

# **First-Order Space-Charge Tuneshift in a Linearly Coupled Lattice<sup>1</sup>**

M. Venturini  
Lawrence Berkeley National Laboratory,  
Berkeley, CA, 94720.

## **Abstract**

Using first-order perturbation theory for maps, we extend the familiar formulae for space-charge tuneshift to encompass the case of a fully coupled lattice.

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# 1 Introduction

The need to accommodate the long bunch trains suitable for a cold linear collider and limitations to the kicker technology will cause the International Linear Collider (ILC) damping rings to be fairly large. A several Km long circumference and small vertical emittance at extraction will combine to produce a sizeable and potentially harmful space-charge tunes shift.

The longest (up to 17 km) 'dogbone' versions of the proposed lattices would be the most vulnerable to space charge effects with the dominant contribution coming from the long straight sections connecting the arcs. The dogbone configuration, however, offers a way to tame the impact of space charge by deliberate introduction of strong linear coupling within the long straight sections to increase the vertical beam size. This scheme, originally proposed for the TESLA DR's is now under consideration for the ILC DR's as well.

To estimate the space-charge tunes shift in a linearly coupled lattice the familiar first-order formulae need to be extended. Carrying out such an extension using first-order perturbation theory for maps [1] is the purpose of these notes. The final result of our calculation is in Eqs. (50) and (51).

## 2 Space Charge Force

The transverse space-charge force on a test particle within an infinitely long bunch with elliptical cross-section and uniform transverse density is linear. Specifically, if the beam cross-section ellipse is upright in a cartesian coördinate frame  $\xi$ - $\eta$  and has semi-axes  $a_\xi$  and  $a_\eta$ , the space charge contribution to the betatron motion is given by [2]

$$\xi'' = \frac{2K}{a_\xi(a_\xi + a_\eta)}\xi = \frac{K}{2\sigma_\xi(\sigma_\xi + \sigma_\eta)}\xi, \quad (1)$$

$$\eta'' = \frac{2K}{a_\eta(a_\xi + a_\eta)}\eta = \frac{K}{2\sigma_\eta(\sigma_\xi + \sigma_\eta)}\eta, \quad (2)$$

with perveance  $K$

$$K = \frac{2\lambda r_e}{\beta^2 \gamma^3}, \quad (3)$$

where  $\lambda$  is the linear density,  $\beta$  and  $\gamma$  the relativistic factors and  $r_e$  the classical electron radius. In the RHS of Eq.'s (1) and (2) the space charge force is expressed in terms of the rms transverse sizes  $\sigma_\xi$  and  $\sigma_\eta$ , related to the semi-axes of a uniform transverse density with elliptical cross-section by  $a_\xi = 2\sigma_\xi$ , and  $a_\eta = 2\sigma_\eta$ .

If the transverse density is not uniform but – say – gaussian, the space charge force is linear only in the limit of infinitely small deviations from

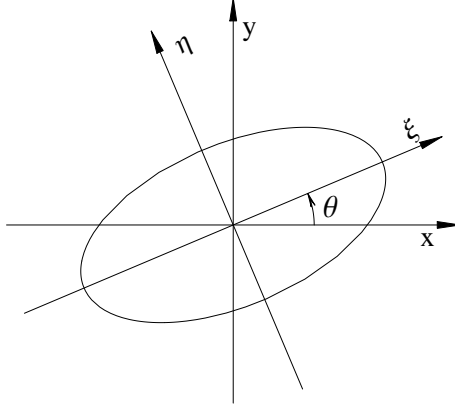


Figure 1: Beam transverse isodensity ellipse and coördinate frames. We define  $\theta \in [-\pi/2, \pi/2]$  to be the angle between the semi-major axis of the isodensity ellipse in the  $x > 0$  semiplane and the  $\hat{x}$ -axis.

the center of the bunch cross-section. As the peak density  $\lambda/2\pi\sigma_\xi\sigma_\eta$  of a transverse gaussian distribution is twice the density of a uniform distribution with the same rms sizes,  $\lambda/\pi a_\xi a_\eta = \lambda/4\pi\sigma_\xi\sigma_\eta$ , the corresponding space-charge contributions to the equations of motion (for  $\xi \ll \sigma_\xi$  and  $\eta \ll \sigma_\eta$ ) read

$$\xi'' = \frac{K}{\sigma_\xi(\sigma_\xi + \sigma_\eta)}\xi \equiv f_\xi, \quad (4)$$

$$\eta'' = \frac{K}{\sigma_\eta(\sigma_\xi + \sigma_\eta)}\eta \equiv f_\eta. \quad (5)$$

The last two equations on the RHS of (4) and (5) define the components  $(f_\xi, f_\eta)$  of the transverse space charge force  $\mathbf{f}$ .

