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Finite Deformation of Materials with an Ensemble of Defects

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ABSTRACT

The theory of large deformations developed here is closely related to continuum mechanics but it differs in several major respects, especially in considering the deformation associated with various types of physical behavior, making it possible to synthesize a general approach to formulating constitutive laws. One goal is to derive general concepts of strain, strain rate, stress, and stress rate that are somewhat more physics-based than in most standard works on continuum mechanics, and to demonstrate some new relations between these quantities. With these concepts it is possible to develop a generalized principle of superposition of strain rates (GSSR) that accounts for damage as well as plastic flow. The traditional superposition of strain rates allows for addition of elastic and plastic strain rates and is commonly thought to be valid only for small strains. The GSSR allows us to compute deformations involving plastic flow and, in addition, brittle failure, fragmentation, high-pressure effects and other types of behavior as necessary, and the theory is valid for arbitrarily large deformations. In fact, GSSR is derived from more basic ideas and has broader application than the standard superposition of strain rates. The physical basis for calculations of complex material response is developed in a separate report. The implementation into the SCRAM computer program is documented separately.

The polar decomposition theorem is taken as a starting point for the theory of large deformation, an approach somewhat different from that usually taken in continuum mechanics. Two sets of orthogonal axes are distinguished, space axes that are fixed in ambient space, and polar axes that are related to material deformation. This clarifies several concepts; for example, it is shown that the Signorini and Green-St. Venant strains are actually measures of the same physical entity, one in space axes and the other in polar axes. It follows that they are not competing measures, as is often implied in traditional continuum mechanics. It also follows that Piola stress is a measure in polar axes, while Cauchy stress is a measure in space axes. Another consequence of polar decomposition is a proof that vorticity is not a measure of the rate of material rotation (as is often stated in the hydrodynamics literature) but that they are related. This allows us to develop an exact approach to computing rates of tensor quantities, called polar rates, that account for material rotation in an exact way. This leads to a simple relation between Signorini strain rate and stretching (the symmetric part of the velocity gradient). It also follows that the polar stress rate is the appropriate measure for the rate of change of Cauchy stress, and that the more traditional stress rate of Zaremba, Jaumann, and Noll is only an approximation, valid at small strains. Examples are described for materials undergoing simple shear, vortex motion, and torsion.

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1. INTRODUCTION AND SUMMARY

Though the mathematical theory of deformation has been studied extensively, the early work assumed continuity of deformation and, in this respect, failed to model real materials, which contain voids, cracks, grain boundaries, dislocations, disclinations, twins, shear bands, inclusions, and combinations of these and other defects that arise in innumerable ways. A completely satisfactory approach to understanding deformation in the presence of real defects is beyond our abilities, but progress toward analyzing the micromechanics of plastic flow, failure, and fragmentation is possible with our current knowledge, and is necessary if we are to take full advantage of the massive computing resources that are becoming available. This monograph is intended to bridge the gap between the classical theory of deformation and a statistical theory of imperfect materials. The computer program in which these ideas are being implemented is SCRAM.

The theory in this volume draws heavily on ideas from continuum mechanics, including, especially, use of the polar decomposition theorem to determine the extent of rotation in a general deformation. (Mathematical notation, definitions and theorems are described in Appendix A.) Various definitions of strain and stress, and the general theory of elastic deformation as a process that depends only on the final state of a material, are taken from continuum mechanics. The emphasis in this work is different from continuum mechanics, however, for the computed deformation is thought of as the average of an ensemble of deformations with different but statistically similar defects. This difference is not emphasized until Section 10, when the principle of *Superposition of Strain Rates* is introduced, but it is the main subject of Volume II. In addition to these topics, we discuss at length material rotation and its rate, which have been imperfectly treated in the literature of continuum mechanics. Whereas many theorists take the view that many different rates that account for material rotation are valid, here we consider such a view to be physically unreasonable, and argue that good measurements would measure only one rate, which is the polar rate. In addition, we do not allow various definitions of strain on the same grounds. The view taken here is that the physical strain is unique, and that Almansi, Green, and Signorini strains are merely different representations of the same physical quantity. Other definitions of strain have no interest. This view is woefully undemocratic, but in writing a computer program to represent physical deformation a definite choice must be made. Furthermore, the physical basis for this view seems compelling. The relation to Hill's general classification is given in Appendix B.

With regard to polar decomposition, the usual emphasis and terminology are somewhat modified. In standard works the deformation gradient is factored with a stretch matrix written either on the right or left of the rotation matrix, and, hence, the stretches are called right and left stretches. Here, the approach is physical rather than typographical, and the stretches have reference axes either fixed in space (left stretch) or rotating with the material (right stretch). Other quantities such as stress and strain can also be referred to either spatial or material axes by a suitable rotation. This distinction clarifies many concepts in the theory of deformation. For example, it becomes clear that Signorini strain and Green strain are related by a rotation from one set of axes to the other, and that Piola stress is defined in polar, not spatial, axes.

