i \leq k$ are such that $m(s)f_1(s)\cdots f_k(s) = 0$, then*

$$m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a_{i_2}^2)\cdots\omega_{g_j}\omega_{h_{i_1}^1}\ldots\omega_{h_{i_{k-1}}^{k-1}}(a_{i_k}^k) = 0$$

for each $1 \leq j \leq n$ and $1 \leq i_r \leq t_i, 1 \leq r \leq k$.

Proof. Suppose $m(s)f_1(s)\cdots f_k(s) = 0$. Then from $m(s)(f_1(s)\cdots f_k(s)) = 0$ we have $m_j\omega_{g_j}(a) = 0$ for each $1 \leq j \leq n$ and each coefficient a of $f_1(s)f_2(s)\cdots f_k(s)$, since M_R is S-skew Armendariz and S-compatible. Thus $(m_jg_jf_1(s))f_2(s)\cdots f_k(s) = 0$ for each $1 \leq j \leq n$. Thus $m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a') = 0$ for each $1 \leq j \leq n$, $1 \leq i_1 \leq t_1$ and each coefficient a' of $f_3(s)\cdots f_k(s)$. By continuing this manner, we see that $m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a_{i_2}^2)\cdots\omega_{g_j}\omega_{h_{i_1}^1}\ldots\omega_{h_{i_{k-1}}^{k-1}}(a_{i_k}^k) = 0$ for each $1 \leq j \leq n$ and $1 \leq i_r \leq t_i, 1 \leq r \leq k$. □

As a consequence of Propositions 1 and 2 we have the following result.

Corollary 6 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Let M_R be a S-compatible reduced module. If $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f_i(s) = a_1^ih_1^i + \cdots + a_{t_i}^i \in R * S$ for $1 \leq i \leq k$ are such that $m(s)f_1(s)\cdots f_k(s) = 0$, then*

$$m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a_{i_2}^2)\cdots\omega_{g_j}\omega_{h_{i_1}^1}\ldots\omega_{h_{i_{k-1}}^{k-1}}(a_{i_k}^k) = 0$$

for each $1 \leq j \leq n$ and $1 \leq i_r \leq t_i, 1 \leq r \leq k$.

It is proved in [18, Theorem 1.6] M_R is reduced if and only if $M[x]_{R[x]}$ is reduced. In the following we extend this result to $M[S]_{R*S}$.

Proposition 3 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Then module M_R is reduced and S-compatible if and only if $M[S]_{R*S}$ is reduced.*

Proof. Assume that M_R is reduced and $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$, $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ with $m(s)f(s) = 0$. Let $g(s) = b_1k_1 + \cdots + b_mk_m \in R * S$ and $k(s) = n_1s_1 + \cdots + n_ps_p \in M[S]$ such that $m(s)g(s) = k(s)f(s) \in m(s)(R * S) \cap M[S]f(s)$. From $m(s)f(s) = 0$ we have $m_i\omega_{g_i}(a_j) = 0 = m_ia_j$ for each $1 \leq i \leq n, 1 \leq j \leq t$, by Proposition 1 and S-compatibility assumption on M_R. Then by Lemma 1 we have $m_ira_j = 0$ for each $r \in R$ which implies that $0 = m(s)g(s)f(s) = k(s)f^2(s)$. Therefore $n_ia_ja_l = 0$ for each $1 \leq i \leq p$ and $1 \leq j, \ell \leq t$ by Proposition 2. Thus $n_ia_j^2 = 0$ and so $n_ia_j = 0$ for each $1 \leq i \leq p$ and $1 \leq j \leq t$ by Lemma 1. Therefore $k(s)f(s) = 0$ which implies that $m(s)(R * S) \cap M[S]f(s) = 0$ and hence $M[S]_{R*S}$ is reduced.