In the following we will be interested in the general case where the bunch density in the transverse plane is tilted by an angle  $\theta$  with respect to the  $x - y$  coördinate frame as shown in Fig. 1. The components  $(f_x, f_y)$  of  $\mathbf{f}$  in the  $x - y$  frame are

$$f_x = f_\xi \cos \theta - f_\eta \sin \theta \quad (6)$$

$$f_y = f_\xi \sin \theta + f_\eta \cos \theta \quad (7)$$

whereas  $\xi = x \cos \theta + y \sin \theta$  and  $\eta = -x \sin \theta + y \cos \theta$ . We have

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \mathbf{V} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (8)$$

with

$$\mathbf{V} = K \begin{pmatrix} \frac{\cos^2 \theta}{\sigma_\xi(\sigma_\xi + \sigma_\eta)} + \frac{\sin^2 \theta}{\sigma_\eta(\sigma_\xi + \sigma_\eta)} & \frac{\cos \theta \sin \theta}{\sigma_\xi(\sigma_\xi + \sigma_\eta)} - \frac{\cos \theta \sin \theta}{\sigma_\eta(\sigma_\xi + \sigma_\eta)} \\ \frac{\cos \theta \sin \theta}{\sigma_\xi(\sigma_\xi + \sigma_\eta)} - \frac{\cos \theta \sin \theta}{\sigma_\eta(\sigma_\xi + \sigma_\eta)} & \frac{\sin^2 \theta}{\sigma_\xi(\sigma_\xi + \sigma_\eta)} + \frac{\cos^2 \theta}{\sigma_\eta(\sigma_\xi + \sigma_\eta)} \end{pmatrix}. \quad (9)$$

We shall denote the entries of this symmetric matrix as

$$\mathbf{V} = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{xy} & V_{yy} \end{pmatrix}. \quad (10)$$

### 3 Second-Order Transfer Matrix

Consider a circular machine with circumference  $C$  and let the observation point be at  $s = 0$ . Denote with  $\mathbf{M}_{0 \rightarrow s}$  the transfer matrix for the unperturbed lattice from the observation point to the point  $s$  along the ring circumference and  $\mathbf{M}_0$  the one-turn transfer matrix at  $s = 0$ .

Next, introduce a segmentation  $s_i = i\Delta s$ , with  $i = 0, 1, \dots, n$  and  $\Delta s = C/n$  and consider a perturbation in the kick approximation applied at each  $s = s_i$  along the ring.

Having denoted with  $\mathbf{z} = (x, p_x, y, p_y)$  the set of dynamical variables, the perturbation kick due to space charge can then be written as

$$\mathbf{z}' = (\mathbf{1} + \Delta s \mathbf{V}_i) \mathbf{z} \quad (11)$$

where  $\mathbf{1}$  is the  $4 \times 4$  identity matrix and  $\mathbf{V}_i = \mathbf{V}(s_i)$  is the matrix

$$\mathbf{V}_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ V_{xx}(s_i) & 0 & V_{xy}(s_i) & 0 \\ 0 & 0 & 0 & 0 \\ V_{xy}(s_i) & 0 & V_{yy}(s_i) & 0 \end{pmatrix}. \quad (12)$$

Because  $\mathbf{V}_i$  is nilpotent ( $\mathbf{V}_i^2 = 0$ ), it is possible to rewrite (11) as

$$\mathbf{z}' = e^{\Delta s \mathbf{V}_i} \mathbf{z}. \quad (13)$$

The one-turn transfer matrix  $\mathcal{M}_0$  including the perturbation can be expressed as

$$\mathcal{M}_0 = e^{\Delta s \mathbf{V}_n} \mathbf{M}_{n-1 \rightarrow n} e^{\Delta s \mathbf{V}_{n-1}} \dots \mathbf{M}_{2 \rightarrow 3} e^{\Delta s \mathbf{V}_2} \mathbf{M}_{1 \rightarrow 2} e^{\Delta s \mathbf{V}_1} \mathbf{M}_{0 \rightarrow 1}, \quad (14)$$

where  $\mathbf{M}_{i \rightarrow i+1}$  is a shorthand for the unperturbed transfer matrix from  $s = s_i$  to  $s = s_{i+1}$ .

Next, we insert identities in between the kicks to move all the exponentials to the right as in the following:

$$\begin{aligned} \mathcal{M}_0 &= \dots \mathbf{M}_{2 \rightarrow 3} e^{\Delta s \mathbf{V}_2} \mathbf{M}_{1 \rightarrow 2} e^{\Delta s \mathbf{V}_1} \mathbf{M}_{0 \rightarrow 1} \\ &= \dots \mathbf{M}_{2 \rightarrow 3} (\mathbf{M}_{0 \rightarrow 2} \mathbf{M}_{0 \rightarrow 2}^{-1}) e^{\Delta s \mathbf{V}_2} \mathbf{M}_{1 \rightarrow 2} (\mathbf{M}_{0 \rightarrow 1} \mathbf{M}_{0 \rightarrow 1}^{-1}) e^{\Delta s \mathbf{V}_1} \mathbf{M}_{0 \rightarrow 1} \\ &= \dots \mathbf{M}_{0 \rightarrow 3} e^{\Delta s \mathbf{M}_{0 \rightarrow 2}^{-1} \mathbf{V}_2 \mathbf{M}_{0 \rightarrow 2}} e^{\Delta s \mathbf{M}_{0 \rightarrow 1}^{-1} \mathbf{V}_1 \mathbf{M}_{0 \rightarrow 1}}, \end{aligned}$$

