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Final Report on subcontract B551021: Optimal AMG interpolation and Convergence theory

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Optimal AMG interpolation and Convergence theory
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The goal of this project is to implement and study various techniques for the construction of Algebraic Multigrid Methods (AMG) for the solution of positive definite linear systems arising from the discretizations of elliptic partial differential equations (PDEs). Both theoretical as well as practical implementation of the methods that we have developed are based on compatible relaxation and energy minimization. The results that we have obtained are reported in several publications resulting from the subcontract (references [1]-[7]).

In [1] we have proposed an adaptive algebraic multigrid that uses compatible relaxation technique to adaptively construct the set of coarse variables. The nonzero supports for the coarse-space basis is then determined by approximation of the so-called "ideal" two-level interpolation operator that gives maximal error reduction per iteration. This approximation uses coarse grid basis, obtained by minimizing a modified Frobenius norm of the coarse-level operator. Our algorithm maintains multigrid-like optimality, without the need for parameter tuning. We have shown the efficacy and the robustness of our approach on several test problems.

In a joint work with Vassilevski [2] we have designed an algorithm for simultaneous approximation of several near null space components, which allows for more aggressive coarsening, and can be applied not only to second order PDEs, but also to higher order (such as discretizations of biharmonic equation) to produce highly accurate interpolation operators and hence efficient multilevel methods.

We have also studied optimal preconditioners for discontinuous Galerkin (DG) methods in [3]. The DG methods have many attractive features, such as being locally conservative and in many cases more accurate that the conventional conforming methods. A drawback is that the number of unknowns in the resulting discrete system of equations is much larger than the number of unknowns for a conforming method. We have developed a unified approach for preconditioning DG methods with the simplest possible discretizations, such as piece-wise constant elements and conforming piece-wise linear elements. The family of preconditioners, that we have proposed and analyzed is based on two level approach and leads to a significant reduction of the problem size. Moreover, our approach leads to coarse grid problems that correspond to conforming methods, thus making possible the use of already developed efficient multilevel methods for their solution.

We have also worked on the iterative methods for singular, nearly singular and semidefinite linear systems ([4] and [6]). The known analysis and convergence results of such methods has been largely based on the sufficient conditions known as P-regularity (or weak regularity) of the corresponding matrix splittings. We have introduced new, more refined conditions that are not only sufficient, but also necessary for energy norm convergence. These conditions basically amount to assuming that the error is damped in a subspace (related to the pseudo-inverse of the approximating matrix), and that the range of the original matrix is contained in the range of the approximating one. Such assumptions are naturally true for most iterative methods, including multigrid and domain decomposition methods. In [6] we have carried over these results to the case of nearly singular systems and we have shown how the techniques developed can be utilized in the analysis of iterative methods for linear systems corresponding to the augmented Lagrangian methods applied to PDEs discretized by mixed and hybrid finite element methods.

In [5] we proposed a new method for the efficient solution of linear systems arising in discretizations of second order elliptic PDEs by a generalized finite element method (GFEM). This has been an open question for some time (last 5-6 years, since GFEM gained popularity), and we were able to deliver solution to it and in particular relate

GFEM discretizations for scalar equations to the discretizations of systems of PDE. The efficient preconditioner that we propose, uses auxiliary (fictitious) space techniques and an additive BPX (Bramble, Pasciak and Xu~1988) preconditioner for the problems in auxiliary space.

The work on Helmholtz equation [7] can be viewed as a preliminary study towards targeting more complicated problems, such as indefinite Maxwell equations. The exact controllability framework (Due R. Glowinski and J. Lions) is based on recasting the Helmholtz equation in its original form (periodic solutions in time of a wave equation). Such solutions are found by minimizing a convex functional using a continuous version of the Conjugate Gradient (CG) algorithm. On every CG iteration it is required to solve a scalar second order PDE (like Laplace equation). The new technique, proposed by us is to use approximate solution instead, provided by few algebraic multigrid V-cycle iterations. Same procedure carries over to the solution of the indefinite Maxwell equations, and our goal is to further implement the inexact CG and the controllability methods to the indefinite Maxwell equation for wide range of wave numbers.

References:

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