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Quadratic Eigenvalue Problems

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Abstract

In this report we will describe some nonlinear eigenvalue problems that arise in the areas of solid mechanics, acoustics, and coupled structural acoustics. We will focus mostly on quadratic eigenvalue problems, which are a special case of nonlinear eigenvalue problems. Algorithms for solving the quadratic eigenvalue problem will be presented, along with some example calculations.

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1 Introduction

In many structural systems, linear eigenvalue analysis is adequate for predicting the modal response. In that case, standard techniques [13] can be used to compute the eigenvalues and eigenvectors. Furthermore, for a symmetric positive definite system, the resulting eigenvalues and eigenvectors are purely oscillatory [14].

However, in many cases a linear eigenvalue formulation is not a sufficient description. This is particularly true in the case of structures with physical damping (i.e. viscoelastic, hysteretic damping, etc), rotating structures, exterior acoustic fluids, and structures that interact with an acoustic fluid. In these cases, nonlinear eigenvalue formulations are required to correctly describe the vibration of the system. The resulting eigenvalues and eigenvectors are usually complex-valued, with the real part describing the damped part of the solution, and the imaginary part describing the oscillatory part. In these cases, special numerical methods are required to compute the complex-valued modes.

In this report we will describe typical nonlinear eigenvalue problems that arise in solid mechanics, acoustics, and coupled structural acoustics. We will focus mostly on quadratic eigenvalue problems, which are a special case of nonlinear eigenvalue problems. Algorithms for solving the quadratic eigenvalue problem will be presented, along with some example calculations.

The linear eigenvalue problem is usually written in the form

$$Ku + \lambda^2 Mu = 0 \tag{1}$$

where K is the tangent stiffness matrix, M is the mass matrix, u is the unknown eigenvector, and λ is the unknown eigenvalue. This eigenvalue formulation is adequate for structures that are stationary, with non-dissipative material models.

The next level of complexity is the quadratic eigenvalue problem, which is usually written in the form

$$Ku + \lambda^2 Mu + \lambda Cu = 0 \tag{2}$$

where C is an additional system matrix, usually referred to as the "damping" matrix. Examples where the quadratic eigenvalue problem is an adequate formulation include structures with proportional damping models, and acoustics with simple absorbing boundary conditions exterior boundary conditions. An interesting special case is when the damping matrix C is *gyroscopic*, or skew-symmetric. In this case, the system is purely conservative. Examples of this occur in rotating structures, and coupled structural acoustics with no exterior boundary conditions.

Both the linear and quadratic eigenvalue problems are special cases of the polynomial eigenvalue problem, which takes the form

$$(A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + ...\lambda^n A_n)u = 0 \tag{3}$$

where $A_0, A_1, ...A_n$ are the system matrices. With the proper choices of $A_0, A_1, ...A_n$, both the linear and quadratic eigenvalue problems can be recovered.

All of the previous formulations are special cases of the more general nonlinear eigenvalue problem

$$f(A_0, A_1, ...A_n, \lambda) = 0 \tag{4}$$

where, again, $A_0, A_1, ...A_n$ are the system matrices, and $f(A_0, A_1, ...A_n, \lambda)$ is some nonlinear function of these matrices, λ, and possibly other parameters as well. Viscoelastic materials, like rubber, require a nonlinear eigenvalue formulation to describe their vibration behaviour. This is because the stiffness properties are not constant (like in the case of the linear and quadratic eigenvalue problems), but instead are frequency-dependent.

In this report, we will focus on the quadratic eigenvalue problem. Specialized algorithms for this problem will be described. Algorithms for computing many of the modes with smallest absolute value are emphasized. To compute the smallest modes first, it is necessary to invert the eigenvalue problems, which involves solving linear systems.

An important property of different formulations of an eigenvalue problem is the character of the resulting linear systems. In particular, formulations that lead to unsymmetric linear systems are not pursued here. A new framework is contributed that make it possible to solve a category of QEPs using a scalable iterative linear solver for SPSD linear systems.

The classic approach to solving the eigenvalue problem is to shift and solve the corresponding linear systems (see [2], §9). The characteristics of linear systems depend on many things, including whether or not the damping matrix is symmetric. With general types of damping, C is unsymmetric and possibly complex. For example, for a rotating structure, C is skew-symmetric, and for coupled structural acoustics, C is also skew-symmetric.

If the damping matrix is symmetric positive semi-definite, then for positive shifts of the quadratic problem, the linear systems are SPD. Furthermore, Cayley transforms are practical (see [2], §9.2.3 page 286). Different formulations of the QEP are studied in [8] . In the Cayley formulation, the right-most eigenvalues are transformed to the largest eigenvalues, but the linear systems may be symmetric

indefinite depending on the transforms. On the other hand if C is unsymmetric, the the shifted linear systems are unsymmetric, and only trivial shifts are available.

Finally, using the algorithms developed in this paper, a number of example problems have been solved. These examples corresponded to several different types of physical damping that are common in engineering applications. These included proportional damping, dashpot elements, and coupled structural acoustics.

2 Formulations for Quadratic Eigenvalue Problems

A quadratic eigenvalue problem is equivalent to a generalized eigenvalue problem or pencil, (A,B). The equivalence transformation is called **linearization**. Four standard linearizations are reviewed in §2.1. For a QEP with some type of structure, different linearizations may also have a related structure. In §2.2, two families of linearizations are discussed that interpolate the four standard linearizations.

If A and B are symmetric and either A or B is positive definite, then (A,B) is a **symmetric definite pencil**. A symmetric definite pencil may be reduced to a symmetric EVP, and has real eigenvalues. Conversely a pencil with unreal eigenvalues is not symmetric definite. An example of a symmetric pencil that has unreal eigenvalues with B symmetric positive semi-definite may be found in [11] on page 48 and equation (75).