Conversely, assume that $M[S]_{R*S}$ is reduced and $m \in M, r \in R$ with $mr = 0$. Also assume that $n \in M, a \in R$ such that $ma = nr \in Mr \cap mR$. Put $m(s) = mg$ and $k(s) = nh$ for some $g, h \in S$. Thus $m(s)a = k(s)r \in M[S]r \cap m(s)(R * S)$. Since $M[S]_{R*S}$ is reduced $M[S]r \cap m(s)(R * S) = 0$ which implies that $ma = nr = 0$. Hence M_R is reduced. Now, assume that $mr = 0$ for some $m \in M$ and $r \in R$. For each $g \in S$ we have $mgr = m\omega_g(r)g \in M[S]r \cap m(R*S)$. Since $M[S]_{R*S}$ is reduced, $M[S]r \cap m(R * S) = 0$. Thus $m\omega_g(r) = 0$. Clearly, if $m\omega_g(r) = 0$ for each $g \in S$ we have $mr = 0$. Therefore M_R is S-compatible. □

Corollary 7 *Let R be a ring and σ an endomorphism of R. Then M_R is reduced and σ-compatible if and only if $M[x]_{R[x;\sigma]}$ is reduced.*

Corollary 8 *Let R be a ring and σ an endomorphism of R. Then R is reduced and σ-compatible if and only if $R[x; \sigma]$ is reduced.*

3 Extensions of Baer and quasi-Baer modules

In this section we study on the relationship between the Baerness and p.p-property of a module M_R and right $R * S$-module $M[S]$.

According to [5] a module M_R is called *quasi-Armendariz* if whenever $m(x)R[x]f(x) = 0$ for $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$, then $m_iRa_j = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Let S be a monoid. According to [12] a ring R is called S-*quasi Armendariz* if for each two elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[S]$ satisfy $\alpha R[s]\beta = 0$, implies that $a_iRb_j = 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$.

Definition 3 *Let* R *be a ring,* S *a monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. A module* M_R *is called* S*-skew quasi-Armendariz, if for any* $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ *and* $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ *satisfy* $m(s)(R*S)f(s) = 0$ *implies that* $m_ig_iRga_jh_j = 0$ *for each* $1 \leq i \leq n$, $1 \leq j \leq t$ *and* $g \in S$.

Clearly a ring R is S-skew quasi-Armendariz if and only if R_R is S-skew quasi-Armendariz.

Birkenmeier and Park in [9, Theorem 1.2] proved that for a u.p. monoid S the monoid ring $R[S]$ is quasi-Baer (resp. right p.q.-Baer) if and only if R is quasi-Baer (resp. right p.q.-Baer). In the following we extend these results to $M[S]$ as a right $R * S$-module.

Theorem 2 *Let* R *be a ring,* S *a u.p. monoid,* $\omega : S \longrightarrow \mathrm{Aut}(R)$ *a monoid homomorphism. If* M_R *is* S*-compatible, then we have the following:*

(i) M_R *is right p.q.-Baer if and only if* $M[S]_{R*S}$ *is right p.q.-Baer.*

(ii) M_R *is quasi-Baer if and only if* $M[S]_{R*S}$ *is quasi-Baer.*

In this case, M_R *is* S*-skew quasi-Armendariz.*

Proof. (i) Assume that R is right p.q.-Baer. Let $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$. There exists $e_i \in S_\ell(R)$ such that $\mathrm{ann}_R(m_iR) = e_iR$ for $1 \leq i \leq n$. Then $e = e_1e_2 \cdots e_n \in S_\ell(R)$ and $eR = \bigcap_{i=1}^n \mathrm{ann}_R(m_iR)$. Since every compatible automorphism is idempotent stabilizing by [3, Theorem 2.14] we have $e(R*S) \subseteq \mathrm{ann}_{R*S}(m(s)R*S)$. Note that $\mathrm{ann}_{R*S}(m(s)R*S) \subseteq \mathrm{ann}_{R*S}(m(s)R)$. Now we show that $\mathrm{ann}_{R*S}(m(s)R) \subseteq e(R*S)$. Let $g(s) = b_1h_1 + \cdots + b_mh_m \in \mathrm{ann}_{R*S}(m(s)R)$. Then $m(s)Rg(s) = 0$. We proceed by induction on n to show that $g(s) \in e(R * S)$. Let $n = 1$. Then $m_1g_1R(b_1h_1 + \cdots + b_th_t) = 0$. Thus $m_1g_1Rb_jh_j = 0$ for each $1 \leq j \leq t$, since S is cancellative, by [9, Lemma 1.1]. Since ω_{g_1} is automorphism $m_1R\omega_{g_1}(b_j) = 0$ and so $\omega_{g_1}(b_j) \in \mathrm{ann}_R(m_1R) = e_1R$ for each $1 \leq j \leq t$. Thus $\omega_{g_1}(b_j) = e_1\omega_{g_1}(b_j)$ and so $b_j = e_1b_j$ for each $1 \leq j \leq t$, since ω_{g_1} is a compatible automorphism of R. Therefore $b_j \in e_1R = eR$. Hence $g(s) = eg(s) \in e(R * S)$, as desired. Now assume that

