A "Hot-Spot" Proof of Normality for the Alpha Constants

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In [2] Richard Crandall and I establish normality base p for the class of constants

$$\alpha_{p,q} = \sum_{k=1}^{\infty} \frac{1}{q^k p^{q^k}}$$

where p and q are co-prime. The proof given in [2] is somewhat difficult and relies on several not-well-known results, including one by Korobov on the properties of certain pseudo-random sequences. In this note I show that normality can be established much more easily, as a consequence of what is sometimes call the "hot spot" theorem. The "hot spot" theorem is as follows. In the following, $\{\cdot\}$ denotes fractional part.

"Hot Spot" Theorem. The real constant α is normal base b if and only if there exists a constant C such that for every subinterval $[c, d) \subset [0, 1)$,

$$\limsup_{n \geq 1} \frac{\#_{0 \leq j < n}(\{b^j \alpha\} \in [c, d))}{n} \leq C(d - c).$$

In other words, normal numbers have no "hot spots", and conversely a non-normal number must have hot spots — there must be digit strings that appear, say, one billion times more often than the frequency they would appear if the number were normal. A proof of the hot spot theorem, using the Birkhoff ergodic theorem [3, pg. 13, 20-29], is given in [1], where it is used to establish that a rational times a normal number is normal. The hot spot theorem is proved by a different (but more difficult) argument in [4, pg. 77].

Here is how the hot spot theorem can be used to establish normality for the α constants studied in [2]. In this note I will use $\alpha = \alpha_{2,3}$, namely

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}},$$

but the proof is very similar for other $\alpha_{p,q}$ constants from [2].

Theorem. α is normal base 2.

Proof: First note that the associated sequence (in the BBP sense) for α is $x_0 = 0$, with $x_n = \{2x_{n-1} + r_n\}$, where $r_n = 1/n$ if $n = 3^k$, and zero otherwise. As above, the notation $\{\cdot\}$ denotes fractional part. Observe that the x sequence has the pattern

$$0, 0, 0, 1/3, 2/3, 1/3, 2/3, 1/3, 2/3,$$
$$4/9, 8/9, 7/9, 5/9, 1/9, 2/9, 4/9, 8/9, 7/9, 5/9, 1/9, 2/9, 4/9, 8/9, 7/9, 5/9, 1/9, 2/9,$$

and so forth. Note that for $n < 3^{k+1}$, each x_n is a multiple of $1/3^k$, and each fraction $j/3^k$, $0 \le j < 3^k$ appears exactly three times in the sequence. Also note that

$$|x_n - \alpha_n| \;=\; \left|\sum_{k=n+1}^{\infty} 2^{n-k} r_k\right| \;<\; \frac{1}{2n}$$

where $\alpha_n = \{2^n \alpha\}$ is the tail of the binary expansion of α after the first n bits.

Given n, let m be the largest power of 3 less than n, and assume that n is large enough so that $n > m > 1/(d-c)$. Now note that the interval $[c - 1/(2n), d + 1/(2n))$ contains exactly $m(d-c)$ (or possibly one more than this number) multiples of $1/m$, and thus can contain at most three times this many occurrences of x_j in the first n elements, assuming $n > m$. Thus for $n > m$ we have

$$\begin{aligned}
\frac{\#_{0\le j<n}(\alpha_j \in [c,d))}{n(d-c)} &\le \frac{\#_{0\le j<n}(x_j \in [c-1/(2n), d+1/(2n))}{n(d-c)} \\
&\le \frac{3[m(d-c)+1]}{n(d-c)} \;<\; \frac{3[m(d-c)+1]}{m(d-c)} \\
&= 3 + \frac{3}{m(d-c)} \;<\; 6
\end{aligned}$$

In other words, for all $n > 3(d-c)$, say, no subinterval of $[0,1)$ has more than six times as many elements of x_j as it "should" in the first n elements, and no binary string appears more than six times as often as it "should" in first n bits of the binary expansion of α. By the hot spot theorem, this establishes that α is normal base 2.

References

[1] David H. Bailey and Daniel J. Rudolph, "An Ergodic Proof that Rational Times Normal is Normal," manuscript, available at the URL
`http://www.nersc.gov/~dhbailey/dhbpapers/ratxnormal.pdf`

[2] David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," to appear in *Experimental Mathematics*, available at the URL
`http://www.nersc.gov/~dhbailey/dhbpapers/bcnormal.pdf`

[3] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.

[4] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.