Rate equations describing deformation can be formulated by differentiating the expression for polar decomposition. It is shown that the rate of polar rotation, $\mathbf{\Omega}$, is different, in general, from vorticity, $\mathbf{W}$, though they can be made to coincide at any particular instant by a suitable choice of reference time (i.e., choosing reference time as current time).

It is also shown that vorticity is not an adequate measure of the rate of rotation for solids. In computing rates of matrix quantities such as stress, strain, and compliance, it is necessary to account for the rate of material rotation when the reference axes are fixed in space, but this is not necessary if polar axes are chosen. Both approaches and their relationship are discussed in detail. Rates of change of matrices referred to space axes are called polar rates here, since the analysis uses polar decomposition as its starting point.[1] If polar axes are used in computing the stress, conversions must be made from spatial axes.

A natural definition of strain $\boldsymbol{\varepsilon}$ in the current context is that of Signorini, though the motivation is quite different from his. It turns out that its polar rate $\hat{\boldsymbol{\varepsilon}}$ is proportional (in a certain sense) to the stretching, $\mathbf{D}$ (the symmetric part of the velocity gradient, using Truesdell's terminology). The stretching is not the rate of change of any path-independent strain, though it is often called *strain rate* in the older literature. When $\mathbf{D}$ is known, a unique strain can be found by integrating $\hat{\boldsymbol{\varepsilon}} = \mathbf{VDV}$, in which the carat accounts for the polar rate of rotation, as discussed in the body of this report. It is shown in Section 10 that $\mathbf{D}$ can be written as the sum of several parts, each one concerned with a separate physical process, such as elastic deformation of the material matrix, plastic flow, opening of voids, and shearing of closed cracks. In view of the proportionality of Signorini strain rate and stretching, the Signorini strain rate can also be written as the sum of parts in the same way. This *Superposition of Strain Rates* goes back to Reuss, but the current treatment is motivated by microphysical considerations rather than by the macroscopic consideration that motivated Reuss, and the application is more general, accounting for crack growth and nonlinear elasticity.

The tensor character of strain and stretching is considered at some length in Appendix C. The tensor concept is used here in a more restrictive sense than in continuum mechanics. For many authors (especially mathematicians), quantities that transform according to the appropriate mathematical laws *in any space* are considered tensors. Here, however, we follow the view common in physics that only quantities that transform like tensors in physical space are considered admissible. It is shown that the usual Green strain[2] is not a tensor in this restricted sense, though Signorini strain is.

The selection of a suitable measure of strain is important for several reasons. First, in formulating constitutive laws it is important to adopt relations between appropriate physical quantities. An example of an inappropriate formulation is the use of the deformation gradient $\mathbf{F}$ as a measure of distortion, since it involves material rotation as well as strain. (This is discussed in Appendix D.) In the analysis of isotropic elastic bodies, it turns out that an analysis using $\mathbf{F}$ may not actually be erroneous, because the constitutive laws are formulated in such a way that the rotation disappears from the constitutive laws, but this approach is misleading since it will not work for anisotropic or

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[1] The names of various tensor quantities vary widely. In a previous version of this report the variables in the frame obtained by rotating to polar axes were called "material quantities" since they tend to follow material rotation. On reflection, however, this is considered a poor choice because axes fixed in a material do not normally remain orthogonal, especially under extreme shear, while polar axes do. Hence, the rotated axes are called "polar axes" in this revised report. Thus, we deal with spatial and polar axes. These are called Eulerian (or Eulerean) by some authors, but this practice is considered confusing since it involves a rather poor analogy to the choice of Eulerian and Lagrangean independent variables in fluid dynamics. These terms, in turn, do not reflect the contributions of Euler and Lagrange, as discussed at length by Truesdell. (Truesdell attributes the Lagrangean description to Euler and the Eulerian description to d'Alembert.)

[2] This is frequently called the right Cauchy-Green tensor, but in the interest of brevity it is called the Green strain here.

inelastic materials. It is preferable to select variables that measure physical behavior directly in the first place, and to adopt an approach that promotes physical thinking rather than abstract formalism.

A second reason for narrowing down the choice of variables in constitutive laws concerns computational methods, which involve much of our time and resources nowadays. Computer programs can be written involving functional relations between any measures of strain, stress, strain rate and stress rate, but it would be unreasonable to suppose that all measures are equally valid and useful. The premise in the current approach is that a combination of mathematical and physical analysis can lead us to a good selection of variables without resorting to experiment or comparisons between computational schemes.