$$(*) \qquad (m_1g_1 + \cdots + m_ng_n)R(b_1h_1 + \cdots + b_th_t) = 0.$$

Since S is u.p. monoid there exist $1 \leq i \leq n, 1 \leq j \leq t$ such that g_ih_j is u.p. element in the product of two subsets $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_t\}$ of S. Without loss of generality we can assume that $i = n, j = t$. Thus $m_ng_nRb_th_t = 0$. That is $\omega_{g_n}(b_t) \in ann_R(m_nR) = e_nR$ and $\omega_{g_n}(b_t) = e_n\omega_{g_n}(b_t)$. Since ω_{g_n} is a compatible automorphism of R, $b_t = e_nb_t$ and $b_t \in e_nR$. Replacing R by Re_n in the equation $(*)$ we have $(m_1g_1 + \cdots + m_{n-1}g_{n-1})R(e_nb_1h_1 + \cdots + e_nb_th_t) = 0$. By induction on n we have $e_nb_j \in e_1R \cap e_2R \cap \cdots \cap e_{n-1}R$ for each $1 \leq j \leq t$. In particular, $b_t \in e_1R \cap \cdots \cap e_{n-1}R$. Therefore $b_t = e_nb_t \in e_1R \cap \cdots \cap e_nR = eR = \bigcap_{i=1}^{n} ann_R(m_iR)$. Since ω_{g_i} is a compatible automorphism of R for each $1 \leq i \leq n$ we have

$$(**) \qquad (m_1g_1 + \cdots + m_ng_n)R(b_1h_1 + \cdots + b_{t-1}h_{t-1}) = 0.$$

Since S is u.p. monoid there exist $1 \leq i \leq n, 1 \leq j \leq t-1$ such that g_ih_j is u.p. element in the product of two subsets $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_{t-1}\}$ of S. Without loss of generality we can assume that $i = n, j = t-1$. Thus $m_ng_nRb_{t-1}h_{t-1} = 0$ which implies that $\omega_{g_n}(b_t) \in ann(m_nR) = e_nR$ and $\omega_{g_n}(b_{t-1}) = e_n\omega_{g_n}(b_{t-1})$. Therefore $b_{t-1} = e_nb_{t-1}$, since ω_{g_n} is an idempotent stabilizing automorphism of R. Replacing R by Re_n in the equation $(**)$ we have $(m_1g_1+\cdots+m_{n-1}g_{n-1})Re_n(b_1h_1+\cdots+b_{t-1}h_{t-1}) = 0$. Then by induction on n we can conclude that $e_nb_j \in ann_R(m_1R) \cap \cdots \cap ann_R(m_{n-1}R)$ for each $1 \leq j \leq t-1$ and hence $b_{t-1} = e_nb_{t-1} \in \cap_{i=1}^{n} ann_R(m_iR) = eR$. Therefore from the equation $(**)$ we have $0 = (m_1g_1 + \cdots + m_ng_n)R(b_1h_1 + \cdots + b_{t-2}h_{t-2})$. By continuing this process we can conclude that $b_j \in \cap_{i=1}^{n} ann_R(m_iR) = eR$ for each $1 \leq j \leq t$ which implies that $g(s) = eg(s)$. Thus $ann_R(m(s)R) \subseteq e(R*S)$. So we have $ann_{R*S}(m(s)(R*S)) \subseteq ann_R(m(s)R) \subseteq e(R*S)$. Hence $ann_{R*S}(m(s)R*S) = e(R*S)$. Therefore $M[S]_{R*S}$ is p.q.-Baer.