If A and B are symmetric and (A,B) is not a symmetric definite pencil, then (A,B) is a symmetric indefinite pencil. Such formulations of the QEP are very popular [10]. A feature of the symmetric indefinite pencil is that the left and right eigenvectors coincide. The primary problem with the symmetric indefinite formulation is that a nonsingular indefinite B defines an indefinite inner product. A B normalized eigenvector may have an arbitrarily large l_2 norm. Another problem with the symmetric indefinite formulation is that the null spaces of the matrices (A,B) may coincide.

In an unsymmetric definite pencil, A is unsymmetric and B is symmetric positive definite. The unsymmetric definite pencil is used here because it requires the solution only of symmetric positive semi-definite linear systems.

2.1 Standard Linearizations

The time-domain version of equation 2 is

$$M\ddot{u} + C\dot{u} + Ku = 0. \tag{5}$$

Since a high order differential equation is equivalent to at least one first order system of differential equations, we can reduce equation 5 to a first order system, which is then amenable to methods that have been developed for linear eigenvalue problems. Approaches I-IV below are the standard linearizations of the state space problem.

Approach I The second order equation equation (5) augmented with the identity $\dot{u} = \dot{u}$ leads to the system

$$\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix} \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I \\ -K & -C \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}. \tag{6}$$

Approach II A symmetric formulation may be derived by multiplying the first row of equation (6) by K and the second row by -1.

$$\begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & -M \end{bmatrix} \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & K \\ K & C \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}. \tag{7}$$

Approach III The construction in Approach I with the rows of the state vectors swapped leads to

$$\begin{bmatrix} \mathbf{0} & I \\ M & C \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -K \end{bmatrix} \begin{bmatrix} \dot{u} \\ u \end{bmatrix}. \tag{8}$$

Approach IV If C is symmetric, then equation (8) may also be symmetrized by multiplying equation (8) row one by M.

$$\begin{bmatrix} \mathbf{0} & M \\ M & C \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & -K \end{bmatrix} \begin{bmatrix} \dot{u} \\ u \end{bmatrix} \tag{9}$$

If the damping matrix C is symmetric, then equations (7) and (9) are symmetric pencils. The difference between a symmetric pencil and a symmetric definite pencil is not well known. A Lemma in the proof that any matrix is similar to a unique Jordan Canonical Form is that any matrix is a product of two symmetric matrices. In other words, any algebraic eigenvalue problem is equivalent to a symmetric pencil. In contrast, there exists a symmetric eigenvalue problem that is equivalent to a given symmetric definite eigenvalue problem.

Symmetric pencils do have at least three useful properties. First, they are perfectly scaled. In general, a symmetric indefinite linear solver may be applied to (A,B), instead of a general sparse linear solver. Third, the left and right eigenvectors coincide. Left eigenvectors may be used in the transient problem to project the initial conditions into the right eigenvectors, for sensitivity calculations. In the absence of symmetry, one may also project using a SVD.

2.2 More Linearizations

Approaches I-IV are instances of families of linearizations. We consider the following two families of linearizations (A,B), which are equivalent to the QEP. The companion linearization is for any nonsingular matrix N,

$$A = \begin{bmatrix} \mathbf{0} & N \\ -K & -C \end{bmatrix}, \quad B = \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}. \tag{10}$$

The justification that the families of linearizations here correspond to the QEP is in §2.3 and [9]. Another linearization is for any nonsingular matrix H,

$$A = \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & H \end{bmatrix}, \quad B = \begin{bmatrix} -C & -M \\ H & \mathbf{0} \end{bmatrix}. \tag{11}$$

Approaches I and III correspond to the linearizations in equation (10) with $N = I$ and equation (11) $H = -I$. If N and H are SPD matrices, then equations (10) and (11) are unsymmetric definite pencils. In finite precision arithmetic, using an extremely ill conditioned matrix occasionally leads to negative mass problems. For this reason, the unsymmetric definite formulation of (11) is avoided.

Approaches II and IV are the forms of the linearizations (10) and (11) that preserve symmetry of C. The Approaches correspond to $N = -K$ and $H = -M$ respectively. These are symmetric indefinite pencils. Similar choices of N and H preserve skew-symmetry of C (see §2.5, 4.4, 5.3).

2.3 One Eigenvalue Problem

The QEP admits the families of linearizations parameterized by nonsingular N and H in equations (10) and (11) respectively. For clarity, let's assume that K is symmetric positive definite. As will be demonstrated below, all of these pencils correspond to one reduced eigenvalue problem. A similar argument justifies the equivalence of equations (11) and (10).

Equation (11) for any nonsingular N admits the factorization

$$A = \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathbf{0} & I \\ -I & -C \end{bmatrix} \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}.$$

The term-wise inverse is

$$A^{-1} = \begin{bmatrix} K^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} -C & -I \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} N^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}.$$

In the reduced problem, N cancels out.

$$A^{-1}B = \begin{bmatrix} K^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} -C & -I \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix} = \begin{bmatrix} K^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} -C & -M \\ I & \mathbf{0} \end{bmatrix}$$

The similar argument runs back to equation (11).

In Arnoldi's method for the eigenvalue problem, the sequence of linear systems $z_i = A^{-1}Bz_{i-1}$ is solved. Only linear systems $Kx_i = b_i$ must be solved. For any nonsingular N or H, the linear systems are

$$\begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} z_i = \begin{bmatrix} -C & -M \\ I & \mathbf{0} \end{bmatrix} z_{i-1}$$

For each i, let $[x_i;y_i] := z_i$. Then x_i is the solution of $-Kx_i = Cx_{i-1} + My_{i-1}$ and $y_i = x_{i-1}$.

2.4 Formulations for Singular K

K may be symmetric positive semi-definite with known null space. In such cases an orthonormal matrix Φ may be constructed such that $K\Phi = 0$. Here we discuss different options that are available if K is singular.

The established algorithms for singular K all involve nontrivial shifts. As mentioned in § 1, there are two ways to think about shifts. One may either linearize and transform the pencil, or transform the QEP and linearize. The transformation may be either a shift (translation) or a Cayley (fractional linear) transformation. If C is unsymmetric, none of the formulations (with nontrivial shifts) generate either SPD or SPSD linear systems.