The plan of this monograph is to develop a self-contained analysis of finite deformation. The theory of polar decomposition is discussed in Section 2. Rates arising from polar decomposition are considered in Section 3, including an important explicit relation between the vorticity $\mathbf{W}$ and the polar rate of rotation, $\boldsymbol{\Omega}$. In Section 4, the form of the most general constitutive law for an elastic deformation is derived. The next four sections give illustrations of the general theory, comparing especially the spatial and polar representations of deformation. Simple shear, discussed in Section 5, involves displacement in a fixed direction, and is a common example of deformation in continuum mechanics. In practice it arises in boundary layers of fluids, in shear bands in solids, and in shear cracks under some conditions. It constitutes an interesting example of finite deformation because both the shear and rotation can be very large. Rigid body motion, Section 6, is trivial, but it is of some interest to see how the theory of deformation reduces to rigid-body motion as a limiting case. Torsion, characterized in Section 7, is quite complex. The stretch and rotation matrices are computed exactly (for the first time, I believe), as well as the strain and strain rate. The full characterization of stress and strain given here is necessary to interpret experiments with large torsion exactly, but no such effort is currently in progress insofar as I know. Though the simple vortex is observed in fluids rather than solids, it illustrates nicely the differences between vorticity, which is zero for an ideal vortex, and rate of polar rotation, which does not vanish. This is discussed in Section 8.

The formulation of constitutive laws is considered next, and it is shown in Section 9 that the general form involves the polar stress rate. The argument is similar to Noll's (1955), but by means of a more general choice of reference axes it is shown that it is the polar stress rate that is appropriate in general, not the more traditional stress rate of Zaremba (1903), Jaumann (1911), and Noll. (Noll assumes that the reference time is the current time.) In this line of argument, the form of the stress rate is derived. This approach is considered stronger and more compelling physically than the usual approach of testing for invariance of the stress rate. The concept of *Superposition of Strain Rates* is discussed in Section 10 in general terms. An alternative, more general derivation is given in Section 11. This superposition principle is fundamental to the analysis of deformation. It is believed to provide a more rigorous and general starting point than the product decomposition sometimes used in finite-deformation plasticity. Details of the physical theory are outlined in Volume II, Physical Theory.[3]

[3] Comments on any aspect of this analysis are most welcome (jkd@lanl.gov).

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2. POLAR DECOMPOSITION

The deformation of a continuum at any point is characterized by

$$\mathbf{F} = \left(F_{ij}\right) = \left(\frac{\partial x_i}{\partial X_j}\right) , \qquad 2.1$$

the deformation gradient, where $x_i(X_j, t)$ denotes the cartesian coordinates at time t of a point with initial cartesian coordinates X_i. It includes both rotation and stretch, which can be characterized by means of the polar decomposition theorem.[4] This theorem makes it possible to write the matrix $\mathbf{F}$[5] as the product of a positive definite matrix $\mathbf{V}$ (stretch) and an orthogonal matrix **R** (rotation), so that

$$\mathbf{F}=\mathbf{VR} \; . \qquad 2.2$$

This equation allows for a mathematical interpretation of the deformation of an infinitesimal line element, ΔX_j, between two material points as if it occurred in two steps: a rotation **R**, followed by a stretch **V**. The notation and relevant aspects of the theory of matrices are summarized in Appendix A. The tensor character of the variables is not an issue in the matrix theory discussed here, and all indexed variables will be described with subscripts. This is not to suggest that the tensor character of the variables is not important; rather, it is considered vital, but it is just not relevant to the current discussion. The tensor character of the variables is addressed in Appendix C. The discussion in this section follows many of the ideas standard in continuum mechanics, but continuity of the deformation is considered to be of a mathematical rather than physical character since the medium of interest may not be continuous. If the physical medium contains defects, we can consider the average of an ensemble of deformations in materials with defects that have a certain statistical character. The mean motion can then be treated as continuous, though individual members of the ensemble are discontinuous in the neighborhood of defects.

The positive-definite character of **V**, which requires that $V_{ij}x_ix_j$ be positive for any choice of the x_i, follows from the positive-definite character of $\mathbf{FF}^T$ (Truesdell's **B**). To show this, we may begin by noting that

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[4] Polar decomposition was first demonstrated by Autonne (1902) and was introduced into thermoelasticity by Signorini (1930). Noll (1945) exploited the theorem in his thesis work, which is a standard reference in continuum mechanics.

[5] **F** can be said to transform as a mixed tensor involving both spatial and reference coordinate systems, but this mathematical description does not have any useful physical interpretation (invariance) in the way the contravariant stress tensor and covariant strain tensor do. This writer finds it misleading to characterize **F** as a tensor.

$$x_i F_{ij} F_{kj} x_k = \sum_j \left(x_i F_{ij}\right)^2 = x_i B_{ik} x_k \quad ,$$

is nonnegative since it can be expressed as a sum of squares. This defines **B**, as discussed further in Appendix C. Now, let **T** denote the matrix of eigenvectors of **B,** and let $\mathbf{\Lambda}$ denote the (diagonal) matrix of its eigenvalues. Then, as indicated in Eq. A.36 (here and in what follows, the A denotes an equation in Appendix A),

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^T \quad . \tag{2.3}$$

The diagonal components of $\mathbf{\Lambda}$ are all positive since **B** is positive definite. It can be readily verified that the stretch can be expressed in the form

$$\mathbf{V} = \mathbf{T}\boldsymbol{\lambda}\mathbf{T}^T \quad , \tag{2.4}$$

where the components λ_i of the diagonal matrix $\boldsymbol{\lambda}$ are the positive square roots of the corresponding elements Λ_i of $\mathbf{\Lambda}$. Since

$$V_{ij} x_i x_j = \sum_k \lambda_k \left(x_i T_{ik}\right)^2 \tag{2.5}$$

is positive, **V** is positive definite.