Conversely assume that $M[S]_{R*S}$ is p.q.-Baer. Take $m \in M$. Then $ann_{R*S}(m(R*S)) = e(s)(R*S)$ for some idempotent $e(s) = e_1s_1 + \cdots + e_ns_n$ in $R*S$. Let $a \in ann_R(mR)$. Since M_R is S-compatible, $ann_R(mR) \subseteq ann_{R*S}(m(R*S)) = e(s)(R*S)$. Therefore $a = e(s)a = (e_1g_1 + \cdots + e_ng_n)a$. Thus there exist $1 \leq i_0 \leq n$ such that $a = e_{i_0}\omega_{g_{i_0}}(a)$ and so $ann_R(mR) \subseteq e_{i_0}R$. Since $e(s) \in ann_{R*S}(m(R*S))$ then $0 = mRe(s) = mR(e_1s_1 + \cdots + e_ng_n)$. Since S is cancellative $mRe_i = 0$ for each $1 \leq i \leq n$. Thus $e_{i_0} \in ann_R(mR)$ and hence $ann_R(mR) = e_{i_0}R$. Also, e_{i_0} is idempotent, since $e_{i_0} \in ann_R(mR), a = e_{i_0}\omega_{g_{i_0}}(a)$ for each $a \in ann_R(mR)$ and $\omega_{g_{i_0}}$ is idempotent stabilizing, we have $e_{i_0} = e_{i_0}\omega_{g_{i_0}}(e_{i_0}) = e_{i_0}^2$. Therefore R is p.q.-Baer.

(ii) Assume that M_R is quasi-Baer. First we show that M_R is S-skew quasi-Armendariz. Suppose that $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f(s) =$

$a_1h_1 + \cdots + a_th_t \in R * S$ such that $m(s)(R * S)f(s) = 0$. Thus $m(s)rgf(s) = 0$ for each $r \in R, g \in S$. We proceed by induction on $\ell(m(s)) + \ell(f(s)) = n + t$. If $\ell(m(s)) = 1$, then $m_1g_1rg(a_1h_1 + \cdots + a_th_t) = 0$. Since S is cancellative $m_1g_1rga_jh_j = 0$, as desired. Also if $\ell(f(s)) = 1$ the result is clear. From

$$(*) \qquad (m_1g_1 + \cdots + m_ng_n)rg(a_1h_1 + \cdots + a_th_t) = 0$$

there exist $1 \le i \le n, 1 \le j \le t$ such that g_ih_j is u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_t\}$ of S. Without loss of generality we can assume that $i = n, j = t$. Then $m_ng_nrga_th_t = 0$ and so $m_n\omega_{g_n}(r)\omega_{g_n}\omega_g(a_t) = 0 = m_nr'\omega_{g_n}\omega_g(a_t)$. Thus $\omega_{g_n}\omega_g(a_t) \in ann_R(m_nR) = eR$ such that $e^2 = e \in R$ and so $\omega_{g_n}\omega_g(a_t) = e\omega_{g_n}\omega_g(a_t)$. Replacing rg by reg in the equation $(*)$ we have

$$(m_1g_1 + \cdots + m_{n-1}g_{n-1})reg(a_1h_1 + \cdots + a_th_t) = 0$$

since ω_g is idempotent stabilizing by [3, Theorem 2.14]. Then by induction we can conclude that $m_ig_irega_jh_j = 0$ for $1 \le i \le n-1, 1 \le j \le t$. Thus $m_ig_irega_th_t = 0$ and so $m_ig_ire\omega_g(a_t)gh_t = 0$ for each $1 \le i \le n-1$. Since $\omega_{g_n}\omega_g(a_t) = e\omega_{g_n}\omega_g(a_t)$ and ω_{g_n} is a compatible automorphism of R, $\omega_g(a_t) = e\omega_g(a_t)$. Thus $0 = m_ig_ire\omega_g(a_t)gh_t = m_ig_ir\omega_g(a_t)gh_t$ for each $1 \le i \le n-1$. On the other hand $m_ng_nrega_th_t = 0$ and hence $m_ig_irga_th_t = 0$ for each $1 \le i \le n$. Thus $0 = m(s)rgf(s) = (m_1g_1 + \cdots + m_ng_n)rg(a_1h_1 + \cdots + a_{t-1}h_{t-1})$. Then by induction hypothesis $m_ig_irga_jh_j = 0$ for each $1 \le i \le n, 1 \le j \le t-1$. Therefore $m_ig_iRga_jh_j = 0$ for each $1 \le i \le n, 1 \le j \le t$. Hence M_R is S-skew quasi-Armendariz. Let V be a submodule of $M[S]$. Let U be a right R-submodule of M generated by all coefficients of elements of V. Since M_R is quasi-Baer $ann_R(U) = eR$ for some $e^2 = e \in R$. Thus $e(R * S) \subseteq ann_{R*S}(V)$, since ω_s is compatible automorphism for each $s \in S$. Suppose that $g(s) = b_1h_1 + \cdots + b_th_t \in ann_{R*S}(V)$. Thus for each $m(s) = m_1g_1 + \cdots + m_ng_n \in V$, $m(s)(R * S)g(s) = 0$ and hence $m_ig_iRgb_jh_j = 0$ for each $1 \le i \le n$, $1 \le j \le t$ since M_R is S-skew quasi-Armendariz. Therefore $\omega_{g_i}\omega_g(b_j) \in ann_R(U) = eR$ which implies that $\omega_{g_i}\omega_g(b_j) = e\omega_{g_i}\omega_g(b_j)$ for each $1 \le i \le n$, $1 \le j \le t$. Since ω_s is compatible automorphism of R for each $s \in S$, $b_j = eb_j$ for each $1 \le j \le t$. That is $g(s) \in e(R * S)$ and so $ann_{R*S}(V) \subseteq e(R * S)$. Hence $M[S]_{R*S}$ is quasi-Baer.