Shift invert formulations with zero shift may be solved as long as the eigenvalue problem and linear systems remain consistent. Consistency is maintained using deflation (see §4).

2.5 Conservative gyroscopic systems

A system in which the coefficient matrix of the velocity vector is skew-symmetric,

$$M\ddot{u} + G\dot{u} + Ku = f(t)$$

is called a **conservative gyroscopic system**. A spinning structure, [14], is an example of a conservative gyroscopic system. An additional example of a conservative

gyroscopic system is a structural acoustics model that neglects the damping mechanisms in the structure and fluid (e.g. inviscid fluid and no exterior boundaries, [6, 7, 15]). In the latter case the gyroscopic matrix represents a coupling between the fluid and solid, but involves no energy dissipation.

Although C appears to be a 'damping' term, the system 2.5 has no damping. In the associated eigenvalue problem, the eigenvalues are all oscillatory (see [14] or [8] for details). There is no physical dissipation in the system, and thus the term conservative.

3 Accuracy

Inaccurate solutions of the QEP are easy to compute and misleading. One of our goals is to develop scalable algorithms and software for complex problems. The scalable algorithms involve inexact solutions to linear systems and eigenvalue problems. This makes finding a good solution to the accuracy problem a requirement. Accuracy is improved in three ways.

First the QEP is scaled to reduce its condition number. In [8] Theorem 7 is the following: If $\|M\| = \|C\| = \|K\| = 1$, then solving the companion matrix linearization with a backward stable algorithm for pencils is backward stable for the QEP. Both the matrices and the eigenvalues are scaled: $K \rightarrow DKD$ and $\lambda \rightarrow \lambda/\mu$. The scaling makes it possible to transform K and M to matrices of unit norm. The norm of the damping matrix changes. The authors numerical results without scaling are inaccurate. With scaling, small residuals imply convergence. Furthermore, the residuals for the scaled problem are comparable the linear system relative residual norms.

It has been observed that in practice, for the scaled problem, $\|C\|$ is smaller than one. §3.1 contains a demonstration that with certain types of damping, if the eigenvalue problem has no real eigenvalues, $\|C\|$ is small.

Second, the null space is deflated from the pencil using a representation based on a Generalized Schur form. The computation is done using orthogonal (symmetric) projections. Oblique projections are used to distinguish whether a null vector of K corresponds to a double root of the pencil, or the null vector corresponds to one zero root and one very small root of the pencil. Here the potential ill conditioning is due to ill conditioning of the QEP. Initial Arnoldi vectors are chosen to mollify the impact of the oblique projections.

Lastly, the eigenvector for the QEP is extracted from the linearization in a way that minimizes the residual norms. The QEP is linearized, and pencil is of twice the order. Both of the components of the eigenvector of the pencil correspond to the same eigenvector of the QEP. A SVD is used to determine the optimal approximate eigenvector from the subspace. If the damping matrix is real, then the eigenvalues are computed in complex conjugate pairs, and this too is exploited.

3.1 Scaling and Lightly Damped Systems

The scaling process has two steps. Step one is a diagonal matrix congruence,

$$(K, C, M) \rightarrow (DKD, DCD, DMD),$$

such that DKD is unit diagonal. Step one is a scaling $(K,C,M),\lambda \rightarrow (K\kappa, C\kappa\mu, M\kappa\mu^2),\lambda/\mu$ for

$$\kappa = 1/\|K\| \quad \mu = \sqrt{\|K\|/\|M\|}.$$

Rayleigh Damping With Rayleigh damping such that there are no real eigenvalues, the norm of the scaled damping matrix is always small.

First, we consider Rayleigh damping. In this case, we start with equation 2, with Rayleigh damping, $C = Ka + Mb$, where a and b are nonnegative numbers less than one. Furthermore, we suppose that the undamped modes, $K\phi = M\phi\omega^2$, satisfy $\omega_{\min} \leq \omega \leq \omega_{\max}$.

All of the eigenvalues of the damped problem are complex if $a < 2/\omega_{\max}$ and $b < 2\omega_{\min}$. Using the definition of C in the quadratic eigenvalue problem, and collecting terms, we see that the eigenvalues of the undamped and damped problems are related through

$$\omega^2 = -(\lambda^2 + be)/(a\lambda + 1)$$

Equivalently

$$\lambda^2 + (a\omega^2 + b)\lambda + \omega^2 = 0.$$

For the modes to be unreal, the discriminant must be negative:

$$0 > (a\omega^2 + b)^2 - 4\omega^2 = ((a\omega^2 + b) - 2\omega)((a\omega^2 + b) + 2\omega)$$

or more simply $0 > a\omega^2 + b - 2\omega$. This implies that

$$b/(1 + \sqrt{1 - ab}) < \omega < (1 + \sqrt{1 - ab})/a.$$

The above expression comes from the approximating $1 + \sqrt{1 - ab}$ by 2.

Second, in the scaled quadratic eigenvalue problem, the norm of the damping matrix is small. If the quadratic eigenvalue problem is scaled

$$K_s = K\kappa, C_s = C\kappa\mu, M_s = M\kappa\mu^2,$$

so that $\kappa = 1/\|K\|$ and $\mu = \sqrt{\|K\|/\|M\|}$.

$$\|C_s\| = \|C\mu\kappa\| = \|K\mu\kappa a + M\mu\kappa b\| \leq$$

$$\|K\mu\kappa a\| + \|M\mu\kappa b\| =$$

$$\|K\|\mu\kappa a + \|M\|\mu\kappa b =$$

$$a\sqrt{\|K\|/\|M\|} + b\sqrt{\|M\|/\|K\|} \geq 2\sqrt{ab}.$$

More useful is an upper bound.