To show that **R** is orthogonal, write, using Eqs. 2.2 and 2.3,

$$\mathbf{R}\mathbf{R}^T = \mathbf{V}^{-1}\mathbf{F}\mathbf{F}^T\mathbf{V}^{-1} = \mathbf{V}^{-1}\mathbf{B}\mathbf{V}^{-1} = \mathbf{I} \quad . \tag{2.6}$$

This completes the proof, for if the product of a matrix and its transpose is the identity matrix **I**, then it is orthogonal, as indicated by Eq. A.44.

Another polar decomposition is also possible in which the stretch occurs second typographically. It can be written

$$\mathbf{F} = \mathbf{Q}\overline{\mathbf{V}} \quad . \tag{2.7}$$

To show the relation to Eq. 2.2, the expression for **F** given therein can be rewritten as

$$\mathbf{F} = \mathbf{R}\mathbf{R}^T\mathbf{V}\mathbf{R} = \mathbf{R}\left(\mathbf{V}^{1/2}\mathbf{R}\right)^T\left(\mathbf{V}^{1/2}\mathbf{R}\right) \quad . \tag{2.8}$$

Now, the product of the two quantities in parentheses is positive definite and, hence, equal to $\overline{\mathbf{V}}$ by comparison with Eq. 2.7, provided polar decomposition is unique, for then $\mathbf{Q} = \mathbf{R}$ and $\overline{\mathbf{V}} = \mathbf{R}^T\mathbf{V}\mathbf{R}$.

To show that the decomposition is indeed unique, assume for the moment that there is another decomposition

$$\mathbf{F} = \tilde{\mathbf{Q}}\tilde{\mathbf{V}} \quad , \tag{2.9}$$

where $\tilde{\mathbf{Q}}$ is orthogonal and $\tilde{\mathbf{V}}$ is positive-definite. Then,

$$\mathbf{F}^T\mathbf{F} = \tilde{\mathbf{V}}\tilde{\mathbf{Q}}^T\tilde{\mathbf{Q}}\tilde{\mathbf{V}} = \tilde{\mathbf{V}}^2 \quad . \tag{2.10}$$

But $\mathbf{F}^T\mathbf{F}$ has a unique, positive-definite square root, by the same argument as above, so $\tilde{\mathbf{V}} = \overline{\mathbf{V}}$, and it follows that $\tilde{\mathbf{Q}} = \mathbf{Q} = \mathbf{R}$.

Using these results (Eqs. 2.2 and 2.4), **F** can be written as the triple product

$$\mathbf{F} = \mathbf{T}\boldsymbol{\lambda}\mathbf{S} \qquad 2.11$$

in a unique way where

$$\mathbf{S} = \mathbf{T}^T\mathbf{R} \quad . \qquad 2.12$$

Thus, any deformation can be represented as a rotation **S**, followed by an orthogonal stretch $\boldsymbol{\lambda}$, followed by another rotation $\mathbf{T}$. These operations can be grouped in two ways:

$$\mathbf{F} = (\mathbf{T}\mathbf{S})\left(\mathbf{S}^T\boldsymbol{\lambda}\mathbf{S}\right) = \mathbf{R}\overline{\mathbf{V}} \qquad 2.13$$

and

$$\mathbf{F} = \left(\mathbf{T}\boldsymbol{\lambda}\mathbf{T}^T\right)(\mathbf{T}\mathbf{S}) = \mathbf{V}\mathbf{R} \quad . \qquad 2.14$$

It is appropriate to describe the grouping of Eq. 2.13 as a material polar decomposition, since $\overline{\mathbf{V}}$ is associated with the stretch ellipsoid whose principal axes (**S**) are measured from rotated axes (i.e., axes that rotate with the material), in view of Eq. 2.12. Then, it is also appropriate to describe the grouping of Eq. 2.14 as a spatial polar decomposition, since **V** is associated with the stretch ellipsoid whose principal axes (**T**) are specified with respect to axes fixed in space. The distinction is discussed in more detail in connection with the example of simple shear in Section 5, and is illustrated in Fig. 5.1, which can be examined without reading Section 5.

It seems natural to call the orthogonal axes specified by **R** "polar axes," since they are uniquely specified by polar decomposition. These axes are orthogonal and rotate with the deformation though they are not fixed in the material. Another set of axes is specified by the principal axes of the stretch ellipsoid. When the stretch is specified with respect to polar axes, which rotate with the deformation as in Eq. 2.13, it is natural to call it $\overline{\mathbf{V}}$, either polar or material stretch. When the stretch is specified with respect to space axes, which are fixed as in Eq. 2.14, it is natural to call it the spatial stretch **V**. *In either case, the stretch is specified by the same ellipsoid.* The difference lies in the reference axes. As mentioned above, though the polar axes rotate with the deformation, they are not fixed in the material. In particular, they are orthogonal, whereas axes fixed in the material become skewed.

