

Final Report for DE-FG02-96ER54370

The work has consisted of three projects. The first one is a continuation of the previous work that was done on the generation of zonal flows due to the four wave modulational instability. In this work, we examined the growth of streamers. This work was done with undergraduate student, and was presented at an APS DPP meeting. A summary of the work is given below. Another project was a study of the stability of curvature driven modes with tied field line geometry. The purpose of this study was to see if this instability was relevant to the observed 'blob' phenomenon in the edge. A summary of this work is given starting in Section II. This work was done with undergraduate student. The final project was an extension of electrostatic work that had been done on the parallel velocity shear instability. In this work, we included electromagnetic effects. We performed the linear stability analysis and discovered a new regime of instability. This work was done in collaboration with undergraduate student, who presented the work at an APS DPP meeting. Details of this are shown in Section III.

I. GENERATION OF STREAMERS DUE TO THE FOUR WAVE MODULATIONAL INSTABILITY

In recent years, there has been much attention given, both theoretically and experimentally, toward the understanding of the generation of zonal flows (or averaged poloidal flows) in fusion plasmas. This is directly related to the fact that zonal flows are known to have a pronounced stabilizing influence on microturbulence, and form 'transport barriers,' regions that effectively block the large anomalous transport that has been characteristic of tokamak discharges. The mechanism for the generation of zonal flows has been a topic of much research and debate. One of the systems that has been studied extensively with regard to zonal flow generation is the Charney-Hasegawa-Mima (CHM) equation, a two-dimensional nonlinear equation that has proven to successfully predict many features of drift wave turbulence in magnetized plasmas as well as in other fluid systems. Since it is the simplest model that retains the essential features of drift wave turbulence, many of the results derived from an analysis of the CHM equation can be related to magnetically confined plasmas. Although the CHM equation cannot describe a tokamak, it does have the advantage that due to its simplicity, it is able to shed light on some of the physical mechanisms that can contribute to the growth of zonal flows.

The main mechanism that exists within the CHM equation to generate zonal flows is referred to as the four wave modulational instability. The essential physics of the instability is as follows. Assume that there exists one large-amplitude, monochromatic wave [known as the pump wave with wave vector $\vec{k}_{pump} = (k_x, k_y)$] and two smaller amplitude waves with shifted radial wave numbers [known as the sidebands with wave vectors $\vec{k}_{side} = (k_x \pm q, k_y)$]. The sidebands interact with the pump wave via a nonlinear convective term (more commonly known as the Reynold's stress) thus generating a wave with no poloidal structure, but a radial structure equal to the difference of the pump and sideband wave numbers i.e. the zonal flow. The zonal flow in turn, directly interacts with the pump to feed energy from the pump back into the sidebands. This cycle continues until the zonal flow and sidebands have depleted the energy from the pump.

Just as zonal flows have proven to be of interest to magnetically confined fusion, so have 'streamers.' Streamers are the radial analog to zonal flows: they are the radially averaged component of the electrostatic potential that have a poloidal structure. Streamers are large scale flows in the radial direction, and can have a profound impact on global transport. Just as zonal flows can be generated via a nonlinear interaction that is described by the CHM equation, streamers can also be generated by a similar four wave interaction. The only difference between the two initial states is that the sidebands for the streamer instability are shifted only in the poloidal direction.

There has been previous work done on the modulational instability of streamers, and as is the case with much of the work done on zonal flows, the results were only valid in the limit where the poloidal wavelength of streamers was much longer than that of the pump and sidebands. In this work it was found that the growth rate of the streamer was considerably slower than that for the zonal flow (about two orders of magnitude slower for typical edge plasma conditions). One concern with only studying this system in the limit of $q \ll k$ where q is the wave vector of the streamer and k is the wave vector of the pump wave, is that this may not necessarily be the region of strongest growth. As will be seen, the growth rate of the streamer is proportional to q^n , where n is a positive exponent, so the weakest modes are necessarily the long wavelength ones. Therefore, the strongest growing modes have been ignored. An argument can be made that this is appropriate because streamers with broad poloidal scales (about on the order of the radial density profile) are typically observed experimentally, however, this implies that the linear dynamics necessarily determines the mode structure of the nonlinear state. Moreover, this also implies that the linear theory necessarily determines the fastest growing modes. Recent work on the dynamics of zonal flows in the CHM system showed that the fastest growing linear modes of the zonal flow (which occurred for $q > k$) interacted nonlinearly to drive the longest wavelength modes at a rate that was more than an order of magnitude greater than the rate predicted by linear theory. Therefore, the longest scale zonal flows, which were the slowest growing linear

modes turned out to be the fastest growing modes when the nonlinear coupling was taken into account. The fundamental physical difference was that the linear theory assumed that the dominant interaction that generates the zonal flow was the pump-sideband interaction. When the unstable sideband-sideband interaction is taken into account, these nonlinearly driven modes greatly exceeded the pump-sideband interaction.

With this point in mind, there is no reason that such an interaction cannot also exist with streamers, so in order to fully understand the stability of streamers, it is necessary to develop a linear theory for streamers that is valid for all sidebands, not just the ones that are slightly shifted from the pump. In fact, there may very well be regimes where the streamers are unstable and the zonal flow stable, in which case, the streamer could dominate. This paper is organized as follows: in section II we develop the analytical theory of the four wave interaction and produce a criterion for instability, and two analytical expressions for the growth rate that are valid in two different parameter regimes; in section III we provide numerical results comparing growth rates from the analytical expressions to the growth rates from the exact dispersion relation; and in section IV, we provide results of the growth of the modes from a nonlinear finite differencing code that evolves the full nonlinear system with the four-wave initial conditions.

A. Linear Analysis

In the limit of $\bar{\phi} = 0$, the CHM equation reads

$$\frac{\partial}{\partial t}[\tilde{\phi} - \rho_s^2 \nabla^2 \tilde{\phi}] - \rho_s^2 \rho_s c_s \hat{z} \times \nabla \tilde{\phi} \cdot \nabla \nabla^2 \tilde{\phi} + V_D \cdot \nabla \tilde{\phi} = 0, \quad (1)$$

where the potential has been normalized to T_e/e and V_D is the diamagnetic velocity given by $V_D = \rho_s c_s / L_N$. In order to examine the growth of streamers (i.e. the $k_x = 0$ component of $\tilde{\phi}$) by wave-wave coupling, we examine the stability of a system with a background wave, which we shall refer to as the "pump wave," interacting with other waves with slightly shifted poloidal wavelength and a small amplitude. The other waves shall be referred to as the "sidebands." Using a notation similar to Ref. [1], the fluctuating potential takes the form

$$\begin{aligned} \tilde{\phi}(x, y, t) = & a_0 \exp[ik_x x + ik_y y - i\omega_0 t] + a_0^* \exp[-ik_x x - ik_y y + i\omega_0 t] \\ & + a_+(t) \exp[ik_x x + i(k_y + q)y - i\omega_0 t] + a_+^*(t) \exp[-ik_x x - i(k_y + q)y + i\omega_0 t] \\ & + a_-(t) \exp[ik_x x + i(k_y - q)y - i\omega_0 t] + a_-^*(t) \exp[-ik_x x - i(k_y - q)y + i\omega_0 t] \\ & + b(t) \exp(iqy) + b^*(t) \exp(-iqy). \end{aligned} \quad (2)$$