We can show that the scaled damping matrix is not large if the eigenvalue problem is scaled so that the mass matrix is well conditioned. Specifically,

$$\|C_s\| \leq 2 + 2\sqrt{\text{cond}(M)}\ \ \omega_{\min}/\omega_{\max}.$$

To prove the claim, we use upper and lower bounds on $\|K\|/\|M\|$. We denote by Ω the diagonal matrix of undamped modes. The generalized eigenvalue problem is equivalent via the Cholesky factorization $M = R^T R$ to the matrix $A = R^{-T} K R^{-1}$ with eigenvalues Ω^2. Expressing K interms of A and R, $K = R'AR$, take norms, and expand:

$$\|K\| = \|R^T AR\| \leq \|R^T\|\ \ \|A\|\ \ \|R\| = \|R\|^2\|\Omega\|^2 = \|M\|\ \ \|\Omega\|^2.$$

Or $\|K\|/\|M\| \leq \|\Omega\|^2$. Similarly for the lower bound,

$$\|\Omega^2\| = \|A\| = \|R^{-T} K R^{-1}\| \leq \|R^{-T}\|\ \ \|K\|\ \ \|R^{-1}\| =$$

$$= \|R^{-1}\|^2\|K\| = \|M^{-1}\|\|K\| = \text{cond}(M)\|K\|/\|M\|.$$

Or

$$\|M\|/\|K\| \leq \text{cond}(M)/\|\Omega\|^2. \tag{12}$$

Substituting the upper bounds in

$$\|C_s\| \leq a\sqrt{\|K\|/\|M\|} + b\sqrt{\|M\|/\|K\|}$$

to find

$$\|C_s\| \leq a\|\Omega\| + b\sqrt{\text{cond}(M)}/\|\Omega\|.$$

Now we substitute the estimates of a and b from the earlier argument: $a\|\Omega\| < 2$ and

$$\sqrt{\text{cond}(M)}b/\|\Omega\| < 2\sqrt{\text{cond}(M)}\omega_{\min}/\omega_{\max}.$$

Add the two terms to establish the claim.

Modal Damping Modal damping is commonly a used approach for introducing damping into the structure. Here again assuming that the damping is light enough all the modes are unreal, and that the condition number of the mass matrix is less than the maximum ratio of eigenvalues, then the scaled damping matrix has small norm. Modal damping refers to

$$C = 2M\Phi\Omega\Gamma\Phi^T M$$

for a diagonal matrix Γ. The diagonal entries (γ_j) are user specified parameters. The eigenvectors of the undamped problem are mass normalized as usual.

Modal damping changes the eigenvalues but not the eigenvectors,

$$(K + C\lambda + M\lambda^2)\phi_j = \phi_j \;\; (\omega_j^2 + 2\gamma_j\omega_j\lambda + \lambda^2)$$

Modal damping changes the mode $i\omega_j$ to

$$\omega_j(i\sqrt{1-\gamma_j^2} + \gamma_j)$$

The eigenvalues are in the left half plane iff $\Gamma \geq 0$, and the eigenvalues are unreal iff $\Gamma < 1$.

Now we argue, as for Rayleigh damping, that

$$\|M\Phi\Omega\Gamma\Phi^T M\| \leq \|M\| \;\; \|\Omega\Gamma\|$$

For the scaled damping matrix C_s, by equation (12),

$$\|C_s\| \leq 2\|\Omega\Gamma\|\sqrt{\|M\|/\|K\|} \leq 2\|\Omega\Gamma\|\sqrt{\text{cond}(M)}/\|\Omega\|$$

In practice, only the mode shapes for the smallest modes are computable, $\|\Omega\Gamma\| \approx \omega_{\min}$, and the bound is similar to the bound for Rayleigh damping.

3.2 Perturbation Theory

The tools developed in this section are useful for showing that, without scaling, the computed solutions of the QEP are highly inaccurate. Differential perturbation theory will be used to compute reference solutions to the eigenvalue problem. Here we assume that it is possible to factor the shifted problem. Continuation from the modes for the undamped problem is used, with the continuation parameter being γ. Refinement is also used ($\gamma = 1$).

We define $Q(,)$ and $F(,,)$ by

$$Q(\lambda,\gamma) = K + C\lambda\gamma + M\lambda^2, \quad F(x,\lambda,\gamma) = Q(\lambda,\gamma)x.$$

The partial derivatives of Q are

$$\partial_\gamma Q = C\lambda, \quad \partial_\lambda Q = C + M\lambda 2.$$

The first order differential of F is

$$F(x+dx,\lambda+d\lambda,\gamma+d\gamma) = F + Qdx + \partial_\lambda Q d\lambda + \partial_\gamma Q d\gamma$$

The standard normalization of $\frac{dx}{d\gamma}$ is $x^*\frac{dx}{d\gamma} = 0$. The differentials are related by

$$\begin{bmatrix} Q & \partial_\lambda Qx \\ x^* & 0 \end{bmatrix} \begin{bmatrix} \frac{dx}{d\gamma} \\ \frac{d\lambda}{d\gamma} \end{bmatrix} = \begin{bmatrix} -\partial_\gamma Qx \\ 0 \end{bmatrix}$$

For refinement ($\gamma = 1$), in the right-hand side replace $-\partial_\gamma Qx$ by $-F$.

If λ is a multiple eigenvalue, then the linear system is singular.

4 Null Space Deflation

The problem of SPSD K with known null space Φ is addressed using deflation. Deflation refers to removing eigenvectors from the problem as they converge. The smallest modes tend to converge first in inverse iteration. In linear problems, delfation transforms the problem so that the smallest unconverged eigenvalues tend to converge next.

4.1 Deflation for (K,M)

A less complicated example follows that illustrates deflation. Deflation is used for the symmetric definite pencil (K,M) if K is singular. The null space of K, the rigid body modes, are determined accurately by the geometry (coordinates) and the partial differential equation. It is possible to deflate the invariant subspace of rigid body modes using an matrix of the form K^+M. Restarts are implemented by deflating a computed invariant subspace.