The stretch $\overline{\mathbf{V}}$ is generally called the right stretch in continuum mechanics because it is written to the right of **R**. The principal axes of the stretch ellipsoid are called a Lagrangean triad by Hill (1978) when measured from the polar axes. The spatial stretch **V** is often called the left stretch to recall its formal location to the left of **R** in Eq. 2.14. The principal axes when defined with respect to spatial axes **T** are called a Eulerian triad by Hill. The relation to Hill's approach is discussed in Appendix B following Eq. B.5.

The matrix **R** is determined algebraically from **F**. It has the properties of a rotation matrix since it is orthogonal, but its geometrical significance is not intuitively obvious. Of course, if a body rotates without stretching, then $\mathbf{V} = \mathbf{I}$, and it follows that **F** = **R**, and the significance is unambiguous. But if the stretching is very large, it is not intuitively obvious that the mapping from an initially orthogonal set of axes to a highly skewed set of axes involves an unambiguous rotation, though it is mathematically straightforward. An interpretation of **R** can be obtained by considering the triple product $\mathbf{T\lambda S}$ of Eq. 2.11. Then, consider another mapping from the deformed state in which the orthogonal stretching does not take place, i.e., $\boldsymbol{\lambda} = \mathbf{I}$. Then **F** = **TS** = **R**. Thus, **R** is the transformation that obtains if the lengths of the line elements were to remain unchanged in the second of the three stages of the transformation of Eq. 2.11.

Since stress is a tensor, constitutive laws should relate stress to a tensor measure of strain, stretch, and/or related tensors. As shown in Appendix C, **B** is a tensor measure of strain, though $\overline{\mathbf{B}}$ is not. The deformation **F** itself does not transform like a tensor, so it should not appear in formulating constitutive laws. (In some simple elastic constitutive laws, **F** appears in the form $\mathbf{F}\mathbf{F}^T$, and this is admissible because the rotation **R** disappears, and this is simply a nonphysical way of writing **B** (Eq. 2.3).

Since **F** involves the deformation of a fiber (dX_i), it might be supposed that **R** is useful only when the underlying physics involves stretching of fibers. However, **R** also appears when shear processes are to be accounted for. To see this, note that the rotation of an element of area in shear processes is characterized by rotation of its normal, $\overline{\mathbf{n}}$, as discussed in deriving Eq. A.66 ($\overline{\mathbf{n}} = \Gamma\mathbf{F}^{-T}\mathbf{N}$). Now $\mathbf{F}^{-T}$ has the polar decomposition $\tilde{\mathbf{V}}\tilde{\mathbf{R}}$ (any matrix has such a decomposition). Let us consider the relation of these matrices to the stretch **V** and rotation **R**. Since polar decomposition is unique,

$$\tilde{\mathbf{V}} = \mathbf{V}^{-1}, \quad \tilde{\mathbf{R}} = \mathbf{R} \quad . \tag{2.15}$$

Thus, **R** also characterizes the rotation of a triad of normals (not necessarily orthogonal) that characterize 3 planes in a general deformation, and it thereby characterizes the rotation of shear defects (such as shear cracks, shear bands, or dislocation loops) as well as lengthening (or shortening) of fibers.

3. RELATIONS BETWEEN VARIOUS RATES

The velocity gradient **G** is related to the deformation gradient **F** through

$$\mathbf{G} = \left(\frac{\partial u_i}{\partial x_k}\right) = \left(\frac{\partial \dot{x}_i}{\partial X_j}\right)\left(\frac{\partial X_j}{\partial x_k}\right) = \dot{\mathbf{F}}\mathbf{F}^{-1} \quad , \tag{3.1}$$

where **F** is given by Eq. 2.1, and the factorization expresses an application of the chain rule. In this expression u_i denotes the velocity

$$u_i(x,t) = \frac{\partial x_i\left(X_j,t\right)}{\partial t} = \dot{x}_i\left(X_j,t\right) \quad , \tag{3.2}$$

that is, the rate of change of position of a point fixed in the material (having a specified X_j). Now, any matrix can be represented as the sum of a symmetric part and an antisymmetric part. Thus, the velocity gradient can be represented as the sum of a symmetric stretching matrix **D** and an antisymmetric vorticity matrix **W**, so that

$$\mathbf{G} = \mathbf{D} + \mathbf{W} \quad . \tag{3.3}$$

By combining Eqs. 3.1, 3.3, and 2.2, it can be shown that

$$\mathbf{D} + \mathbf{W} = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}\mathbf{V}^{-1} \quad , \tag{3.4}$$

where

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^{-1} \tag{3.5}$$

is an antisymmetric matrix representing the rate of polar rotation. By premultiplying and postmultiplying Eq. 3.4 by **V** and adding the former to the transpose of the latter, it can be shown that

$$\dot{\mathbf{B}} + \mathbf{B}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{B} = 2\mathbf{VDV} \quad , \tag{3.6}$$

where **B**, the square of **V**, is defined by Eq. 2.3.