and find

$$\mathcal{M}_0 = M_0 e^{\Delta s \tilde{V}_n} e^{\Delta s \tilde{V}_{n-1}} \dots e^{\Delta s \tilde{V}_2} e^{\Delta s \tilde{V}_1}, \quad (15)$$

where

$$\tilde{V}_i = M_{0 \rightarrow i}^{-1} V_i M_{0 \rightarrow i}. \quad (16)$$

We then use the Hausdorff formula and combine the exponentials in (15)

$$e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon A + \varepsilon B + \varepsilon^2 \frac{1}{2} [A, B]} + \mathcal{O}(\varepsilon^3) \quad (17)$$

to find through second order

$$e^{\Delta s \tilde{V}_n} e^{\Delta s \tilde{V}_{n-1}} \dots e^{\Delta s \tilde{V}_2} e^{\Delta s \tilde{V}_1} \simeq \exp \left( \Delta s \sum_{i=1}^n \tilde{V}_i + \frac{(\Delta s)^2}{2} \sum_{j=1}^n \sum_{i=j+1}^n [\tilde{V}_i, \tilde{V}_j] \right). \quad (18)$$

Taking the limit  $n \rightarrow \infty$  finally yields

$$\mathcal{M}_0 = M_0 e^{\int_0^C ds \tilde{V}(s) + \frac{1}{2} \int_0^C ds \int_{t=s}^C dt [\tilde{V}(t), \tilde{V}(s)]}. \quad (19)$$

## 4 Normal Form

Reduction of the map (19) to normal form (*i.e.* a rotation) can be achieved by successive transformations order by order. We shall limit the calculation to first order.

Let  $A_s$  be the normalizing transformation for the one-turn unperturbed map  $M_s$ . That is

$$A_s^{-1} M_s A_s = R(2\pi\nu_{x0}, 2\pi, \nu_{y0}) \quad (20)$$

where  $R(2\pi\nu_{x0}, 2\pi\nu_{y0})$  is the diagonal-block rotation

$$R(2\pi\nu_{x0}, 2\pi\nu_{y0}) = \begin{pmatrix} \cos(2\pi\nu_{x0}) & \sin(2\pi\nu_{x0}) & 0 & 0 \\ -\sin(2\pi\nu_{x0}) & \cos(2\pi\nu_{x0}) & 0 & 0 \\ 0 & 0 & \cos(2\pi\nu_{y0}) & \sin(2\pi\nu_{y0}) \\ 0 & 0 & -\sin(2\pi\nu_{y0}) & \cos(2\pi\nu_{y0}) \end{pmatrix} \quad (21)$$

and  $\nu_{x0}$  and  $\nu_{y0}$  are the two transverse tunes.

In the rest of this section it will be understood that if its arguments are not indicated the symbol  $R$  alone means  $R \equiv R(2\pi\nu_{x0}, 2\pi\nu_{y0})$ .

For later use notice that a diagonal-block rotation matrix can be generated by exponentiation  $R(\phi_{x0}, \phi_{y0}) = \exp(\phi_{x0} \tau_2 + \phi_{y0} \tau_5) = \exp(\phi_{x0} \tau_2) \exp(\phi_{y0} \tau_5) = \cos(\phi_{x0}) \mathbf{1}_x + \sin(\phi_{x0}) \tau_2 + \cos(\phi_{y0}) \mathbf{1}_y + \sin(\phi_{y0}) \tau_5$  where

$$\mathbf{1}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (22)$$

$$\mathbf{1}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (23)$$

Also, recall that a transfer map between two locations (say  $s = 0$  and  $s$ ) along the lattice, *i.e.*  $\mathbf{M}_{0 \rightarrow s}$ , can be reduced to diagonal-block rotation form as well. This can be achieved by means of the matrices  $\mathbf{A}_0$  and  $\mathbf{A}_s$ , (which normalize the one-turn matrices at the points  $s = 0$  and  $s$  respectively):

$$\mathbf{A}_s^{-1} \mathbf{M}_{0 \rightarrow s} \mathbf{A}_0 = \mathbf{R}(\mu_x, \mu_y) \equiv \mathbf{R}_{0 \rightarrow s}, \quad (24)$$

where  $\mu_x = \psi_x(s) - \psi_x(0)$  and  $\mu_y = \psi_y(s) - \psi_y(0)$  are the phase differences between the points  $s$  and  $s = 0$ .