where the coefficients a_0 and a_0^* are the amplitudes of the pump wave and are assumed to be constant in time, and a_+ , a_+^* , a_- , a_-^* , b , and b^* are the time dependent amplitudes of the sidebands and the streamer respectively. The frequency ω_0 , is the natural drift frequency of the streamer. Since we wish to find an evolution equation for the amplitude of $b(t)$, the wave-wave couplings that we need to retain are those with a factor of $\exp(iqy)$. This occurs in the interaction of a_-^* with a_0 and when a_+ interacts with the complex conjugate of the pump wave. The evolution equations for the sidebands and the streamer can be obtained by inserting Eq. (2) into Eq. (1) and only keeping terms with perfect wave number and frequency matching. This leaves us with the three coupled equations

$$\frac{da_+}{dt} - i\omega_0 a_+ + i\omega_+ a_+ = -\Omega_0 \Lambda_+ a_0 b, \quad (3)$$

$$\frac{da_-^*}{dt} + i\omega_0 a_-^* - i\omega_- a_-^* = \Omega_0 \Lambda_- a_0^* b, \quad (4)$$

$$\frac{db}{dt} + i\omega_0 b = -\Omega_0 \Lambda_0 \rho_s^2 [(k_+^2 - k^2) a_+ a_0^* - (k_-^2 - k^2) a_-^* a_0], \quad (5)$$

where

$$k^2 = k_y^2 + k_x^2, \quad \text{and} \quad k_\pm^2 = (k_y \pm q)^2 + k_x^2$$

$$\omega_\pm = \frac{(k_y \pm q)V_D}{1 + k_\pm^2 \rho_s^2}, \quad \text{and} \quad \omega_0 = \frac{qV_D}{1 + q^2 \rho_s^2}$$

$$\Lambda_\pm = \frac{(k^2 - q^2) \rho_s^2}{1 + k_\pm^2 \rho_s^2}, \quad \text{and} \quad \Lambda_0 = \frac{1}{1 + q^2 \rho_s^2},$$

and $\Omega_0 = \rho_s c_s k_x q$.

We now assume that all of the time dependent variables behave as $\sim \exp(-i\omega t)$. Linearization of Eqs. (3) - (5) yields the following dispersion relation

$$(\omega - \omega_0)(\omega - \omega_0 + \omega_-)(\omega + \omega_0 - \omega_+) = \frac{\Omega_0^2 \rho_s^2 (k^2 - q^2) \rho_s^2 |a_0|^2}{(1 + q^2 \rho_s^2)(1 + k_+^2 \rho_s^2)(1 + k_-^2 \rho_s^2)} \times$$

$$[(k^2 - k_-^2)(\omega + \omega_0 - \omega_+)(1 + k_+^2 \rho_s^2) - (k_+^2 - k^2)(\omega - \omega_0 + \omega_-)(1 + k_-^2 \rho_s^2)] \quad (6)$$

At this point, it is beneficial from an analytical point of view to define dimensionless variables, and to rewrite the dispersion relation in dimensionless form. Defining the normalized frequency $z = \omega/qV_D$, the ratio $\kappa = k_y/q$, and normalizing all wave vectors to the Larmor radius, Eq. (6) reads

$$(z - z_0)(z - z_0 + (\kappa - 1)z_-)(z + z_0 - (\kappa + 1)z_+) =$$

$$\frac{\lambda^2 z_0 z_+ z_-}{\hat{q}^2} [(\hat{k}^2 - \hat{k}_-^2) \frac{z + z_0 - (\kappa + 1)z_+}{z_+} - (\hat{k}_+^2 - \hat{k}^2) \frac{z - z_0 + (\kappa - 1)z_-}{z_-}] \quad (7)$$

where $\hat{k}, \hat{q}, \dots = k\rho_s, q\rho_s, \dots$, $z_0 = 1/(1 + \hat{q}^2)$, $z_{\pm} = 1/(1 + \hat{k}_{\pm}^2)$, and $\lambda^2 = 2k_x^2 L_N^2 q^2 \rho_s^4 (k^2 - q^2) |a_0|^2$. As this equation is cubic in z , there is no simple analytical expression for the growth rate, and therefore, no obvious criterion for instability. In order to gain some insight into the actual physics that determines the stability of streamers, we take the limit where the Larmor radius is much smaller than the wavelength of the fluctuations, which is the fluid limit for a magnetized plasma. In the $k\rho_s \ll 1$ limit, which for our dimensionless terms means $z_0, z_{\pm} = 1$ the dispersion relation reads

$$(z - 1)(z - 2 + \kappa)(z - \kappa) = -\lambda^2(z - 1 - 2\kappa + 2\kappa^2). \quad (8)$$

The two parameters that determine the stability of this mode are λ^2 and κ . This equation can be readily reduced to the standard form

$$z^2 + a_2 z^2 + a_1 z + a_0 = 0$$

where

$$a_0 = -\lambda^2 - 2\kappa(1 + \lambda^2) + \kappa^2(1 + 2\lambda^2)$$

$$a_1 = 2 + \lambda^2 + 2\kappa - \kappa^2$$

$$a_2 = -3 \quad (9)$$

Using standard methods we can use these forms to write an analytical expression for the growth rate, and we can also find the condition for instability. By defining the quantities $p = a_1/3 - a_2^2/9$ and $r = (a_1 a_2 - 3a_0)/6 - a_2^3/27$, we can write the discriminant which is defined as $p^3 + r^2$. In order for the cubic equation to have a pair of complex roots the discriminant must be positive. Using the coefficients from Eq. (8), this condition reads

$$(\lambda^2 - (1 - \kappa)^2)^3 + 27\lambda^4 \kappa^2 (1 - \kappa)^2 > 0. \quad (10)$$

Figure 1a shows a plot of this inequality as a function of λ^2 for $\kappa \leq 1$. A number of interesting conclusions regarding the stability of this mode can be drawn from this graph. First of all, if $|\lambda| < 1$, the inequality is at best only marginally satisfied, but in general is not satisfied and the mode is stable. The one exception to this is for $\kappa \approx 1$. In this case, the condition for instability is that $\lambda > 1 - \kappa$, however, this is a marginal case, and as will

be seen later, does not produce strong growth. Therefore, for wavelengths of the streamer that are considerably different from that of the pump wave, $|\lambda| \geq 1$ for instability. We can also draw conclusions about the stability for various wavelengths of the streamer. For $\kappa > 1$, (i.e. $q < k$), λ^2 can be either positive or negative, however the condition $\kappa > 1$ necessarily implies that $\lambda^2 > 0$. For $\kappa < 1$, λ^2 must be positive for instability, however, for homogeneous background turbulence (i.e. $k_x \sim k_y$), this is problematic. If $\kappa < 1$, q must be larger than k_y , which will generally force λ^2 to be negative. The one exception to this is if the background turbulence is not homogeneous, but rather has more elongated poloidal structures i.e. $k_x > k_y$. In this situation, λ^2 will be positive and if $q > k_y$, $\kappa < 1$.