It will prove useful to define the Signorini (1931) strain

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}(\mathbf{B} - \mathbf{I}) \quad . \tag{3.7}$$

This spatial strain (based on the spatial stretch **V**) satisfies the relation

$$\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{\varepsilon} = \mathbf{VDV} \tag{3.8}$$

that follows readily from Eqs. 3.6 and 3.7. The relation of the Signorini strain to other strain measures commonly used in continuum mechanics is discussed in Appendices B and C. The quantity on the left side of Eq. 3.8 I call the polar strain rate. The stretching **D** is called the strain rate in some works on continuum mechanics, but this seems misleading since it is not the rate of change of any well-defined, history-independent quantity. On the other hand, the left side of Eq. 3.8 is the rate of change of a material state $\boldsymbol{\varepsilon}$ involving a rate (the polar rate) appropriate to spatial tensors. This rate will be discussed further in this section and examples given in subsequent sections. For small deformations, **V** is nearly equal to **I**, and the strain rate can be approximated as **D**.

Matrices of the form $\dot{\mathbf{A}} + \mathbf{A}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{A}$ appearing in Eq. 3.8 represent the rate of change of **A** in axes having fixed orientation, accounting for the effect of polar rotation on the components of **A**. These may be called polar rates, since $\boldsymbol{\Omega}$ arises from polar decomposition. Alternatively, it may be useful to work in polar axes $\bar{\mathbf{x}}$, which are related to the fixed axes by

$$\bar{\mathbf{x}} = \mathbf{R}^T\mathbf{x} \quad . \tag{3.9}$$

(In an earlier paper I called these “material axes.” This is not a particularly good name, since axes specified in a material at some reference time may also be reasonably called material axes but they become skewed at later times. Such axes are different from the axes characterized by Eq. 3.9, which remain orthogonal as a material deforms. “Polar axes” seems a natural term for the axes of Eq. 3.9, since determination of these axes requires polar decomposition. It is consistent with the phrase “polar rate” introduced above.) The polar axes represent a set of coordinates from which the deformation can be considered a pure stretch (that is, without rotation). As in Section 2, quantities represented in polar axes are denoted with bars. In particular, matrix quantities in polar axes are related to those in fixed axes by

$$\overline{\mathbf{A}} = \mathbf{R}^T\mathbf{A}\mathbf{R} \;\; , \;\; \mathbf{A} = \mathbf{R}\overline{\mathbf{A}}\mathbf{R}^{\mathrm{T}} \quad , \tag{3.10 a,b}$$

$$\mathbf{R}\left(\frac{d}{dt}\overline{\mathbf{A}}\right)\mathbf{R}^T = \dot{\mathbf{A}} + \mathbf{A}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{A} = \hat{\mathbf{A}} \quad . \tag{3.11}$$

This rate, denoted by a carat and called (here) a polar rate, has some of the properties of an ordinary derivative. For, if **Z** = **X + Y**, then

$$\hat{\mathbf{Z}} = \hat{\mathbf{X}} + \hat{\mathbf{Y}} \quad . \tag{3.12}$$

Also, if **Z** = **XY**, then it can be readily shown that

$$\hat{\mathbf{Z}} = \hat{\mathbf{X}}\mathbf{Y} + \mathbf{X}\hat{\mathbf{Y}} \quad . \tag{3.13}$$

With this notation, Eqs. 3.6 and 3.8 become

$$\hat{\mathbf{B}} = 2\mathbf{V}\mathbf{D}\mathbf{V} \tag{3.14}$$

and

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{V}\mathbf{D}\mathbf{V} \quad . \tag{3.15}$$

If we define the Green metric $\overline{\mathbf{B}}$ as

$$\overline{\mathbf{B}} = \mathbf{R}^{\mathbf{T}}\mathbf{B}\mathbf{R} = \mathbf{F}^{\mathbf{T}}\mathbf{F} \quad , \tag{3.16a,b}$$

and polar strain as

$$\overline{\boldsymbol{\varepsilon}} = \mathbf{R}^T\boldsymbol{\varepsilon}\mathbf{R} \quad , \tag{3.17}$$

then Eqs. 3.14 and 3.15 become

$$\dot{\overline{\mathbf{B}}} = 2\overline{\mathbf{V}}\overline{\mathbf{D}}\overline{\mathbf{V}} \tag{3.18}$$

and

$$\dot{\overline{\boldsymbol{\varepsilon}}} = \overline{\mathbf{V}}\overline{\mathbf{D}}\overline{\mathbf{V}} \quad , \tag{3.19}$$

with the dot denoting an ordinary time derivative. In using these relations, it should be kept in mind that the polar axes are constantly changing as deformation proceeds.