Denote with  $\mathcal{N}^{(0)}$  the matrix obtained from  $\mathcal{M}_0$  by a similarity transformation with  $\mathbf{A}_0$ :

$$\mathcal{N}^{(0)} = \mathbf{A}_0^{-1} \mathcal{M}_0 \mathbf{A}_0. \quad (25)$$

Upon use of (19) and appropriate insertion of  $\mathbf{A}_0 \mathbf{A}_0^{-1}$ , through first order, we have

$$\begin{aligned} \mathcal{N}^{(0)} &= \mathbf{A}_0^{-1} \mathcal{M}_0 \mathbf{A}_0 \\ &= \mathbf{A}_0^{-1} \mathbf{M}_0 e^{\int_0^C ds \tilde{\mathbf{V}}(s)} \mathbf{A}_0 \\ &= \mathbf{A}_0^{-1} \mathbf{M}_0 \mathbf{A}_0 \mathbf{A}_0^{-1} e^{\int_0^C ds \tilde{\mathbf{V}}(s)} \mathbf{A}_0 \\ &= \mathbf{R} e^{\int_0^C ds \mathbf{A}_0^{-1} \tilde{\mathbf{V}}(s) \mathbf{A}_0} \\ &= \mathbf{R} e^{\mathbf{P}}. \end{aligned} \quad (26)$$

In the last line we introduced the definition of the first-order perturbation matrix

$$\mathbf{P} = \int_0^C ds \hat{\mathbf{V}}(s) \quad (27)$$

with  $\hat{\mathbf{V}}(s)$  given by

$$\hat{\mathbf{V}}(s) = \mathbf{A}_0^{-1} \tilde{\mathbf{V}}(s) \mathbf{A}_0 \quad (28)$$

$$= \mathbf{A}_0^{-1} \mathbf{M}_{0 \rightarrow s}^{-1} \mathbf{V}(s) \mathbf{M}_{0 \rightarrow s} \mathbf{A}_0 \quad (29)$$

$$= \mathbf{R}_{0 \rightarrow s}^{-1} \mathbf{A}_s^{-1} \mathbf{V}(s) \mathbf{A}_s \mathbf{R}_{0 \rightarrow s}. \quad (30)$$

In (29) and (30) we made use of (16) and (24) respectively.

The first-order normal form  $\mathcal{N}^{(1)}$  is obtained from  $\mathcal{N}^{(0)}$  by a similarity transformation with the symplectic matrix  $e^{\mathbf{T}}$  (with  $\mathbf{T}$  to be determined):

$$\mathcal{N}^{(1)} = e^{-\mathbf{T}} \mathcal{N}^{(0)} e^{\mathbf{T}}. \quad (31)$$

We have

$$\begin{aligned}
\mathcal{N}^{(1)} &= e^{-\mathbf{T}} \mathcal{N}^{(0)} e^{\mathbf{T}} \\
&= e^{-\mathbf{T}} \mathbf{R} e^{\mathbf{P}} e^{\mathbf{T}} \\
&= \mathbf{R} e^{-\mathbf{R}^{-1} \mathbf{T} \mathbf{R}} e^{\mathbf{P}} e^{\mathbf{T}} \\
&= \mathbf{R} e^{-\mathbf{R}^{-1} \mathbf{T} \mathbf{R} + \mathbf{P} + \mathbf{T}},
\end{aligned} \tag{32}$$

with the last equality valid through first order. The exponent in (32) can be rewritten in the more expressive form  $\mathbf{P}' = -\mathbf{R}^{-1} \mathbf{T} \mathbf{R} + \mathbf{P} + \mathbf{T} = -\mathbf{R}^{-1} [\mathbf{T}, \mathbf{R}] + \mathbf{P}$ . From the requirement that  $\exp(\mathbf{P})$  and  $\exp(\mathbf{T})$  be symplectic it follows that the matrices  $\mathbf{P}$  and  $\mathbf{T}$  belong to the 10-dim symplectic Lie algebra  $\mathfrak{sp}(4)$ . Think of  $\mathbf{P}$  as represented in terms of some basis  $\boldsymbol{\tau}_i$  of  $\mathfrak{sp}(4)$ :  $\mathbf{P} = \sum_{i=1}^{10} p_i \boldsymbol{\tau}_i$ , with the two matrices  $\boldsymbol{\tau}_2$  and  $\boldsymbol{\tau}_5$  defined in (22) and (23) belonging to this basis (see Appendix A for a suitable choice of  $\boldsymbol{\tau}_i$ ).

The goal is to determine  $\mathbf{T}$  so as to reduce  $\mathbf{P}'$  to some minimal - ideally vanishing - form. However, it can be verified that the combination  $\mathbf{R}^{-1} [\mathbf{T}, \mathbf{R}]$  cannot contain either a  $\boldsymbol{\tau}_2$  or  $\boldsymbol{\tau}_5$  component and therefore  $p_2 \boldsymbol{\tau}_2$  and  $p_5 \boldsymbol{\tau}_5$  cannot be removed from  $\mathbf{P}'$ . With  $\mathbf{T}$  determined by the equation  $\mathbf{R}^{-1} [\mathbf{T}, \mathbf{R}] = \sum_{i \neq 2,5}^{10} p_i \boldsymbol{\tau}_i$  we find,

$$\begin{aligned}
\mathcal{N}^{(1)} &= \mathbf{R}(2\pi\nu_{x0}, 2\pi\nu_{y0}) e^{p_2 \boldsymbol{\tau}_2 + p_5 \boldsymbol{\tau}_5} \\
&= \mathbf{R}(2\pi\nu_{x0}, 2\pi\nu_{y0}) \mathbf{R}(p_2, p_5) \\
&= \mathbf{R}(2\pi\nu_{x0} + p_2, 2\pi\nu_{y0} + p_5).
\end{aligned} \tag{33}$$

The last line identifies  $p_2/2\pi$  and  $p_5/2\pi$  as the horizontal and vertical first order tunes.