Recall the generalized symmetric eigenvalue problem with decomposition is $K\Phi = M\Phi\Omega^2$ and $\Phi^T M\Phi = I$. In exact arithmetic the Lanczos algorithm determines a M-orthogonal V and a symmetric tridiagonal T such that $K^{-1}MV = VT$ via the three term recurrence

$$v_{k+1}\beta_{k+1} = K^{-1}Mv_k - v_k\alpha_k - v_{k-1}\beta_k.$$

The eigenvalue decomposition $TW = W\Omega^{-2}$ determines the eigenvalues and eigenvectors $\Phi = VW$.

Given an M-orthogonal basis for the null space of K, the Lanczos algorithm proceeds with a specific definition of K^+. The proof that $V_k^T M\Phi_1 = 0$ by induction on k explains the issues. The consistency of linear system that defines v_2 is guaranteed by multiplying the initial Lanczos vector by $I - \Phi_1\Phi_1^T M$. Suppose that $V_k^T M\Phi_1 = 0$ for an arbitrary $k \geq 1$. By construction $Ku_k = Mv_k$ is consistent. The least squares solution is characterized by $\Phi_1^T u_k = 0$. In contrast the Lanczos algorithm ensures that $\Phi^T Mv_{k+1} = 0$, by proceeding with

$$v_{k+1}\beta_{k+1} = (I - \Phi_1\Phi_1^T M)u_k. \tag{13}$$

Eigenvalue problem restarts are implemented in the same way. Any M-orthogonal subspace Φ_1 that is invariant, $K\Phi_1 = M\Phi_1\Omega_1^2$, is deflated using equation (13).

4.2 Null Space Deflation in QEP

The companion linearization, (6), leads to the pencil (A,B) for

$$A=\begin{bmatrix} \mathbf{0} & I \\ -K & -C \end{bmatrix}, \quad B=\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}. \tag{14}$$

Deflation in the pencil is based on the generalized Schur form. A generalized Schur form of (A,B) is

$$(A,B) = Z(T_A,T_B)Q^* \quad Z,Q \text{ unitary}, \;\; T_A,T_B \text{ triangular.}$$

Φ spans the null space of K and is orthonormal. For

$$\Psi=\begin{bmatrix} \Phi \\ \mathbf{0} \end{bmatrix} \quad \tilde{\Psi}=\begin{bmatrix} \mathbf{0} \\ \Phi \end{bmatrix}$$

there holds

$$A\Psi=\mathbf{0},\; \Psi^T A=\tilde{\Psi}^T,\; B\Psi=\Psi,\; \Psi^T B=\Psi^T.$$

This suggests that it may be possible for the generalized Schur form to use orthogonal $[\Psi,Z]$ and $[\Psi,Q]$. The potential real Schur form is

$$A=[\Psi,Z]\begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}[\Psi,Q]^T \quad B=[\Psi,Z]\begin{bmatrix} B_{11} & B_{12} \\ \mathbf{0} & B_{22} \end{bmatrix}[\Psi,Q]^T$$

If this holds, then

$$A_{11}=\mathbf{0},\; \Psi A_{12}Q^T=\Psi\tilde{\Psi}^T,\; B_{11}=I,\; B_{12}=\mathbf{0}.$$

The reason for mentioning generalized Schur forms forms is that the previous equation is equivalent to

$$A=\Psi\tilde{\Psi}^T+ZA_{22}Q^T, \quad B=\Psi\Psi^T+ZB_{22}Q^T$$

The pencil $(ZA_{22}Q^T, ZB_{22}Q^T)$ is unsymmetric definite over $\Psi^{\perp}$. Computations are done in the original coordinate system to preserve sparsity.

For $\Pi = I - \Phi\Phi^T$, the deflated pencil is

$$A_d = \begin{bmatrix} \mathbf{0} & \Pi \\ -K & -C \end{bmatrix}, \quad B_d = \begin{bmatrix} \Pi & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}. \tag{15}$$

A solution of the QEP corresponds to a solution of the pencil by construction, but not conversely. Suppose that

$$(A_d - B_d\lambda) \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}.$$

If $\lambda \neq 0$, then λ and y solve the QEP. The equation $\Pi y = \Pi x\lambda$ indicates that in practice the eigenvector that corresponds to λ will be in the span of $[x, y, \Phi]$. The case $\lambda = 0$ is subtle. If $C\Phi = \mathbf{0}$, then $A_d\tilde{\Psi} = \mathbf{0}$. If C is skew, then

$$A_d \begin{bmatrix} K^+C\Phi \\ -\Phi \end{bmatrix} = \mathbf{0}.$$

In addition to the space spanned by the columns of Ψ, the null space of A_d contains all $y = \Phi a$ for which there exists an x such that $Kx + Cy = \mathbf{0}$. The dimension of the set of eigenvectors of the deflated problem is

$$\text{nullity}(A_d) = \text{nullity}(K) + \text{nullity}(\Phi^T C\Phi).$$

If $\Phi^T C\Phi$ is singular, then some zero eigenvectors of the deflated problem are not zero eigenvectors of the QEP.

The intersection the null spaces of K and C are the internally damped rigid body modes. Other rigid body modes are externally damped. The externally damped rigid body modes may have gyroscopic parts R, such that $R'CR = 0$. The three kinds of linear damping, internal, external gyroscopic and external non-gyroscopic are each treated differently.

4.3 ARPACK Operations

The sparse matrix algorithm applies ARPACK with SPSD mass matrix B_d. Multiplication by B_d includes a redundant projection. The additional projection reduces the inconsistency of linear systems in finite precision arithmetic.

Arnoldi's method: $A_dVH = B_dV$ and $V^TB_dV = I$

1. Find u: $A_d u = B_d v_{j+1}$ and $\Phi^T u = 0$.
2. $H(1:j,j) = V_j^T B_d u$
3. $u = u - V_j H(1:j,j)$
4. $v_{j+1} H(j+1,j) = u$

A sequence of linear systems of the form

$$\begin{bmatrix} \mathbf{0} & \Pi \\ -K & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \quad \phi^T b = \phi^T x = \mathbf{0}.$$

are solved. $-Ku_i = Cx_{i-1} + My_{i-1}$ is solved such that $\Phi^T u_i = 0$.