Relation of Polar-Rotation Rate and Vorticity

A relation between rate of polar rotation $\boldsymbol{\Omega}$ and vorticity **W** can be obtained by postmultiplying Eq. 3.4 by **V** and subtracting the transpose of the result, leading to

$$\mathbf{V}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{V} - \mathbf{V}\mathbf{W} - \mathbf{W}\mathbf{V} = \mathbf{Z} \quad , \tag{3.20}$$

where **Z** denotes the antisymmetric commutator matrix

$$\mathbf{Z} = \mathbf{DV} - \mathbf{VD} \quad . \tag{3.21}$$

The matrix equation 3.20 can be considered as a set of three scalar equations in three unknowns, the independent components of $\boldsymbol{\Omega}$. Since Eq. 3.20 involves three antisymmetric matrices, $\boldsymbol{\Omega}$, $\mathbf{W}$, and $\mathbf{Z}$, it is natural to consider its form when expressed in terms of the associated vectors $\boldsymbol{\omega}$, $\mathbf{w}$, and $\mathbf{z}$. The necessary relations are

$$\Omega_{ik} = e_{ijk}\omega_j \quad , \tag{3.22}$$

$$W_{ik} = e_{ijk}w_j \quad , \tag{3.23}$$

and

$$Z_{ik} = e_{ijk}z_j \quad , \tag{3.24}$$

where e_{ijk} is defined following Eq. A.15. To simplify the calculations it is convenient to define a relative rate

$$\mathbf{H} = \boldsymbol{\Omega} - \mathbf{W} \tag{3.25}$$

and the associated vector h_j by

$$H_{ik} = e_{ijk}h_j \quad . \tag{3.26}$$

Writing out Eq. 3.20 in this notation, we find

$$h_\ell\left(V_{ij}e_{j\ell k} + e_{i\ell j}V_{jk}\right) = Z_{ik} \quad . \tag{3.27}$$

This set of equations can be put in the equivalent form using Eq. A.30,

$$h_\ell V_{ii} - h_i V_{i\ell} = z_\ell \quad , \tag{3.28}$$

or in matrix notation, treating **h** and **z** as vectors

$$(\mathbf{I}\ \mathrm{tr}\ \mathbf{V}\text{-}\mathbf{V})\ \mathbf{h} = \mathbf{z} \quad , \tag{3.29}$$

where tr **V** denotes the trace of **V**, that is,

$$tr\,\mathbf{V} = V_{ii} \tag{3.30}$$

summed on i. When Eq. 3.29 is solved for **h**, and **h** is expressed in terms of $\boldsymbol{\omega}$ and **w**, an explicit expression for the rate of polar rotation is obtained:

$$\boldsymbol{\omega} = \mathbf{w} + \left(\mathbf{I}\ tr\mathbf{V} - \mathbf{V}\right)^{-1}\mathbf{z} \quad . \tag{3.31}$$

It is sometimes convenient to separate the deformation into an isotropic part and a deviatoric part (denoted by primes) whose trace vanishes, so that

$$\mathbf{D} = \mathbf{D}' + \frac{1}{3}\dot{\theta}\ \mathbf{I} \tag{3.32}$$

and

$$\mathbf{V} = \mathbf{V}' + \frac{1}{3} v\, \mathbf{I} \quad , \qquad 3.33$$

where $\dot{\theta}$ is the rate of change of compression and v is the dilatation. Then,

$$\mathbf{Z} = \mathbf{DV} - \mathbf{VD} = \mathbf{D'}\,\mathbf{V'} - \mathbf{V'}\,\mathbf{D'} \quad , \qquad 3.34$$

indicating that the isotropic part of the deformation does not influence **Z**. **Z** vanishes for rigid body motion **D** = **0**, as well as for pure dilatation, **V** = a **I**. It also vanishes if the rate of deformation is isotropic, so that $\mathbf{D} = \dot{\theta}\, \mathbf{I} / 3$. More generally, as pointed out by Zuo (2001), Z vanishes when **D** and **V** are parallel (in the sense that they have the same eigenvectors).

The rate of polar rotation is expressed in Eq. 3.31 as the sum of two parts, the vorticity and a historical part, the former independent of previous motion and the latter involving **V** as well as **D**. It is shown in Appendix A that the rotation **R** can be expressed in a more physical manner by Eq. A.45 ($\mathbf{R} = e^{\phi \mathbf{Q}}$), where ϕ is the angle of rotation and **Q** represents the axis of rotation, through Eq. A.47 ($Q_{ik} = e_{ijk} r_j$) , with $\mathbf{r} = (r_1, r_2, r_3)$ a unit vector along the axis of rotation. This leads to a representation of Ω in terms of ϕ and **Q** given as Eq. A.59, which can be expressed in vector form as

$$\boldsymbol{\omega} = \dot{\phi}\, \mathbf{r} + \sin\phi\;\dot{\mathbf{r}} + (1 - \cos\phi)\dot{\mathbf{r}}\, x\, \mathbf{r} \quad . \qquad 3.35$$