## 5 Case with no Coupling

For an uncoupled lattice the normalizing matrix  $\mathbf{A}_s$  is block diagonal with the form of the two  $2 \times 2$  blocks being the same in both the horizontal and vertical plane - so we can restrict the discussion to only one degree of freedom - say the horizontal motion. The space-charge kick matrix is simply

$$\mathbf{V}(s) = \begin{pmatrix} 0 & 0 \\ V_x & 0 \end{pmatrix} \tag{34}$$

with  $V_x = K/[\sigma_x(\sigma_x + \sigma_y)]$ , while the normalizing matrix  $\mathbf{A}_s$  can be expressed in terms of the Twiss functions in the usual way

$$\mathbf{A}_s = \begin{pmatrix} \sqrt{\beta_x(s)} & 0 \\ -\frac{\alpha_x(s)}{\sqrt{\beta_x(s)}} & \frac{1}{\sqrt{\beta_x(s)}} \end{pmatrix}, \tag{35}$$

and

$$\mathbf{R}_{0 \rightarrow s} = \begin{pmatrix} \cos \mu_x & \sin \mu_x \\ -\sin \mu_x & \cos \mu_x \end{pmatrix}, \quad (36)$$

with  $\mu_x = \psi_x(s) - \psi_x(0)$ . A basis for the 3-dimensional Lie algebra  $\mathfrak{sp}(2)$  is given by the matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (37)$$

It is just a matter of carrying out matrix multiplications to find  $\mathbf{A}_s^{-1} \mathbf{V}(s) \mathbf{A}_s = \frac{1}{2} \beta_x(s) V_x(s) (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$ , and hence the quantity defined in (28)

$$\begin{aligned} \hat{\mathbf{V}}(s) &= \mathbf{R}_{0 \rightarrow s}^{-1} \mathbf{A}_s^{-1} \mathbf{V}(s) \mathbf{A}_s \mathbf{R}_{0 \rightarrow s} \\ &= \frac{1}{2} \beta_x(s) V_x(s) \mathbf{R}(\mu_x) (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \mathbf{R}(-\mu_x), \end{aligned} \quad (38)$$

yielding the first-order perturbation matrix  $\mathbf{P}$

$$\mathbf{P} = \int_0^C ds \beta_x(s) V_x(s) \begin{pmatrix} -\frac{1}{2} \sin(2\mu_x) & -\sin^2 \mu_x \\ \cos^2 \mu_x & \frac{1}{2} \sin(2\mu_x) \end{pmatrix}, \quad (39)$$

or  $\mathbf{P} = \sum_{i=1}^3 p_i \boldsymbol{\sigma}_i$  with

$$p_1 = \int_0^C ds \beta_x(s) V_x(s) \left( \frac{1}{2} - \sin^2 \mu_x \right) \quad (40)$$

$$p_2 = -\frac{1}{2} \int_0^C ds \beta_x(s) V_x(s) \quad (41)$$

$$p_3 = -\frac{1}{2} \int_0^C ds \beta_x(s) V_x(s) \sin(2\mu_x) \quad (42)$$

and finally the familiar formula for the tuneshift

$$\Delta \nu_x = \frac{p_2}{2\pi} = -\frac{1}{4\pi} \int_0^C ds \beta_x(s) V_x(s). \quad (43)$$

## 6 First-Order Space-Charge Tunesift with Coupling.

Coupling adds to the algebra involved in the calculation as the normalizing matrix  $\mathbf{A}$  is now fully  $4 \times 4$ . There is, however, a residual freedom in setting the entries that we can use to impose  $A_{12} = A_{34} = 0$  [1, 3]. From the entries

$$\begin{aligned} \hat{V}_{12} &= -V_{xx} A_{11} \sin^2 \mu_x - 2V_{xy} (A_{31} \sin^2 \mu_x + \sin \mu_x \cos \mu_x A_{32}) A_{11} \\ &\quad - V_{yy} (A_{31} \sin \mu_x + A_{32} \cos \mu_x)^2, \end{aligned} \quad (44)$$

$$\begin{aligned} \hat{V}_{21} &= V_{xx} A_{11} \cos^2 \mu_x + 2V_{xy} (A_{31} \cos^2 \mu_x - \sin \mu_x \cos \mu_x A_{32}) A_{11} \\ &\quad + V_{yy} (A_{31} \cos \mu_x - A_{32} \sin \mu_x)^2, \end{aligned} \quad (45)$$



of the matrix  $\hat{\mathbf{V}}$  and inspection of the  $\boldsymbol{\tau}_i$  reported in Appendix A we conclude that the  $\hat{v}_2$  component of  $\hat{\mathbf{V}} = \sum_{i=1}^{10} \hat{v}_i \boldsymbol{\tau}_i$  reads

$$\hat{v}_2 = \frac{1}{2}(\hat{V}_{12} - \hat{V}_{21}) = -\frac{1}{2}[V_{xx}A_{11}^2 + 2V_{xy}A_{11}A_{31} + V_{yy}(A_{31}^2 + A_{32}^2)]. \quad (46)$$