With these points in mind, we will seek analytical forms for the growth rate that are valid for these two regimes. Using the coefficients given in Eq (8), we can write an analytical expression for the growth rate of this mode by using the fact that the imaginary part of the complex root is given by

$$\gamma = \frac{\sqrt{3}}{2}(s_1 - s_2) \quad (11)$$

where $s_1 = \sqrt[3]{r + \sqrt{(p^3 + r^2)}}$ and $s_2 = \sqrt[3]{r - \sqrt{(p^3 + r^2)}}$. Although this form is considerably simpler than the exact expression, nonetheless it is still quite complicated and not particularly illuminating from an analytical point of view. Therefore, we consider two different limits, one in which p^3/r^2 is small, and one in which this factor is large. Therefore, the dispersion relation can take two different analytical forms:

$$\gamma = \frac{\sqrt{3}}{2} \sqrt[3]{2r} \left(1 + \sqrt[3]{\frac{p^3}{r^2}}\right) \quad \text{for } p^3/r^2 \ll 1 \quad (12)$$

$$\gamma = \sqrt{3p} \quad \text{for } p^3/r^2 \gg 1 \quad (13)$$

Using the coefficients from Eq. (8), we can find an expression for p^3/r^2

$$\frac{p^3}{r^2} = \frac{(\lambda^2 - (1 - \kappa)^2)^3}{27\lambda^4\kappa^2(1 - \kappa)^2}. \quad (14)$$

Figure 2 shows plots of this factor as a function of λ for various values of κ . As can be seen quite clearly, in Fig 2(a) ($\kappa \lesssim 1$), p^3/r^2 is generally much greater than one, and only gets larger as λ increases. Therefore when the streamer has a wavelength near that of the pump wave or shorter than the pump wave, the dispersion relations reads

$$\gamma = qV_D[\lambda^2 - (1 - \kappa)^2]^{1/2} \quad (15)$$

which for $\lambda \gg 1$ reduces to $\gamma \simeq qV_D\lambda$.

From Fig. 2(b) when $\kappa \geq 2$, the factor p^3/r^2 is much smaller than one. In fact, for larger λ , it is so small ($\sim 10^{-3}$), that the term $\sqrt[3]{p^3/r^2}$ can be neglected too. Therefore, when the streamer has a wavelength longer than that of the pump wave, the dispersion relation reads

$$\gamma = \frac{3^{1/2}}{2^{2/3}} q V_D [\lambda^2 \kappa (\kappa - 1)]^{1/3} \quad (16)$$

B. Numerical Results

Although we have produced two analytical expressions for the growth rate, these expressions were derived with some approximations, so it is useful to check as to how well these expressions compare to the actual growth rates. We have the exact dispersion relation given by Eqs. (6) and (7), an approximate dispersion relation given by Eq. (8) and the two analytical dispersion relations. Shown in Figure 3(a) is a representative comparison of the three growth rates as a function of λ for $\hat{k}_x = 0.3$, $\hat{k}_y = 0.3$ and $\hat{q}_x = 0.06$. For these parameters, $\kappa > 1$ so the analytic form for the growth rate that is used in this figure is given by Eq. (16). Not only does the analytic form show excellent qualitative agreement with the numerical solutions, it shows very good quantitative agreement, and for some peculiar reason, agrees even better with exact solution than it does with the approximate solution. Shown in Figure 3(b) is a similar plot for $\hat{k}_x = 0.1$, $\hat{k}_y = 0.1$ and $\hat{q}_x = 0.3$. In this case, the analytic expression for the growth rate is given by Eq. (15) as $\kappa < 1$. Again, the qualitative and quantitative agreement between the analytic expression and the exact solution is excellent. It should be noted that these results are quite typical for a wide range of parameters. Even when the wave vectors become larger ~ 0.5 , the only thing that suffers is the quantitative agreement. Qualitatively, the analytical forms still do a very good job of reproducing the true growth rates.

Now that we have a general analytical result for the growth rate of the streamers, we can now compare these to the growth rates of the zonal flow and see if there are any situation in which the streamer growth could exceed that of the zonal flow.

II. NONLINEAR EQUATIONS AND LINEAR THEORY FOR CURVATURE DRIVEN EDGE INSTABILITY

For many years, the conventional picture of transport in the scrape off layer of a magnetically confined fusion device was as follows: plasma transported from the core across the separatrix would stream along the magnetic field until it reached the divertor plate. Recombination occurred at the divertor, and the recombined neutral particles would then make their way back into the core plasma, where ionization would occur. Recent experimental data from a wide variety of magnetically confined devices, however, paints a different

picture, suggesting that a majority of the recycling actually occurs at the wall of the main chamber [1-3] as opposed to at the divertor. The implications of these data are twofold. First, because the density profile is exponentially decaying only near the separatrix, but then becomes essentially flat, this implies that that transport is non-diffusive, but rather convective in nature. The second implication from these results, is that the transport is considerably faster than the sound transit time from the outboard midplane to the divertor plate.

There are quite a few other interesting characteristics of these experimental results. Probe measurements of the ion saturated current (I_{sat}), which is used to calculate the value of the electron density, show that the particle transport is characterized by large intermittent bursts. These bursts, or 'blobs' as they have been come to be called, even exist in the region where the density profile is relatively flat, and in that region, the density fluctuations show a tendency to favor positive events. Examination of the probability distribution function (pdf) of the fluctuations closer to the separatrix show a more Gaussian distribution, but the pdf's of the fluctuations as the probes move further from the separatrix observe a distributed that is decidedly positively skewed. Perhaps the most remarkable result is that this type of transport appears to be ubiquitous among magnetically confined fusion devices. Such events have been observed in tokamaks (JET, DIII-D), stellarators (ATF, W7-AS), and even in the linear device PISCES, therefore indicating that the reason for this convective physics is due to the physics of the scrape off layer, as opposed to a toroidal effect.

Naturally, all of this compelling experimental evidence has motivated some recent theoretical work. D'Ippolito et. al [?] studied the dynamics of a blob-like perturbation (without speculating as to what the actual source of the blob was) and found that the blobs had the following features: large fluctuation levels for the density, temperature and vorticity; a faster decay of the temperature and vorticity perturbations than the density perturbations; convective radial transport of the particles; and a finite density flux in a region where the density profiles are flat. In another paper, a qualitative physical model was presented that showed that such blob-like structures could be produced due to a drift-like instability that arises from a $\nabla \mathbf{B}$ polarization current coupled to the associated $\mathbf{E} \times \mathbf{B}$ drift.

In this paper, we extend the work of Ref. 5 and perform a linear stability analysis of

A. Nonlinear equations

In our model, we have a system with an equilibrium density gradient and retain the gradient of the magnetic field at the outboard edge ($\theta = 0$). We assume that the electrons have a finite temperature but that the ions are cold. Since the basic equations that we will solve are the continuity equation and the vorticity equation ($\nabla \cdot \vec{j} = 0$), we start by identifying the currents. In the scrape off layer (SOL), there is a net flow of plasma to the

divertor plates. The ions are accelerated by an ambipolar electric field that causes them to produce an averaged current proportional to the sound speed $j_{\parallel i} \sim en\sqrt{T_e/M}$, while the contribution of the electrons is given by $j_{\parallel e} \sim en\sqrt{T_e/m} \exp(-e\phi/T_e)$, where the exponential factor represents the fraction of Maxwellian electrons that reach the plate.