Conversely, assume that $M[S]_{R*S}$ is quasi-Bear and U is a right R-submodule of M_R. Then as in the proof of the sufficiently of (i), one can show that $ann_R(U)$ is generated as a right R-submodule, by an idempotent of R.Therefore M is quasi-Baer. $\square$

Now we obtain the following results as a corollary of Theorem 2.

Corollary 9 *Let* R *be a ring,* S *a u.p. monoid,* $\omega : S \longrightarrow \mathrm{Aut}(R)$ *a monoid homomorphism and* M_R *is a* S*-compatible module. Then we have the following:*

(i) M_R *is a reduced p.p.- module if and only if* $M[S]_{R*S}$ *is a reduced p.p.- module.*

(ii) M_R *is a reduced Baer module if and only if* $M[S]_{R*S}$ *is a reduced Baer module.*

Proof. (i) Clearly reduced p.p.- modules are p.q.-Baer. Then the result follows from Theorem 2 and Proposition 3.

(ii) The result follows from Theorem 2 and the fact that a reduced quasi-Baer module is Baer. □

Corollary 10 *Let* R *be a ring and* S *a u.p. monoid. Then we have the following:*

(i) [6, Theorem 1.2] R *is quasi-Baer (resp. right p.q.-Baer) if and only if* $R[S]$ *is quasi-Baer (resp. right p.q.-Baer).*

(ii) [6, Corollary 1.3] R *is reduced Baer (resp. p.p.- ring) if and only if* $R[S]$ *is a reduced Baer (resp. p.p.- ring).*

Corollary 11 *Let* M_R *be a module. Then the following are equivalent:*

(i) M_R *is quasi-Baer (resp. p.q.-Baer).*

(ii) $M[x]_{R[x]}$ *is quasi-Baer (resp. p.q.-Baer).*

(iii) $M[x, x^{-1}]_{R[x,x^{-1}]}$ *is quasi-Baer (resp. p.q.-Baer).*

Corollary 12 *Let* R *be a* σ*-compatible ring for an automorphism* σ *of* R*. Then the following are equivalent:*

(i) R *is quasi-Baer (resp. p.q.-Baer).*

(ii) $R[x; \sigma]$ *is quasi-Baer (resp. p.q.-Baer).*

(iii) $R[x, x^{-1}; \sigma]$ *is quasi-Baer (resp. p.q.-Baer).*

(iv) $R[x]$ *is quasi-Baer (resp. p.q.-Baer).*

(v) $R[x, x^{-1}]$ *is quasi-Baer (resp. p.q.-Baer).*

Birkenmeier et al. [6, Example 1.5] showed that the "u.p. monoid" condition on S in Theorem 2 is not superfluous.

The next example shows that the "S-compatibility" assumption on R_R in Theorem 2 is not superfluous.