Similarly

$$\begin{aligned} \hat{V}_{34} = & -V_{xx}(A_{13} \sin \mu_y + A_{14} \cos \mu_y)^2 \\ & -2V_{xy}(A_{13} \sin^2 \mu_y + \sin \mu_y \cos \mu_y A_{14})A_{33} - V_{yy}A_{33} \sin^2 \mu_y, \end{aligned} \quad (47)$$

$$\begin{aligned} \hat{V}_{43} = & +V_{xx}(A_{13} \cos \mu_y - A_{14} \sin \mu_y)^2 \\ & +2V_{xy}(A_{13} \cos^2 \mu_y - \sin \mu_y \cos \mu_y A_{14})A_{33} + V_{yy}A_{33} \cos^2 \mu_y, \end{aligned} \quad (48)$$

and

$$\hat{v}_5 = \frac{1}{2}(\hat{V}_{34} - \hat{V}_{43}) = -\frac{1}{2}[V_{xx}(A_{13}^2 + A_{14}^2) + 2V_{xy}A_{13}A_{33} + V_{yy}A_{33}^2]. \quad (49)$$

Consequently, the tuneshifts ( $\Delta\nu_x = p_2/2\pi$ ,  $\Delta\nu_y = p_5/2\pi$  with  $p_i = \int ds \hat{v}_i(s)$  and  $i = 2, 5$ ) read

$$\Delta\nu_x = -\frac{1}{4\pi} \int_0^C ds [V_{xx}A_{11}^2 + 2V_{xy}A_{11}A_{31} + V_{yy}(A_{31}^2 + A_{32}^2)], \quad (50)$$

$$\Delta\nu_y = -\frac{1}{4\pi} \int_0^C ds [V_{xx}(A_{13}^2 + A_{14}^2) + 2V_{xy}A_{13}A_{33} + V_{yy}A_{33}^2]. \quad (51)$$

## 7 Application to the ILC DR Dogbone Lattice.

As an example of application of Eq.'s (50) and (51) we study one of the two long straight sections in a dogbone lattice proposed for ILC [6]. Each straight is approximately 6 km long and consists of a sequence of  $\simeq 100$  m FODO cells and two short matching sections at each end. At one end three skew quadrupoles create strong linear coupling which is then undone by another set of skew quads symmetrically placed in the opposite end. As a result, the beam acquires a nearly round transverse cross section - in contrast to the rest of the ring where it is largely flat.

We consider a simplified periodic lattice consisting of this one long straight section and compare the space-charge tuneshifts calculated in three different ways: *i*) by application of the first-order formulae derived in Sec. 6; *ii*) by Fourier analysis of a test-particle orbit; *iii*) by analysis of the total transfer matrix including space-charge effects in the kick approximation.

We assumed a model of beam with gaussian distribution in phase-space (see also [5]). Because the resulting transverse charge density is nonuniform we recall that the analytical formulae Eq.'s (50) and (51) for the tuneshift

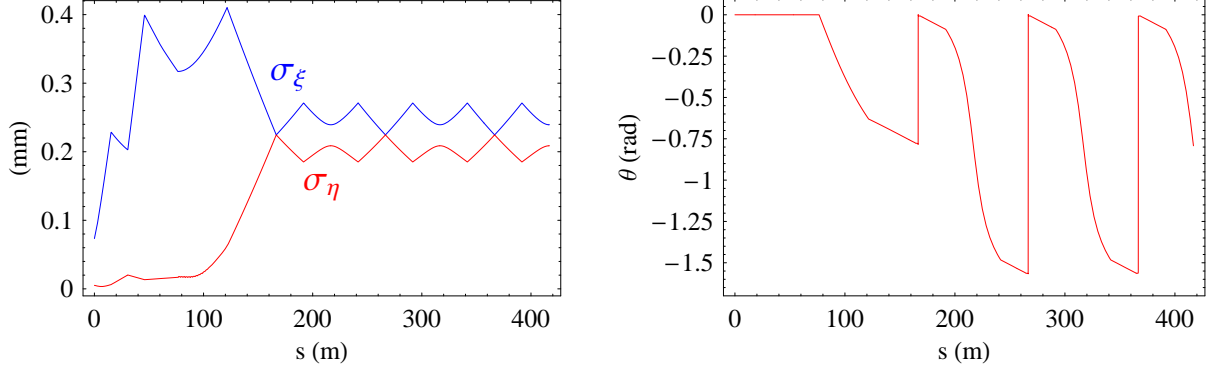


Figure 2: Beam transverse rms ‘eigen-sizes’ (left figure) and tilt-angle  $\theta$  (right figure);  $\theta$  is discontinuous where  $\sigma_\xi = \sigma_\eta$ .

will only apply to particles experiencing infinitely small-amplitude betatron and no synchrotron oscillations.