To identify the perpendicular currents, we write the cold ion and electron perpendicular momentum equations in the flute limit ($\partial/\partial z = 0$):

$$nm_i \frac{d\mathbf{v}_i}{dt} = -ne\nabla\phi + \frac{ne}{c}\mathbf{v}_i \times \mathbf{B} \quad (17)$$

$$0 = ne\nabla\phi - T_e\nabla n - \frac{ne}{c}\mathbf{v}_e \times \mathbf{B} \quad (18)$$

We can obtain an expression for the lowest order ion response by assuming that $\Omega_i = eB/m_i c \gg \omega$, which means that we can neglect the time derivative in Eq. (1). Therefore, to lowest order the ion velocity is given by

$$v_{\perp i}^{(0)} = \frac{cT_e}{eB^2}\mathbf{B} \times \nabla\phi \quad (19)$$

where we have normalized the potential to T_e/e . We can now insert $v_{\perp i}^{(0)}$ into the left hand side of Eq. (1) to obtain the next order ion response

$$v_{\perp i}^{(1)} = -\frac{cT_e}{eB\Omega_i} \frac{d}{dt} \nabla\phi \quad (20)$$

which is the polarization drift.

We can find the lowest order electron response from Eq. (2)

$$v_{\perp e}^{(0)} = \frac{cT_e}{eB^2}\mathbf{B} \times \nabla\phi - \frac{cT_e}{eB^2}\mathbf{B} \times \nabla \ln n \quad (21)$$

The first term is the $\mathbf{E} \times \mathbf{B}$ drift and the second is the electron diamagnetic drift. We now want to evaluate the divergence of the quantity $j_{\perp} = ne(v_{\perp i} - v_{\perp e})$. Since the $\mathbf{E} \times \mathbf{B}$ drift is the same for both species, these lowest order terms produce no contribution to the current. The diamagnetic drift, which is usually divergence free, has a finite divergence when the radial dependence of the magnitude of the magnetic field is taken into account. To lowest order, the magnetic field behaves as $B = B_0(R/r)$ where R is the major radius, and r is the radial coordinate, therefore, the divergence of the electron diamagnetic drift will equal

$$\nabla \cdot \left[\left(\frac{cT_e}{eB^2} \mathbf{B} \right) \times \nabla n \right] = \nabla n \cdot \nabla \times \left(\frac{cT_e}{eB^2} \mathbf{B} \right) = -\nabla n \cdot \left[\nabla \left(\frac{cT_e}{eB^2} \right) \times \mathbf{B} \right] = -\frac{2cT_e}{eB^2 R} \frac{\partial n}{\partial y}$$

where we have used the fact that $\nabla(1/B^2) = 2\hat{r}/RB^2$ and $\nabla \times \mathbf{B} = 0$. Therefore, our perpendicular current consists of the ion polarization drift, and the electron ∇B drift. If we now impose the condition that the current be divergence free. Integrating over the parallel

direction which has a length $L/2$ where L is the connection length, we arrive at the following equation

$$\rho_s^2 \nabla \cdot [n \frac{d\nabla \phi}{dt}] + \frac{2\rho_s c_s}{R} \frac{\partial n}{\partial y} = \frac{2nc_s}{L} [1 - \sqrt{M/m_e} \exp(-\phi)]. \quad (22)$$

where $c_s = \sqrt{T_e/M}$ and $\rho_s = c_s/\Omega_i$. In order to complete the model of the scrape off layer, we must account for the fact that the divertor plates act as a particle sink. The rate for this is equal to the sound speed divided distance to the target. Therefore the continuity equation reads

$$\frac{\partial n}{\partial t} + \nabla \cdot [nv_{\perp e}^{(0)}] = -\frac{2c_s}{L}(n - n_0). \quad (23)$$

This equation, coupled with Eq. (6) closes the system.

B. Linear theory

If we assume that the perturbations have the form $\tilde{\phi} = \phi_0 \exp(ik_y y + ik_x x - i\omega t)$, Eqs. (6) and (7) have the following linear forms

$$(\omega k^2 \rho_s^2 + i\nu_{\parallel}) \tilde{\phi} = -\nu_{\parallel} \left(\frac{L}{R}\right) (k_y \rho_s) \frac{\tilde{n}}{n_0} \quad (24)$$

$$(\omega + i\nu_{\parallel}) \frac{\tilde{n}}{n_0} = \omega_* \tilde{\phi} \quad (25)$$

where we have defined $k^2 = k_x^2 + k_y^2 - ik_x/L_N$, $\nu_{\parallel} = 2c_s/L$, and $\omega_* = -k_y \rho_s c_s/L_N$ where L_N is the density scale length defined as $1/L_N = d(\ln n^{(0)})/dx$. When these two relations are combined, we are left with a quadratic equation for ω

$$\omega^2 k^2 \rho_s^2 + i\omega \nu_{\parallel} (1 + k^2 \rho_s^2) - \nu_{\parallel}^2 + \omega_* \hat{\omega} = 0, \quad (26)$$

where we have defined a new frequency $\hat{\omega} = (L/R)k_y \rho_s \nu_{\parallel}$. Factoring this equation gives the expression

$$\omega(2k^2 \rho_s^2) = -i\nu_{\parallel} (1 + k^2 \rho_s^2) \pm [-\nu_{\parallel}^2 (1 + k^2 \rho_s^2)^2 + 4\nu_{\parallel}^2 k^2 \rho_s^2 - 4\omega_* \hat{\omega} k^2 \rho_s^2]^{1/2}, \quad (27)$$

which can be simplified to

$$\omega = i \frac{\nu_{\parallel}}{2k^2 \rho_s^2} [-(1 + k^2 \rho_s^2) + (1 - k^2 \rho_s^2)(1 + \frac{\Delta}{(1 - k^2 \rho_s^2)^2})^{1/2}], \quad (28)$$

where we have defined the dimensionless parameter

$$\Delta = \frac{4\omega_*\hat{\omega}k^2\rho_s^2}{\nu_{\parallel}^2}. \quad (29)$$

At this point, it is useful to estimate the magnitude of Δ so we can obtain a simplified expression for the growth rate. Rewriting Δ in terms of the characteristic lengths of the system gives us the expression

$$\Delta = \left(\frac{2L^2}{-RL_N}\right)k_y^2\rho_s^2k^2\rho_s^2.$$

In order for the mode to be unstable, the real part of Δ must be positive, which implies that L_N must be negative. This result is typical of ballooning modes which are unstable on the outboard side where the density gradient is in the same direction as the gradient of the magnitude of the magnetic field. Typical values for the lengths in a SOL plasma are $L = 3000\text{cm}$, $R = 100\text{cm}$, $L_N = -2\text{cm}$, and $0.3 \geq k_y\rho_s \geq 0.1$, therefore giving a value of $1000 \geq \Delta \geq 10$. Taking the limit $\Delta \gg 1$, we can write Eq. (12) as

$$\omega = i\frac{\nu_{\parallel}}{2k^2\rho_s^2}(\Delta^{1/2} - (1 + k^2))$$

which reduces to

$$\omega = i\frac{\nu_{\parallel}}{2}\left[\sqrt{\frac{2L^2}{-RL_N}}\frac{k_y}{k} - \left(1 + \frac{1}{k^2\rho_s^2}\right)\right]. \quad (30)$$