If $\Phi^T C \Phi$ is well conditioned, then $x_i = u_i + \Phi^T v_i$. The v_{i-1} term ensures that the linear system that defines x_i is consistent,

$$(\Phi^T C \Phi) v_i = -\Phi^T C u_i - \Phi^T M y_i.$$

In general $\Phi^T y$ is nonzero. The component of y along Φ is determined by the condition that x exists.

- $y := b$
- $a := Cy + c$
- Repeat twice
  1. $d := \Phi^T a$
  2. $e := \Phi d$
  3. $f := Ce$
  4. $y := y - e$
  5. $a := a - f$
- Solve $Kx = -a$

The right-hand side for the linear system is computed using an oblique projection with condition number $\|C\|/\lambda_{\min}(\Phi^T C \Phi)$.

The first author has found that in his experience, for a problem when $\Phi^T C \Phi$ is well conditioned, an initial Arnoldi vector from the space spanned by $[0;\Phi]$ is preferable. C is usually small in norm, and $[0;\Phi]$ is near to a null vector of A_d,

$$A_d \begin{bmatrix} K^+ C \Phi \\ -\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \Phi^T C \Phi \end{bmatrix}.$$

Iterative methods for eigenvalue problems amplify the displacements in the direction of the eigenvectors of the eigenvalues with smallest modulus. In the sequence of linear systems $Kx_i = b_i$, $\|\Phi^T b_i\|$ grows exponentially with i without reorthogonalization. One motivation for using the B_d inner product is that it provides an extra opportunity to orthogonalize the Arnoldi vectors to Ψ. ARPACK does not provide users access to the Arnoldi vectors, but, as it happens, the current Arnoldi vector is actually stored in the `resid` vector.

4.4 Deflation with gyroscopic damping

In a conservative gyroscopic system (§2.5) C is skew-symmetric and $\Phi^T C\Phi = \mathbf{0}$. The implementation of the deflation differs from (§4.3). The details of deflation in a conservative gyroscopic system are worked out under the assumption that

$$\text{rank}(C\Phi) = \text{rank}(\Phi).$$

The null space of A_d (c.f. equation (15)) contains a second set of left and right vectors

$$L_2 = \begin{bmatrix} C^T\Phi \\ \Phi \end{bmatrix}, \quad R_2 = \begin{bmatrix} K^+C\Phi \\ -\Phi \end{bmatrix}.$$

A conservative gyroscopic system admits a generalized Schur form (see §4.2)

$$Q^H AZ = [0, 0, *] \quad Q^H BZ = \begin{bmatrix} I & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

Alas, the additional null vectors of A_d are not eigenvectors of B.

To preserve sparsity, it is necessary to use instead an oblique projection. Multiplying out terms shows that

$$L_2^T BR_2 = \Phi^T CK^+C\Phi - \Phi^T M\Phi.$$

Note that $-L_2^T BR_2$ is SPD.

The oblique projection $I - R_2(L_2^T B_d R_2)^{-1} L_2^T B_d$ is applied to Arnoldi vectors along with Π. The oblique projection may be ill conditioned. In the authors experience, the condition number of the projection has never been larger than four. A user would not know about the projection, but might observe that Arnoldi's method converges slowly or that the eigenvalues are inaccurate. It is easy to check the condition

number of the oblique projection. A user must be prepared for the extremely remote possibility that a simulation is not possible due to an ill conditioned projection.

Another significant implementation issue is the selection of the initial Arnoldi vector. The goal is to choose v_1 so that $L_2^T B_d A_d^{-1} B_d v_i$ is very small. As $A_d^T v_1$ approaches $B_d L_2$, the B_d-orthogonality of the Arnoldi vectors meets the goal. For simplicity, assume that Φ has rank one. For some scalar α determined by normalization, $v_1 \alpha = [X;Y]$ for X and Y determined as follows.

1. Find G such that $(R_2^T R_2)G = L_2^T B R_2$.
2. Find Y such that $KY = K^+ C\Phi G - C^T\Phi$.
3. Find J such that $(\Phi^T C C^T \Phi)J = \Phi^T C M \Phi + \Phi^T M Y$.
4. $F = C^T \Phi J$.
5. $X = \Pi M \Phi - F$

The correction term F ensures that v_2 exists, $\Phi^T C X + \Phi^T M Y = 0$.

4.5 Deflation with internal damping

If in a conservative gyroscopic system $\text{rank}(C\Phi) < \text{rank}(\Phi)$, the construction of §4.4 still works. In the authors experience though, the only case in which $\text{rank}(C\Phi) < \text{rank}(\Phi)$ is the internally damped case, $C\Phi = 0$. The much simpler choice $L_2 = R_2 = \tilde{\Psi}$ leads to a projection that is symmetric with respect to M. Performance is less sensitive to the choice of initial vector.

5 Numerical Examples

In this section we present results from several examples that were designed to test the algorithms presented in the preceding sections. All of the examples have a damping matrix, and the main interest is how the eigenvalues are changed by the presence of the damping. The examples are proportional damping, dashpot damping, and coupled structural acoustics.

5.1 Proportional Damping

For the one dimensional case, the exact solution for an elastic bar that is given stiffness or mass proportional damping follows. After that, a more general procedure is presented that allows one to compute the complex eigenvalues resulting from a proportionally damped structure, given a closed form expression for the oscillatory eigenvalues of the undamped structure.

Stiffness Proportional Damping We follow a similar derivation as was given in [5]. In order to facilitate the derivation, we define a complex modulus as

$$E^* = E(1+\beta s)$$

where E is the elastic modulus, β is the stiffness proportional damping coefficient, and s is the complex-valued eigenvalue. The equations of motion of the bar are

$$-E^* u_{xx} + \rho s^2 u = 0$$

where ρ is the density, and the ends of the bar are subjected to homogeneous boundary conditions at $x = 0$ and $x = L$

$$u'(0) = 0 \tag{16}$$
$$u'(L) = 0 \tag{17}$$
$$\tag{18}$$

We note that the system defined above with complex modulus is equivalent to a system with real-valued modulus but with added stiffness proportional damping.