The tuneshift calculation requires knowledge of the normalizing matrix  $\mathbf{A}(s)$  and matrix (9) at each location  $s$  along the lattice. Matrix (9) depends on the rms sizes  $\sigma_\xi$  and  $\sigma_\eta$  of the bunch density in the frame where the beam isodensity ellipses are up-right, and the tilt angle  $\theta$ .  $\sigma_\xi$  and  $\sigma_\eta$  are calculated starting from the transverse rms sizes  $\sigma_x = \sqrt{\langle x^2 \rangle}$ ,  $\sigma_y = \sqrt{\langle y^2 \rangle}$  and the correlation  $\langle xy \rangle$  of the transverse charge density. For a matched beam these quantities depend on the lattice functions (*i.e.* the matrix  $\mathbf{A}(s)$ ) and the rms (unnormalized) eigenemittances  $\varepsilon_I$ ,  $\varepsilon_{II}$  according to<sup>2</sup>

$$\sigma_x^2 = A_{11}^2 \varepsilon_I + (A_{13}^2 + A_{14}^2) \varepsilon_{II}, \quad (52)$$

$$\sigma_y^2 = (A_{31}^2 + A_{32}^2) \varepsilon_I + A_{33}^2 \varepsilon_{II}, \quad (53)$$

$$\langle xy \rangle = A_{11} A_{31} \varepsilon_I + A_{13} A_{33} \varepsilon_{II}. \quad (54)$$

The rms ‘eigen-sizes’  $\sigma_\xi$  and  $\sigma_\eta$  and tilt angle are then obtained by diagonalization of the matrix

$$\begin{pmatrix} \langle x^2 \rangle & \langle xy \rangle \\ \langle xy \rangle & \langle y^2 \rangle \end{pmatrix}. \quad (55)$$

We choose  $\sigma_\xi^2$  ( $\sigma_\eta^2$ ) to be the larger (smaller) of the two eigenvalues. The tilt-angle  $\theta$  is defined as in Fig. 1.

A plot of the rms ‘eigen-sizes’ and tilt-angle  $\theta$  in the first 400 m of the dogbone long straight-section is shown in Fig. 2, whereas Fig. 3 reports the relevant lattice functions (*i.e.* combinations of matrix  $\mathbf{A}$  entries entering the expressions for the first-order tunes shift formulae) together with the beam transverse isodensity ellipses at selected points along the lattice.

A summary of the tunes shift estimates obtained using the three methods is reported in Table: 1. In the calculation we considered a 5 GeV infinitely

<sup>2</sup>Again, we are assuming the gauge  $A_{12} = A_{34} = 0$ .