Example 4 [15, Example 2] *Let K be a field, $A = K[s,t]$ a commutative polynomial ring, and consider the ring $R = A/(st)$. Then R is reduced. Let $\bar{s} = s + (st)$ and $\bar{t} = t + (st)$ in $R = A/(st)$. Define an automorphism σ of R by $\sigma(\bar{s}) = \bar{t}$ and $\sigma(\bar{t}) = \bar{s}$. Hirano in [15] showed that $R[x;\sigma]$ is quasi-Baer but R is not quasi-Baer. Since $\sigma(\bar{s}\bar{t}) = 0$ but $\bar{s}\sigma(\bar{t}) = \bar{s}^2 \neq 0$ (since R is reduced), hence σ is not compatible. Therefore the "compatibility" assumption on σ is not superfluous.*

Theorem 3 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. If M_R is a S-compatible and S-skew Armendariz module, then M_R is Baer if and only if $M[S]_{R*S}$ is Baer.*

Proof. The proof is similar to that of Theorem 2. □

Corollary 13 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. Let M_R is S-compatible reduced module. Then M_R is Baer if and only if $M[S]_{R*S}$ is Baer.*

Proof. This follows from Proposition 1 and Theorem 3. □

Corollary 14 *Let R be a σ-compatible ring for an automorphism σ of R. If R is σ-skew Armendariz, then the following are equivalent:*

(i) *R is Baer.*

(ii) *$R[x;\sigma]$ is Baer .*

(iii) *$R[x,x^{-1};\sigma]$ is Baer.*

(iv) *$R[x]$ is Baer.*

(v) *$R[x,x^{-1}]$ is Baer.*

Theorem 4 *Let R be a ring, S a monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. If M_R is S-compatible and S-skew quasi-Armendariz, then M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[S]_{R*S}$.*

Proof. Assume that M_R satisfies the ascending chain condition on annihilator of submodules. Let $V_1 \subseteq V_2 \subseteq \ldots$ be a chain of annihilator of submodules of $M[S]_{R*S}$. Then there exist submodules K_i of $M[S]_{R*S}$ such that $ann_{R*S}(K_i) = V_i$ and $K_i \supseteq K_{i+1}$ for each $i \geq 1$. Let U_i be a submodule of M generated by all coefficients of elements of K_i. Clearly $U_1 \supseteq U_2 \supseteq \cdots$. Then $ann_R(U_1) \subseteq ann_R(U_2) \subseteq \cdots$ is a chain of annihilator of submodules of M_R. Since M_R satisfies the ascending chain condition on annihilator of submodules there exists $n \geq 1$ such that $ann_R(U_n) = ann_R(U_i)$ for all $i \geq n$. We show that $ann_{R*S}(K_n) = ann_{R*S}(K_i)$ for all $i \geq n$. Let $f(s) = a_1h_1 + a_2h_2 + \cdots + a_th_t \in ann_{R*S}(K_i)$. For each $m(s) = m_1g_1 + \cdots + m_ng_n \in K_i$, $m(s)(R * S)f(s) = 0$. Therefore $m_ig_iRga_ph_p = 0$ for each $1 \leq j \leq n, 1 \leq p \leq t$ since $M[S]$ is S-skew quasi-Armendariz. Thus $m_jR\omega_{g_j}\omega_g(a_p) = 0$ and so $m_jRa_p = 0$, since M_R is S-compatible. Therefore $a_p \in ann(U_i) = ann(U_n)$ for each $1 \leq p \leq t$ and hence $f(s) \in ann_{R*S}(K_n)$. Thus $ann_{R*S}(K_n) = ann_{R*S}(K_i)$. Now assume that $M[S]_{R*S}$ satisfies the ascending chain condition on annihilator of submodules. Let $U_1 \subseteq U_2 \subseteq \cdots$ be a chain of annihilator of submodules of M_R. Then there exist submodules M_i of M such that $ann_R(M_i) = U_i$. Thus $M_1 \supseteq M_2 \supseteq \cdots$. Hence $M_i[S]$ is a submodule of $M[S]_{R*S}$, $M_i[S] \supseteq M_{i+1}[S]$ and $ann_{R*S}(M_i[S]) \subseteq ann_{R*S}(M_{i+1}[S])$ for all $i \geq 1$. Thus $ann_{R*S}(M_1[S]) \subseteq ann_{R*S}(M_2[S]) \subseteq \cdots$ is a chain of annihilator of submodules of $M[S]$ and so there exists $n \geq 1$ such that $ann_{R*S}(M_n[S]) = ann_{R*S}(M_i[S])$. We show that $ann_R(M_n) = ann_R(M_i)$ for $i \geq n$. Assume that $r \in ann_R(M_i)$. Since M is S-compatible, $r \in ann_{R*S}(M_i[S]) = ann_{R*S}(M_n[S])$ for all $i \geq n$. For each $m(s) \in M_n[S]$ and $r \in R$, $m(s)(R * S)r = 0$ which implies that $m_pg_pRgr = 0$ for each $1 \leq p \leq t, g \in S$, since M_R is S-skew quasi-Armendariz. Thus $m_pR\omega_{g_p}\omega_g(r) = 0 = m_pRr$, since M_R is S-compatible, and so $r \in ann_R(M_n)$. Therefore $ann_R(M_i) = ann_R(M_n)$. ☐