The general solution to these equations of motion is

$$u = Ae^{\sqrt{\frac{\rho}{E^*}}sx} + Be^{-\sqrt{\frac{\rho}{E^*}}sx}$$

where A and B are arbitrary constants. The boundary condition $u^{'}(0) = 0$ leads to the condition $A = B$. Applying the second boundary condition, $u^{'}(L) = 0$, leads to the condition

$$\sqrt{\frac{\rho}{E^*}} sL = in\pi$$

or

$$s_n = \frac{in\pi}{L}\sqrt{\frac{E^*}{\rho}} \tag{19}$$

$$= \frac{in\pi}{L}\sqrt{\frac{E(1+\beta s_n)}{\rho}} \tag{20}$$

$$\tag{21}$$

This equation can be solved exactly to obtain the exact complex-valued frequencies of the system with stiffness proportional damping.

Figure 1 shows a comparison between the exact and computed eigenvalues for the case of stiffness proportional damping. Two mesh refinements were performed on the original mesh in order to confirm the convergence of the computed eigenvalues to the exact ones. As seen in Figure 1, the computed eigenvalues converge to the exact ones.

Mass Proportional Damping In this case, the equations of motion are

$$-Eu_{xx} + \alpha\rho s u + \rho s^2 u = -Eu_{xx} + \rho s\left[\alpha + s\right]u = 0$$

where α is the amount of mass proportional damping, and E is a *real-valued* elastic modulus, in contrast to the complex-valued one in the case of stiffness proportional damping.

The general solution to this equation is

$$u = Ae^{\sqrt{\frac{\rho s[\alpha+s]}{E}}sx} + Be^{-\sqrt{\frac{\rho s[\alpha+s]}{E}}sx}$$

Applying the boundary conditions in the same way as before leads to the condition $A = B$, and the condition

$$s_n = = \frac{in\pi}{L}\sqrt{\frac{E}{\rho s\left[\alpha+s\right]}} \tag{22}$$

$$\tag{23}$$

This is a different nonlinear equation that may be solved to obtain the exact complex-valued eigenvalues of the system with mass proportional damping.

For combined mass and stiffness proportional damping, the same procedures can be followed to define the following nonlinear equation for the complex eigenvalues

$$s_n = \quad = \quad \frac{in\pi}{L}\sqrt{\frac{E(1+\beta s_n)}{\rho s\,[\alpha + s]}} \tag{24}$$

$$\tag{25}$$

When solving this nonlinear equation, it is necessary to take the square root of complex numbers. This can be done as follows. If $z = re^{i\theta}$ is a complex number, then

$$z^{\frac{1}{2}} = r^{\frac{1}{2}}\left[\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right]$$

5.2 Dashpot

The dash pot example uses the same mesh and geometry and the proportional damping example. The two differences in this case are that there is no proportional damping, and that dashpots are added to the model.

The previous example showed that proportional damping imparted a real component on *all* of the eigenvalues, though higher frequency modes received larger real components than the lower frequency modes. The dashpots provide a more nonuniform type of damping, as shown by the results in Figure 2, in that only some of the modes are affected by the dashpot. Others only receive a very small real component in the eigenvalues. This can be explained by examining the mode shapes. It is seen that some modes do not involve extension along the points connecting the dashpots, and thus in this case the dashpots do not introduce dissipation into these modes.

5.3 Coupled Structural Acoustics

In this example, the modes of a coupled structural acoustic system are computed. The problem consists of a steel bar next to a tube filled with air. The system is closed at both ends with no physical mechanisms of energy dissipation, and thus

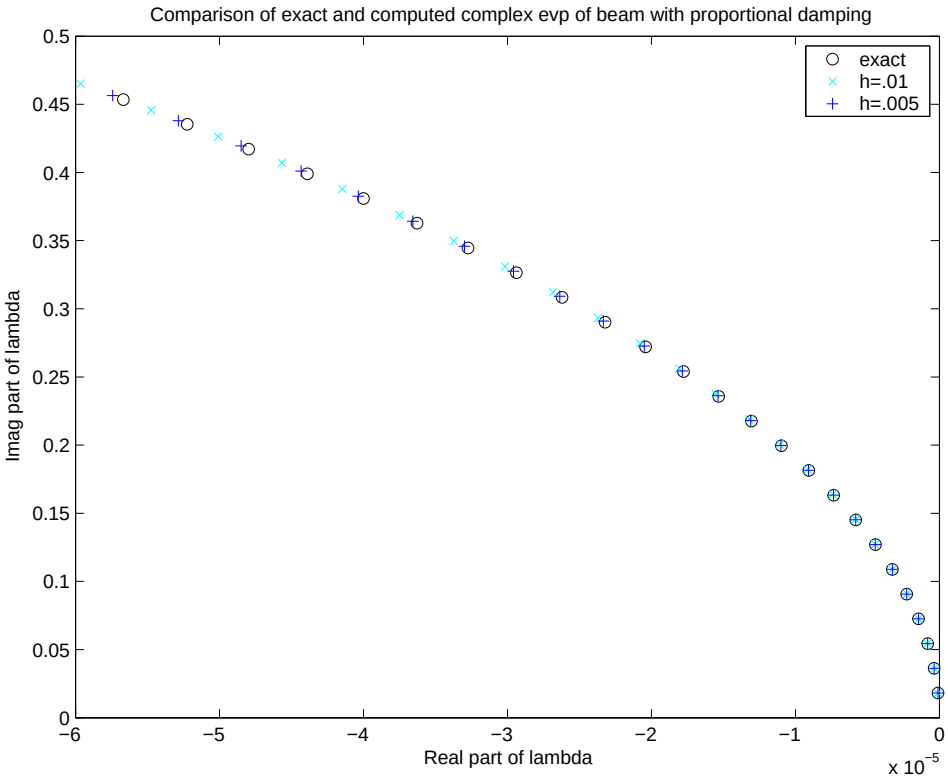

Figure 1. Convergence of complex eigenvalues for a beam with stiffness proportional damping, with respect to spacial discretization.