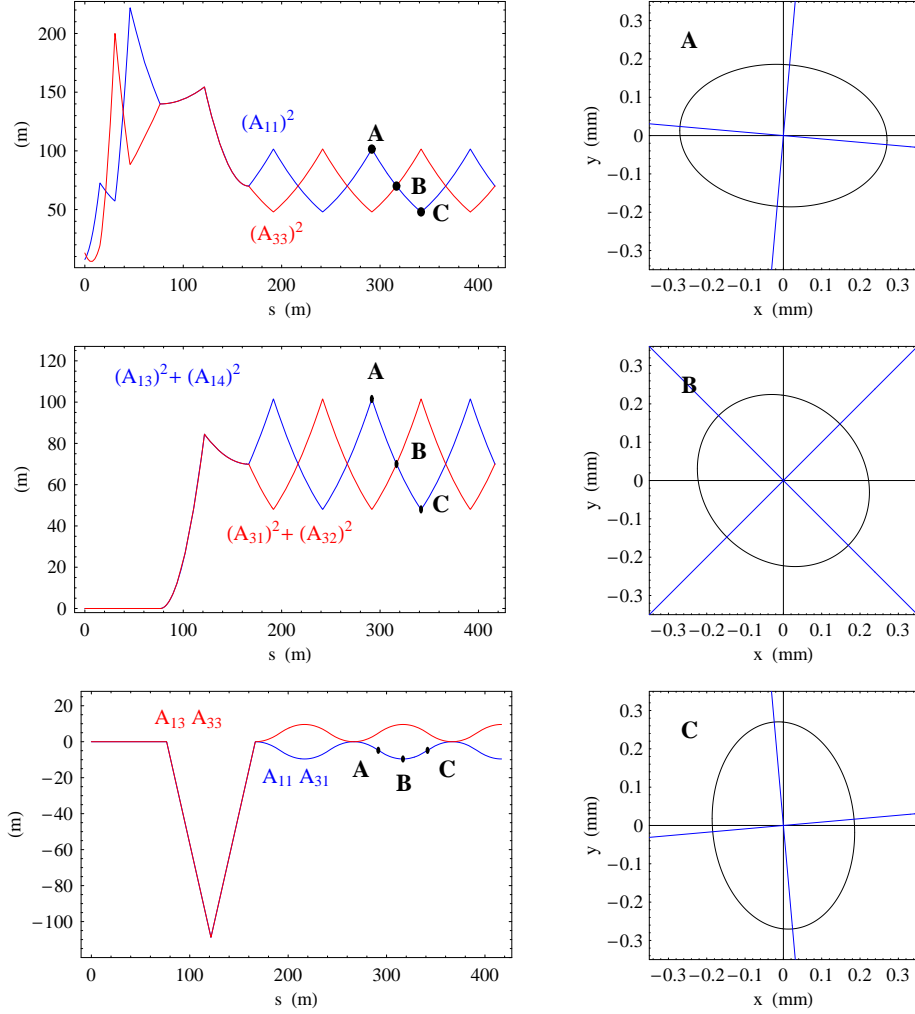


Figure 3: Lattice functions (left figures) and beam transverse isodensity ellipse at selected points along the lattice (right figures).

long electron beam with longitudinal density  $\lambda = N/\sqrt{2\pi}\sigma_z$ ;  $N = 2 \times 10^{10}$  particles,  $\sigma_z = 6$  mm;  $\varepsilon_I = 0.72$  nm and  $\varepsilon_{II} = 2$  pm.

In method *ii*) the Fourier analysis was applied to a particle orbit through 100 lattice periods using interpolated FFT with Hanning filter [7].

Particle orbits, lattice functions, and maps were calculated using the code MaryLie/Impact [4, 5].

Table 1: Space-charge tunes shifts using different methods of calculation.

	$\Delta\nu_x$	$\Delta\nu_y$
first-order theory	$-5.150 \times 10^{-3}$	$-7.876 \times 10^{-3}$
Fourier analysis	$-5.152 \times 10^{-3}$	$-7.892 \times 10^{-3}$
map analysis	$-5.151 \times 10^{-3}$	$-7.903 \times 10^{-3}$

## 8 Appendix A: A Basis for the $\mathfrak{sp}(4)$ Lie Algebra and Commutators

$$\tau_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (56)$$

$$\tau_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (57)$$

$$\tau_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (58)$$

$$\tau_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \tau_8 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (59)$$

$$\tau_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (60)$$

Commutation rules <sup>3</sup>:

$$[\tau_2, \tau_1] = 2\tau_3, \quad [\tau_3, \tau_1] = 2\tau_2 \quad (61)$$

$$[\tau_7, \tau_1] = -\tau_8, \quad [\tau_8, \tau_1] = -\tau_7 \quad (62)$$

$$[\tau_9, \tau_1] = \tau_{10}, \quad [\tau_{10}, \tau_1] = \tau_9 \quad (63)$$

$$[\tau_3, \tau_2] = 2\tau_1, \quad [\tau_7, \tau_2] = -\tau_9 \quad (64)$$

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<sup>3</sup>Commutators that do not appear in the list, i.e.  $[\tau_4, \tau_1] = 0$ , are understood to vanish.

$$[\tau_8, \tau_2] = -\tau_{10}, \quad [\tau_9, \tau_2] = \tau_7 \quad (65)$$

$$[\tau_{10}, \tau_2] = \tau_8, \quad (66)$$

$$[\tau_7, \tau_3] = -\tau_{10}, \quad [\tau_8, \tau_3] = -\tau_9 \quad (67)$$

$$[\tau_9, \tau_3] = -\tau_8, \quad [\tau_{10}, \tau_3] = -\tau_7 \quad (68)$$

$$[\tau_5, \tau_4] = 2\tau_6, \quad [\tau_6, \tau_4] = 2\tau_5 \quad (69)$$

$$[\tau_7, \tau_4] = \tau_8, \quad [\tau_8, \tau_4] = \tau_7 \quad (70)$$

$$[\tau_9, \tau_4] = \tau_{10}, \quad [\tau_{10}, \tau_4] = \tau_9 \quad (71)$$

$$[\tau_6, \tau_5] = 2\tau_4, \quad [\tau_7, \tau_5] = \tau_9 \quad (72)$$

$$[\tau_8, \tau_5] = -\tau_{10}, \quad [\tau_9, \tau_5] = -\tau_7 \quad (73)$$

$$[\tau_{10}, \tau_5] = \tau_8, \quad (74)$$

$$[\tau_7, \tau_6] = \tau_{10}, \quad [\tau_8, \tau_6] = -\tau_9 \quad (75)$$

$$[\tau_9, \tau_6] = -\tau_8, \quad [\tau_{10}, \tau_6] = \tau_7 \quad (76)$$

$$[\tau_8, \tau_7] = -2\tau_1 + 2\tau_4, \quad [\tau_9, \tau_7] = -2\tau_2 + 2\tau_5 \quad (77)$$

$$[\tau_{10}, \tau_7] = -2\tau_3 + 2\tau_6 \quad (78)$$

$$[\tau_9, \tau_8] = 2\tau_3 + 2\tau_6, \quad [\tau_{10}, \tau_8] = 2\tau_2 + 2\tau_5 \quad (79)$$

$$[\tau_{10}, \tau_9] = 2\tau_1 + 2\tau_4. \quad (80)$$

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