At first glance, it appears as if this is a purely growing mode, however, it must be remembered that k^2 is complex. Defining the quantity $k_{\perp}^2 = k_x^2 + k_y^2$, we can rewrite k^2 as $k^2 = k_{\perp}^2(1 - ik_x/L_N k_{\perp}^2)$, and since $k_x/L_N \ll k_{\perp}^2$, we can use the imaginary part of k^2 as an expansion parameter. Separating the real and imaginary parts of the frequency, we have the relation

$$\omega = \frac{\nu_{\parallel}}{2}\left[\left(\sqrt{\frac{L^2}{-2RL_N}}\frac{k_y}{k_{\perp}} - \frac{1}{k_{\perp}^2\rho_s^2}\right)\left(\frac{-k_x}{L_N k_{\perp}^2}\right) + i\left(\sqrt{\frac{2L^2}{-RL_N}}\frac{k_y}{k_{\perp}} - \left(1 + \frac{1}{k_{\perp}^2\rho_s^2}\right)\right)\right]. \quad (31)$$

Examination of this dispersion relation shows that the ratio of the real frequency to the growth rate (for typical SOL parameters) is

$$\omega_r/\gamma \approx -k_x/2L_N k_{\perp}^2 \ll 1,$$

so we can assume that the mode has saturated before it has had time to propagate.

An interesting feature of this dispersion relation is that to leading order in k_x (for typical edge parameters), the real frequency is proportional to $-k_x/L_N$, therefore, every mode will propagate opposite the direction of the density gradient at a similar velocity. We can obtain an expression for this velocity by dividing the real part of ω by k_x

$$v_x = \omega_r/k_x \approx \frac{\nu_{\parallel}}{2} \sqrt{\frac{L^2}{-2RL_N}} \frac{1}{(-L_N k_{\perp})} = \frac{c_s}{\sqrt{-2RL_N}} \frac{\rho_s^2}{(-L_N k_{\perp}^2 \rho_s^2)}. \quad (32)$$

Similarly, to leading order in k_y , the real frequency is linear in k_y , so we would expect a fairly uniform poloidal propagation. We can now obtain an estimate for the value of this radial group velocity by inserting the same numerical values that we used previously and take $\rho_s = 0.03$ cm, which give $v_x \approx c_s/400$. Another interesting feature of the real frequency is that for both directions of the group velocity, $v_g \sim 1/k_{\perp}^2$ i.e. big blobs move faster. In order to get a better feel of the true behavior of the dispersion relation, shown in Figure 1 are the real frequency [taken from Eq. (15)] and as a function of both k_x and k_y plotted along with the group velocities defined as $v_{x,y} = \partial\omega_r/\partial k_{x,y}$. The first set of graphs are for a mode with scale size $L = 1$ cm, and the second set of graphs are for a larger blob with $L = 2$ cm. Clearly the larger blob propagates faster, particularly in the poloidal direction.

It should be noted that the only reason that this mode has a real frequency is due to the fact that polarization drift term was evaluated properly. Typically, the term $\nabla \cdot [n(d\nabla\phi/dt)]$ is written as $n(d\nabla^2\phi/dt)$. The fact is that for the edge, this is a bad assumption for the reason that the density has a large equilibrium gradient, and when this term is linearized, the term $ik_x(dn^{(0)}/dx)\tilde{\phi}$ appears, and it is this term that gave us the real part of the frequency. Therefore, it is likely that a similar sort of result will occur (i.e. a finite radial group velocity opposite the direction of the density gradient) for any sort of a flute mode in the SOL.

C. Numerical results

Eqs. (6) and (7) were solved numerically using a finite differencing fluid code with an explicit time stepping scheme. In a general sense, Eq. (6) which is the vorticity equation can be thought of as the evolution equation for the potential, and Eq. (7), which is the continuity equation, evolves the density. Evolution of the continuity equation is straightforward, however, in its current form, however the leading term in Eq. (6), $\nabla \cdot [n(d\nabla\phi/dt)]$, is in a form that is very difficult to invert for ϕ . The most common way of simplifying this term is to assume that the density commutes through the divergence operator and also that the divergence commutes with the convective derivative thus producing an evolution equation for $\nabla^2\phi$ (which readily inverted for ϕ). The latter assumption is correct if the perpendicular velocity is divergence free (which to lowest order it is), however, the former assumption is not good at all, particularly in the edge where the equilibrium density gradients are large. In order to retain the lowest order effect of the equilibrium density gradient, yet put the equations into a readily solvable form, we set the density in that term equal to its equilibrium value. Therefore that term now reads

$$\nabla \cdot [n_0(x) \frac{d\nabla\phi}{dt}] = \frac{\partial \nabla \cdot [n_0 \nabla\phi]}{\partial t} + \nabla \cdot [n_0 \mathbf{v}_{\perp} \cdot \nabla \nabla\phi].$$

From a computational point of view, it is advantageous to put these term into the form $\partial\Omega/\partial t + \nabla \cdot [v\Omega]$, or what is known as conservative form. If we define $\Omega = \nabla \cdot [n_0 \nabla \phi]$, we can write this term as

$$\nabla \cdot [n_0(x) \frac{d\nabla \phi}{dt}] = \frac{\partial \Omega}{\partial t} + \nabla \cdot [v\Omega] - \frac{dn_0}{dx} v_x \nabla^2 \phi - \frac{d^2 n_0}{dx^2} v_x \frac{\partial \phi}{\partial x}. \quad (33)$$

We now proceed to reduce these equations to dimensionless form. If we normalize the time to the parallel sound transit time $t \rightarrow t(2c_s/L)$ and normalize the x and y directions to a box size a , the nonlinear equations reduce to

$$\frac{\partial \Omega}{\partial t} + \nabla \cdot [v_{\perp} \Omega] - v_x \left(\frac{dn_0}{dx} \nabla^2 \phi + \frac{d^2 n_0}{dx^2} \frac{\partial \phi}{\partial x} \right) + \frac{\beta}{\hat{\rho}^2} \frac{\partial n}{\partial y} = \frac{1}{\hat{\rho}^2} n [1 - \exp(-\phi)] \quad (34)$$

$$\frac{\partial n}{\partial t} + \nabla \cdot [nv_{\perp}] = -(n - n_0). \quad (35)$$

where $v_{\perp} = \alpha \hat{z} \times \nabla \phi$ and we have defined the dimensionless parameters

$$\alpha = \frac{L\rho_s}{2a^2} \quad \beta = \frac{L\rho_s}{aR} \quad \hat{\rho} = \frac{\rho}{a}. \quad (36)$$

If we choose our computational box to have a size $a = 5$ cm and use typical values for the SOL ($\rho_s \sim 0.03 - 0.05$ cm, $L \sim 3000$ cm, $R \sim 100$ cm), these parameters have the values $\alpha = 2 - 3$, $\beta = 0.2 - 0.3$, and $\hat{\rho} = 0.006 - 0.01$.