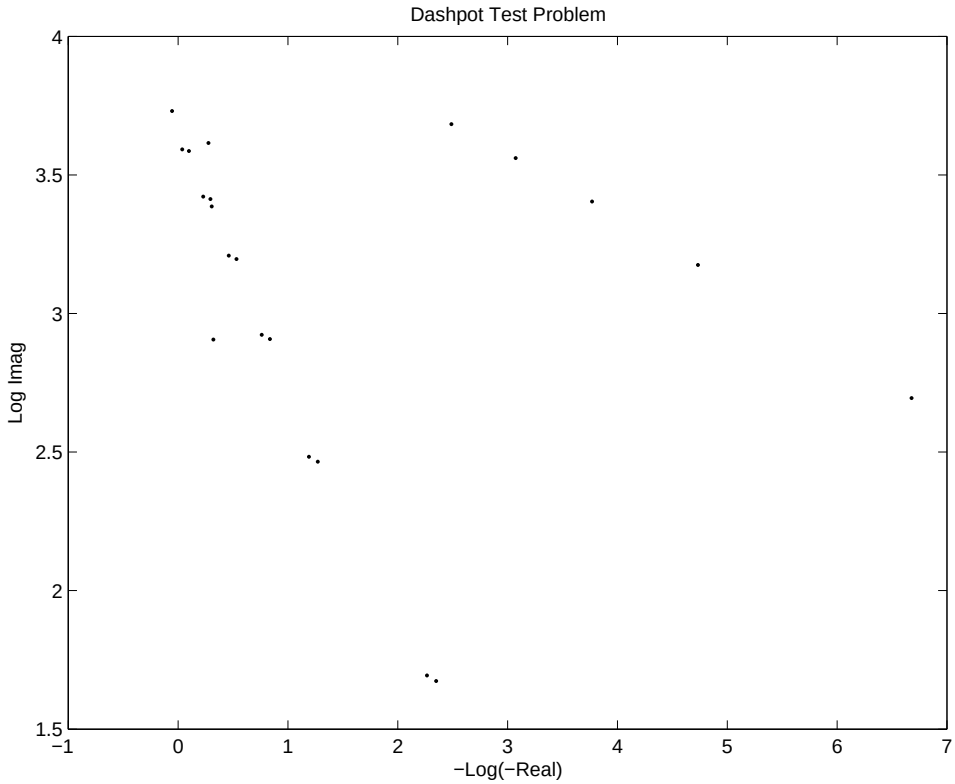

Figure 2. Complex spectrum for coupled dashpot-beam problem.

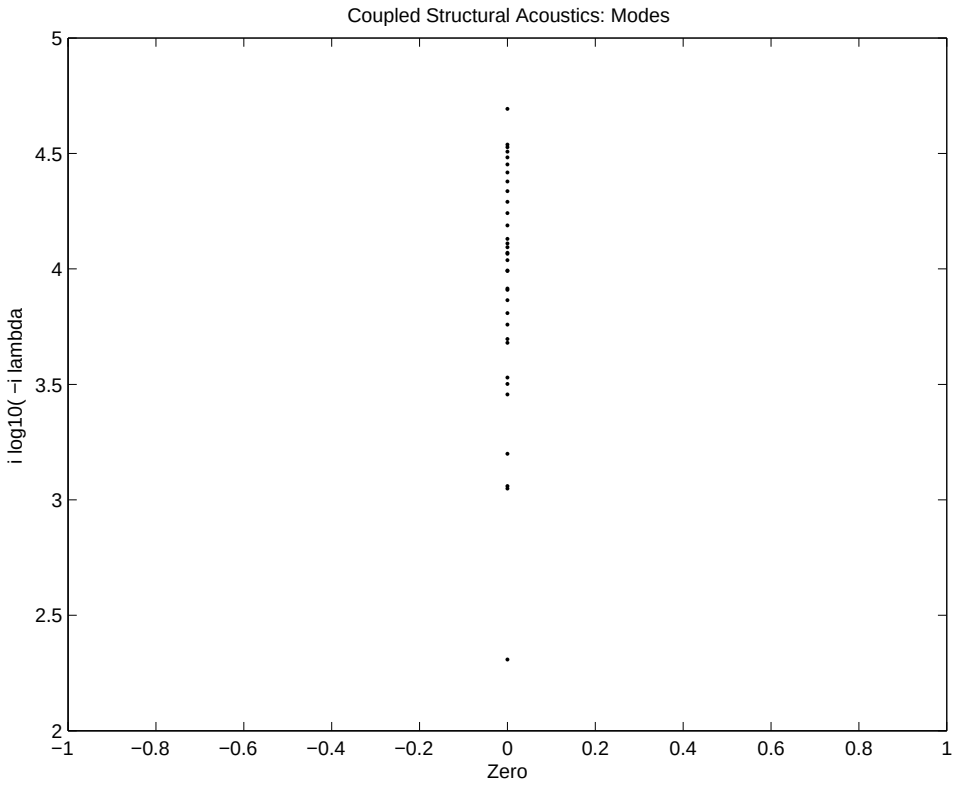

Figure 3. Spectrum for coupled structure acoustic problem. This problem has an interior fluid, but no exterior boundary conditions.

all of the eigenvalues are imaginary, with no real components. The semidiscrete equations of motion follow.

$$\begin{bmatrix} M_s & 0 \\ 0 & M_a \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} C_s & L \\ -L^T & C_a \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix} \begin{bmatrix} u \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

where subscripts a and s refer to acoustic and solid, respectively, and the matrices L and L^T arise from fluid-structure coupling.

Figure 3 shows the computed spectrum in this case. As expected, the eigenvalues are purely imaginary.

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