Corollary 15 *Let* M_R *be a module and* σ *a compatible automorphism of* R. *The following are equivalent:*

(i) M_R *satisfies the ascending chain condition on annihilator of submodules.*

(ii) $M[x]_{R[x;\sigma]}$ *satisfies the ascending chain condition on annihilator of submodules.*

(iii) $M[x, x^{-1}]_{R[x,x^{-1};\sigma]}$ *satisfies the ascending chain condition on annihilator of submodules.*

References

[1] E. P. Armendariz, A note on extensions of Baer and p.p.-rings, *J. Austral. Math. Soc.*, **18** (1974), 470–473.

[2] A. Alhevaz and A. Moussavi, On skew Armendariz and skew quasi-Armendariz modules, *Bull. Iranian. Math. Soc.*, **38** (2012), 55–84.

[3] A. Alhevaz, D. Kiani, McCoy property of skew Laurent polynomials and power series rings, *J. Algebra Appl.*, **13** (2014), 1350083.

[4] S. Annin, Associated Primes Over ore extension Rings, *J. Algebra Appl.*, **3** (2004), 193–205.

[5] M. Baser, On Armendariz and quasi-Armendariz modules, *Not. Mat.*, **26** (1) (2006), 173–177.

[6] G. F. Birkenmeier, Idempotents and completely semiprimes ideals, *Comm. Algebra*, **11** (1983), 567–580.

[7] G. F. Birkenmeier, J. Y. Kim and J. K. Park, Principally quasi-Baer rings, *Comm. Algebra*, **29** (2001), 639–660.

[8] G. F. Birkenmeier, J. Y. Kim and J. K. Park, Polynomial extensions of Baer and quasi-Baer rings, *J. Pure Appl. Algebra*, **159** (2001), 24–42.

[9] G. F. Birkenmeier, J. K. Park, Triangular matrix representations of ring extension, *J. Algebra*, **265** (2003), 457–477.

[10] W. E. Clark, Twisted matrix units semigroup algebras, *Duke math. J*, **34** (1967), 417–424.

[11] N. J. Groenewald, A note on extensions of Baer and p.p.-rings, *Publ. Inst. Math.*, **34** (1983), 71–72.

[12] E. Hashemi, Quasi-Armendariz rings relative to a monoid, *J. Pure Appl. Algebra*, **211** (2007), 374–382.

[13] E. Hashemi, Compatible ideals and radicals of Ore extensions, *New York J. Math.*, **12** (2006), 349–356.

[14] E. Hashemi and A. Moussavi, Polynomial extension of Baer and quasi-Baer rings, *Acta Math. Hungar.*, **107** (2005), 207–224.

[15] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, *J. Pure Appl. Algebra*, **168** (2002), 45–52.

[16] I. Kapslanky, *Rings of operators*, Benjamin, New York, 1965.

[17] N. H. Kim and Y. Lee, Armendariz rings and reduced rings, *J. Algebra*, **223** (2000), 477–488.

[18] T. K. Lee and Y. Zhou, *Reduced Modules in: Rings, Modules, Algebras, and Abelian Groups*, Lecture Notes in Pure and Appl. Math., vol. 236, Marcel Dekker, New York, 2004, pp. 365–377.

[19] Z. Liu, Armendariz rings relative to a monoid, *Comm. Algebra*, **33** (2005), 649–661.

[20] S. T. Rizvi and C. S. Roman, Baer and quasi-Baer modules, *Comm. Algebra*, **32** (2004), 103–123.

[21] C. P. Zhang and J. L. Chen, α-skew Armendariz modules and α-semicommutative modules, *Taiwanese J. Math*, **12** (2008), 473–486.

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