- [1] H. L. Berk, D. D. Ryutov, and Yu. A. Tsidulko, Phys. Fluids B **3** (6), 1346 (1991).
- [2] D. Farina, R. Pozzoli, and D. Ryutov, Plasma Phys. Control. Fusion **35**, 1271 (1993).
- [3] K. Lotov, D. Ryutov, and J. Weiland, Physica Scripta, **50** 153-160, (1994).
- [4] B. B. Kadomstev and O. P. Pogutse, in *Reviews of Plasma Physics* edited by M. A. Leontovich (Consultants Bureau, New York, 1970), Vol. **5** p. 249.
- [5] R. Prater, Phys. Fluids **17**, 193 (1974).
- [6] M. A. Lieberman and S. L. Wong, Plasma Phys. **19**, 745 (1977).
- [7] X. Q. Xu, R. H. Cohen, Contrib. Plasma Phys. **38** 1/2 158 (1998).
- [8] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th Ed. Cambridge: Cambridge University Press, 342-343 (1927).
- [9] Wolfram Research, Inc., *Mathematica*, Ver. 4.0 (Wolfram Research, Inc., Champaign, IL, 1999).

III. ELECTROMAGNETIC PARALLEL VELOCITY SHEAR INSTABILITY

We start with the nonlinear equations that describe the evolution of this mode. These are the plasma continuity equation, the ion parallel momentum equation, Ampère's Law, the parallel electron force balance (or Ohm's Law), and the vorticity equation. These equations couple the density n , electrostatic potential ϕ , the parallel component of the ion velocity v_z , the parallel current j_{\parallel} , and the parallel component of the vector potential A_{\parallel} . The geometry we are using is a uniformly magnetized 3-D slab, and the equilibrium has radial gradients of the parallel velocity, the density, and the current. Electron and ion temperatures are finite, but constant. With these points in mind, the nonlinear equations read:

$$\frac{\partial n}{\partial t} + \nabla \cdot [n\vec{v}] = 0 \quad (37)$$

$$nM\left[\frac{\partial v_z}{\partial t} + \vec{v} \cdot \nabla v_z\right] = -T\nabla_{\parallel} n \quad (38)$$

$$\nabla_{\perp}^2 A_{\parallel} = -\frac{4\pi}{c} j_{\parallel} \quad (39)$$

$$\eta j_{\parallel} = \frac{1}{n} \nabla_{\parallel} n - \frac{e}{T_e} \nabla_{\parallel} \phi - \frac{e}{T_e c} \frac{\partial A_{\parallel}}{\partial t} \quad (40)$$

$$e\nabla_{\perp} \cdot [nv_{\perp}] + \nabla_{\parallel} j_{\parallel} = 0 \quad (41)$$

where $T = T_e + T_i$, M is the ion mass and η is the parallel resistivity. We now proceed to find an expression for the perpendicular velocity. Assuming that the ion gyrofrequency is much faster than the frequencies of the system, we can separate our velocity into a lowest order and first order terms.

$$v_{\perp} = v_{\perp}^{(0)} + v_{\perp}^{(1)}.$$

The lowest order velocity is given by the $E \times B$ drift and the ion diamagnetic drift (both equilibrium and perturbed).

$$v_{\perp}^{(0)} = \frac{c}{B} \hat{z} \times \nabla \tilde{\phi} + \frac{cT_i}{neB} \hat{z} \times \nabla \tilde{n} + \frac{cT_i}{neB} \hat{z} \times \nabla n^{(0)}.$$

The next order terms is the polarization drift, which balances the inertial response of the both terms of the lowest order response with the Lorentz force

$$v_{\perp}^{(1)} = \frac{1}{\Omega_i} \frac{d}{dt} \hat{z} \times v_{\perp}^{(0)}.$$

Defining the temperature ration $\tau = T_e/T_i$ and normalizing the potential as $\tilde{\phi} = e\tilde{\phi}/T_e$, the velocities have the following forms:

$$v_{\perp}^{(0)} = \rho_i v_i [\tau \hat{z} \times \nabla \tilde{\phi} + \frac{1}{n} \hat{z} \times \nabla n] \quad (42)$$

$$v_{\perp}^{(1)} = -\rho_i^2 \frac{d}{dt} [\tau \nabla \phi + \frac{\nabla n}{n}]. \quad (43)$$

We now proceed to find the linearized form of each terms from Eqs. (1) - (5). Starting with the divergence of the lowest order perpendicular velocity from Eq. (1),

$$\nabla \cdot [n v_{\perp}^{(0)}] = \rho_i v_i \tau \nabla \cdot [n \hat{z} \times \nabla \tilde{\phi}] = \rho_i v_i \tau \hat{z} \times \nabla \tilde{\phi} \cdot \nabla n^{(0)} = -\frac{ik_y \rho_i v_i \tau n_0 \tilde{\phi}}{L_N} = -i\tau n_0 \omega_{*n} \tilde{\phi}$$

where $\omega_{*n} = (k_y \rho_i v_i)/L_N$ and $\frac{1}{L_N} = \frac{1}{n_0} \frac{dn_0}{dx}$

The next term is the divergence of the polarization current. We retain the radial dependence of the density (not what is done in gs2).

$$\begin{aligned} \nabla \cdot [n v_{\perp}^{(1)}] &= -\rho_i^2 \nabla \cdot [n \frac{d}{dt} (\tau \nabla \phi + \frac{\nabla n}{n})] \\ &= -\rho_i^2 (-i\omega) \nabla \cdot [n_0 (\tau \nabla \tilde{\phi}) + (\nabla \tilde{n} - \frac{\tilde{n}}{n_0} \nabla n_0)] \\ &= -\rho_i^2 (-i\omega) [\tau \frac{dn_0}{dx} (ik_x \tilde{\phi}) - \tau k_{\perp}^2 n_0 \tilde{\phi} - k_{\perp}^2 \tilde{n} - ik_x \frac{\tilde{n}}{n_0} \frac{dn_0}{dx} - \frac{\tilde{n}}{n_0} \frac{d^2 n_0}{dx^2} + \frac{\tilde{n}}{n_0^2} (\frac{dn_0}{dx})^2] \\ &= i\omega \rho_i^2 [(-k_{\perp}^2 + \frac{ik_x}{L_N}) \tau n_0 \tilde{\phi} + (-k_{\perp}^2 - \frac{ik_x}{L_N} - \frac{1}{L_{N2}} + (\frac{1}{L_N})^2) \tilde{n}] \\ &= -i\omega \rho_i^2 n_0 [\tau (k_{\perp}^2 - \frac{ik_x}{L_N}) \tilde{\phi} + (k_{\perp}^2 + \frac{ik_x}{L_N} + \frac{1}{L_{N2}} - (\frac{1}{L_N})^2) \frac{\tilde{n}}{n_0}] \end{aligned} \quad (44)$$

The next term that we consider is the divergence of the parallel momentum. Since this is an electromagnetic system, we must also account for the fact that the magnetic field can fluctuate, and that this variation can give a higher order contribution to the parallel derivatives. If we consider the parallel derivative operating on a generic variable, X , we have

$$\nabla_{\parallel} X = \hat{b} \cdot \nabla X = \hat{b}^{(0)} \cdot \nabla \tilde{X} + \hat{b}^{(1)} \cdot \nabla X_0$$

$$= ik_z \tilde{X} + \frac{\nabla \times \tilde{A}_{\parallel}}{B_0} \cdot \frac{dX_0}{dx}$$

$$= ik_z \tilde{X} + \frac{ik_y}{B_0} \tilde{A}_{\parallel} \frac{dX_0}{dx}$$

$$\begin{aligned}
&= ik_z \tilde{X} + ik_y \tau \frac{cT_i}{eB} \frac{dX_0}{dx} \left(\frac{e\tilde{A}_{\parallel}}{cT_e} \right) \\
&= ik_z \tilde{X} + i\omega_{*X} \tau X_0 \left(\frac{e\tilde{A}_{\parallel}}{cT_e} \right)
\end{aligned} \tag{45}$$

With these points in mind, the linearized versions of Eq. (1) reads

$$\begin{aligned}
&-i\omega \tilde{n} - i\omega_{*n} \tau \tilde{\phi} n_0 - i\omega \rho_i^2 n_0 \left[\tau (k_{\perp}^2 + \frac{ik_x}{L_N}) \tilde{\phi} + \hat{k}^2 \frac{\tilde{n}}{n_0} \right] + \\
&ik_z n_o \tilde{v}_z + ik_z v_{z0} \tilde{n} + i\omega_{*v} n_o \tau v_i \left(\frac{e\tilde{A}_{\parallel}}{cT_e} \right) + i\omega_{*n} n_0 \tau v_z^{(0)} \left(\frac{e\tilde{A}_{\parallel}}{cT_e} \right) = 0
\end{aligned}$$

where we have defined $\hat{k}^2 = k_{\perp}^2 + \frac{ik_x}{L_N} + \frac{1}{L_{N2}} - (\frac{1}{L_N})^2$. Grouping like terms, this equation reduces to

$$\omega(1 + \hat{k}^2 \rho_i^2 - k_z v_{z0}) \frac{\tilde{n}}{n_0} + [\omega_{*n} \tau + \omega \tau \rho_i^2 (k_{\perp}^2 - \frac{ik_x}{L_N})] \tilde{\phi} = k_z v_i \frac{\tilde{v}_z}{v_i} + (\omega_{*v} \tau + \omega_{*n} \tau \frac{v_{z0}}{v_i}) \left(\frac{ev_i \tilde{A}_{\parallel}}{cT_e} \right) \tag{46}$$

We now examine each term in the ion parallel momentum equation. The convective term contains four individual contributions

$$\vec{v} \cdot \nabla v_z = v_{\perp}^{(0)} \cdot \nabla \tilde{v}_z + \tilde{v}_{\perp}^{(0)} \cdot \nabla v_z^{(0)} + \tilde{v}_{\perp}^{(1)} \cdot \nabla v_z^{(0)} + \tilde{v}_z^{(0)} \cdot \nabla_{\parallel} \tilde{v}_z,$$

and each of these terms have the following linearized forms:

$$v_{\perp}^{(0)} \cdot \nabla \tilde{v}_z = \frac{\rho_i^2 v_i}{n_0} \frac{dn_0}{dx} (ik_y \tilde{v}_z) = i\omega_{*n} \tilde{v}_z \tag{47}$$

$$\tilde{v}_{\perp}^{(0)} \cdot \nabla v_z^{(0)} = \rho_i v_i [-\tau ik_y \tilde{\phi} - ik_y \frac{\tilde{n}}{n_0}] \frac{dv_z^{(0)}}{dx} = -i\omega_{*v} v_i (\tau \tilde{\phi} + \frac{\tilde{n}}{n_0})$$

$$\tilde{v}_{\perp}^{(1)} \cdot \nabla v_z^{(0)} = -\rho_i^2 (-i\omega) [\tau ik_x \tilde{\phi} + ik_x \frac{\tilde{n}}{n_0}] \frac{dv_z^{(0)}}{dx} = -\frac{k_x \rho_i}{k_y} \omega \omega_{*v} (\tau \tilde{\phi} + \frac{\tilde{n}}{n_0})$$

$$v_z^{(0)} \cdot \nabla_{\parallel} \tilde{v}_z = v_z^{(0)} [ik_z \tilde{v}_z + i\omega_{*v} \tau v_i \left(\frac{e\tilde{A}_{\parallel}}{cT_e} \right)] = iv_i \frac{v_z^{(0)}}{v_i} [k_z v_i \frac{\tilde{v}_z}{v_i} + \omega_{*v} \tau \left(\frac{ev_i \tilde{A}_{\parallel}}{cT_e} \right)]$$

Inserting these expressions into Eq. (2) and grouping terms yields the following linearized equation

$$\begin{aligned}
& [\omega - \omega_{*n} - k_z v_i v_{z0}] \tilde{v}_z + [\omega_{*v} (1 + \frac{i\omega k_x \rho_i}{k_y v_i}) - (1 + \tau) k_z v_i] \frac{\tilde{n}}{n_0} + \\
& \omega_{*v} (1 + \frac{i\omega k_x \rho_i}{k_y v_i}) \tau \tilde{\phi} - [v_{z0} \omega_{*v} + (1 + \tau) \omega_{*n}] \tau (\frac{ev_i \tilde{A}_{\parallel}}{cT_e}) = 0
\end{aligned} \tag{48}$$

The next equation to consider is Ampère's Law. This has the straightforward form:

$$-k_{\perp}^2 \rho_i^2 \tilde{A}_{\parallel} = -\frac{4\pi \rho_i^2}{c} \tilde{j}_{\parallel}.$$

We can now rearrange this equation so that the variables are in dimensionless form,

$$k_{\perp}^2 \rho_i^2 \tau (\frac{ev_i \tilde{A}_{\parallel}}{cT_e}) = \beta (\frac{\tilde{j}_{\parallel}}{nev_i}) \tag{49}$$

where $\beta = 4\pi n T_i / B^2$.

The linearized forms of Eqs. (4) and (5) follow directly from the previous derivations.

$$i\hat{\eta} (\frac{\tilde{j}_{\parallel}}{nev_i}) = -k_z v_i \frac{\tilde{n}}{n_0} - \omega_{*n} \tau (\frac{ev_i \tilde{A}_{\parallel}}{cT_e}) + k_z v_i \tilde{\phi} - \omega (\frac{ev_i \tilde{A}_{\parallel}}{cT_e}) \tag{50}$$

$$\omega \rho_i^2 [\tau \hat{k}^2 \frac{\tilde{n}}{n_0}] = k_z v_i (\frac{\tilde{j}_{\parallel}}{nev_i}) + \omega_{*j} \tau (\frac{ev_i \tilde{A}_{\parallel}}{cT_e}) \tag{51}$$

We now rewrite all linear equations by defining the following dimensionless variables :

$$\frac{\tilde{n}}{n_0} \rightarrow \tilde{n}; \quad (\frac{\tilde{j}_{\parallel}}{nev_i}) \rightarrow \tilde{j}_{\parallel}; \quad (\frac{ev_i \tilde{A}_{\parallel}}{cT_e}) \rightarrow \tilde{A}_{\parallel}; \quad \frac{\tilde{v}_z}{v_i} \rightarrow \tilde{v}_z$$

Therefore, the final linear forms of Eqs. (1) - (5) read

$$\begin{aligned}
& [\omega (1 + \hat{k}_{\perp}^2 \rho_i^2) - k_z v_z^{(0)}] \tilde{n} + [\omega_{*n} \tau + \omega (k_{\perp}^2 - \frac{ik_x}{L_N}) \tau \rho_i^2] \tilde{\phi} \\
& = k_z v_i \tilde{v} + (\omega_{*v} \tau + \omega_{*n} \tau v_z^{(0)}) \tilde{A}_{\parallel}
\end{aligned} \tag{52}$$

$$\begin{aligned}
& [\omega - \omega_{*n} - k_z v_i v_z^{(0)}] \tilde{v}_z + [\omega_{*v} (1 + \frac{i\omega k_x \rho_i}{k_y v_i}) - (1 + \tau) k_z v_i] \tilde{n} + \\
& (\omega_{*v} (1 + \frac{i\omega k_x \rho_i}{k_y v_i}) \tau) \tilde{\phi} = [v_z^{(0)} \omega_{*v} + (1 + \tau) \omega_{*n}] \tau \tilde{A}_{\parallel}
\end{aligned} \tag{53}$$

$$\tilde{j}_{\parallel} = \frac{k_{\perp}^2 \rho_i^2 \tau}{\beta} \tilde{A}_{\parallel} \tag{54}$$

$$(\omega + \omega_{*n} \tau) \tilde{A}_{\parallel} + i\hat{\eta} \tilde{j}_{\parallel} = -k_z v_i (\tilde{n} - \tilde{\phi}) \tag{55}$$

$$\omega \rho_i^2 [\hat{k}^2 \tilde{n} + \tau (k_{\perp}^2 - \frac{ik_x}{L_N}) \tilde{\phi}] = k_z v_i \tilde{j}_{\parallel} + \omega_{*j} \tau \tilde{A}_{\parallel} \tag{56}$$

A. The simplest form of the v'_\parallel mode - electrostatic limit

The simplest limit of this system that retains the essential physics of the v'_\parallel mode is when the density gradient and the parallel viscosity are neglected. We shall assume that we are in the moving frame of the plasma i.e. $v_z^{(0)} = 0$. In the electrostatic limit ($\beta = 0$) and in the limit of small Larmor radius ($k\rho < 1$), the linearized equations read

$$\omega \tilde{n} = k_z v_i \tilde{v} \quad (57)$$

$$\omega \tilde{v}_z + [\omega_{*v} - (1 + \tau)k_z v_i] \tilde{n} + \omega_{*v} \tau \tilde{\phi} = 0 \quad (58)$$

Employing the adiabatic condition from Eq. (4), the dispersion relation reduces to the simple expression

$$\omega^2 = (1 + \tau)(k_z^2 v_i^2 - k_z v_i \omega_{*v}) \quad (59)$$

B. Same system, but finite β

When we add the electromagnetic effects, the system of linear equations becomes considerably more involved:

$$\omega(1 + k_\perp^2 \rho_i^2) \tilde{n} + \omega k_\perp^2 \rho_i^2 \tau \tilde{\phi} = k_z v_i \tilde{v} + \omega_{*v} \tau \tilde{A}_\parallel \quad (60)$$

$$\omega \tilde{v}_z + [\omega_{*v} - (1 + \tau)k_z v_i] \tilde{n} + \omega_{*v} \tau \tilde{\phi} = 0 \quad (61)$$

$$\tilde{j}_\parallel = \frac{k_\perp^2 \rho_i^2 \tau}{\beta} \tilde{A}_\parallel \quad (62)$$

$$\omega \tilde{A}_\parallel = -k_z v_i (\tilde{n} - \tilde{\phi}) \quad (63)$$

$$\omega \rho_i^2 [k_\perp^2 \tilde{n} + \tau k_\perp^2 \tilde{\phi}] = k_z v_i \tilde{j}_\parallel + \omega_{*j} \tau \tilde{A}_\parallel \quad (64)$$

Note that the existence of electromagnetic effects impacts this system in three ways: the term $\omega_{*v} \tau \tilde{A}_\parallel$ which arises due to the parallel gradient of the parallel velocity; in the parallel electron response, which is no longer adiabatic; and in the vorticity equation, in which there is an added effect due to the radial gradient of the current. Solving this system yields a quartic equation in ω .

$$\begin{aligned} \beta \omega^4 - [(1 + (1 + \tau)(\beta + k_\perp^2 \rho_i^2))k_z^2 v_i^2 + \Omega_0 k_z v_i (1 + (1 + \tau)k_\perp^2 \rho_i^2) - \beta(1 + \tau)k_z v_i \omega_{*v}] \omega^2 \\ (1 + \tau)k_z^2 v_i^2 (k_z v_i - \omega_{*v})(k_z v_i + \Omega_0) = 0 \end{aligned} \quad (65)$$

where $\Omega_0 = \beta \omega_{*j} / k_\perp^2 \rho_i^2$. Since this system is actually quadratic in ω^2 , all four roots can be readily found analytically. In order to elucidate the real effect of finite β on this system, we will solve this system in two limits. We will assume throughout that $\beta \ll 1$, and that $k_\perp^2 \rho_i^2 \ll 1$ but place no restriction on the ordering of Ω_0 compared to the other frequencies.

1. *Ordering 1:* $\omega \sim k_z v_i \sim \omega_{*v} \sim \Omega_0$

In this limit, the quartic term can be neglected, thus leaving just only the quadratic and constant term. Remarkably, this limit exactly reproduces Eq. (23). The only contributions due to finite β are corrections to the growth rate of the order of β , which for edge plasmas are insignificant

2. *Ordering 2:* $\omega \sim k_z v_i / \sqrt{\beta}$ and $k_z v_i \sim \omega_{*v} \sim \Omega_0$

In this limit, the constant term in Eq. (29) can be neglected and we are left with the following expression:

$$\omega^2 = k_z^2 v_A^2 + \Omega_0 k_z v_i. \quad (66)$$

This instability is one that is driven purely by the radial gradient of the current, and the criteria for instability is that. The physics of this new mode has a number of parallels to the traditional v'_{\parallel} mode, yet there are some noted differences. The v'_{\parallel} mode exists when there is a parallel compression of the ions, and a corresponding fluctuation of the parallel velocity. Due to the adiabatic response of the electrons, the compression results in an electric field. With the proper poloidal wave structure, the electric field then produces a radial $E \times B$ drift. If there is a radial gradient in the parallel velocity, the faster/slower flowing plasma can convect and can enhance the fluctuation in the parallel velocity. At this simple level, the mode is stabilized by the sound wave, which flattens the original compression. Note that the mode is completely electrostatic.

The j'_{\parallel} mode, on the other hand, is electromagnetic, and involves more of an interaction of the fields. If there is a fluctuation of the electrostatic potential, there will be an ion polarization drift which will produce a net divergence of the perpendicular current. From the vorticity equation, which states that the total current in the quasineutral limit must be divergence-free, we can readily see that there will be two terms that will respond: the parallel compression of the parallel current, as well a variation of the magnetic field itself which is strictly an electromagnetic effect. Therefore, according to Ampère's Law, the current is produced by a fluctuating magnetic field, and from the electron force balance relation, the electrostatic potential will also produce a time varying magnetic field. Therefore, comparing this mode to the v'_{\parallel} mode, we have the following scenario. The j'_{\parallel} mode exists when there is parallel fluctuation in the parallel current. This fluctuation produces a fluctuation of the parallel component of the vector potential which in turn produces a fluctuation of the electrostatic potential. This potential produces a polarization drift, which in the presence of a radial gradient of the parallel current, can convect plasma so as to enhance the perturbation

of the parallel current. This mode is stabilized by the Alfvén wave, which flattens the magnetic perturbation.