The rate of rotation can be expressed explicitly in terms of $\boldsymbol{\omega}$ by solving Eq. 3.35 for $\dot{\mathbf{r}}$. That is, Eq. 3.35 can be written as

$$\mathbf{A}\dot{\mathbf{r}} = \mathbf{p} \quad , \qquad 3.36$$

where

$$\mathbf{A} = \begin{pmatrix} \sin\phi & (1-\cos\phi) r_3 & -(1-\cos\phi) r_2 \\ -(1-\cos\phi) r_3 & \sin\phi & (1-\cos\phi) r_1 \\ (1-\cos\phi) r_2 & -(1-\cos\phi) r_1 & \sin\phi \end{pmatrix} \qquad 3.37$$

and

$$\mathbf{p} = \boldsymbol{\omega} - \dot{\phi}\mathbf{r} \quad . \qquad 3.38$$

Then,

$$\dot{\mathbf{r}} = \mathbf{A}^{-1}\mathbf{p} \quad . \qquad 3.39$$

The angular rate can be expressed in terms of $\boldsymbol{\omega}$ by taking the dot product of Eq. 3.35 with **r**, so that

$$\dot{\phi} = \mathbf{r} \bullet \boldsymbol{\omega} \quad . \qquad 3.40$$

Now, since $\boldsymbol{\omega}$ can be expressed as the sum of two parts, according to Eq. 3.31, so can $\dot{\mathbf{r}}$. This is done by writing

$$\boldsymbol{\omega} = \mathbf{w} + \mathbf{h} \quad , \quad \mathbf{h} = \left(\mathbf{I}\; tr\mathbf{V} - \mathbf{V}\right)^{-1} \mathbf{z} \quad . \qquad 3.41a,b$$

Then,

$$\mathbf{p} = \mathbf{w} + \mathbf{h} - (\dot{\phi}_w + \dot{\phi}_h)\mathbf{r} = \mathbf{p}_\mathrm{w} + \mathbf{p}_\mathrm{h} \quad , \qquad 3.42,a,b$$

where

$$\dot{\phi}_\mathrm{w} = \mathbf{r} \bullet \mathbf{w} \quad , \ \dot{\phi}_\mathrm{h} = \mathbf{r} \bullet \mathbf{h} \quad . \qquad 3.43a,b$$

Thus,

$$\dot{\mathbf{r}}_\mathrm{w} = \mathbf{A}^{-1}\mathbf{p}_\mathrm{w} = \mathbf{A}^{-1}(\mathbf{w} - \mathbf{w} \bullet \mathbf{r}\,\mathbf{r}) \qquad 3.44a,b$$

$$\dot{\mathbf{r}}_\mathrm{h} = \mathbf{A}^{-1}\mathbf{p}_\mathrm{h} = \mathbf{A}^{-1}(\mathbf{h} - \mathbf{h} \bullet \mathbf{r}\,\mathbf{r}) \qquad 3.45a,b$$

The stress rate that arises in computing the normal component of traction on a crack is slightly different. (For cracks with reasonably simple shapes, such as the plane two-dimensional crack or the penny-shaped crack, this quantity determines the crack opening). To compute the rate of change of crack opening, we require the rate of change of the normal component of traction, σ^n, which can be written

$$\dot{\sigma}^n = \frac{d}{dt}\left(\sigma_{ij} n_i n_j\right) \quad . \qquad 3.46$$

Now, as the material in which the crack is imbedded rotates, the crack normal rotates. To compute the rate of rotation, we can make use of Eq. A.66 written in index form as

$$n_i = L_{ik} N_k \quad , \qquad 3.47$$

where N_k represents the normal at t = 0. Then,

$$\dot{n}_i = \tilde{\Omega}_{ik} n_k \quad , \qquad 3.48$$

where, using Eq. A.72,

$$\tilde{\mathbf{\Omega}} = \dot{\Gamma}\,\mathbf{I} / \Gamma - \mathbf{G}^T \qquad 3.49$$

is different from $\mathbf{\Omega}$ given by Eq. 3.5 and, in particular, is not antisymmetric. Thus, the expression for $\dot{\sigma}^n$ becomes, after some algebra,

$$\dot{\sigma}^n = \tilde{\sigma}_{ij} n_i n_j \quad , \qquad 3.50$$

where the operation (~) defines a "planar rate"

$$\tilde{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A}\tilde{\mathbf{\Omega}} + \tilde{\mathbf{\Omega}}^T\mathbf{A} \quad . \qquad 3.51$$

Thus, in the calculation of invariants, it may be necessary to account for polar-like rates. If σ^n represents a "constant" normal stress, this requires that $\tilde{\sigma}_{ij} = 0$, and not that $\dot{\sigma}_{ij} = 0$, since the latter does not account for changing axes.

4. ENERGETICS AND EQUILIBRIUM ELASTICITY

The rate at which quasistatic work is done on a mass M of material bounded by a surface S, as illustrated in Fig. 4.1, is given by

$$\dot{W} = \int_S \sigma_{ij} n_j u_i dS \quad . \tag{4.1}$$

Here, σ_{ij} denotes the Cauchy stress tensor, which can be written in matrix form as

$$\boldsymbol{\sigma} = \left(\sigma_{ij}\right) \quad , \tag{4.2}$$

the components of velocity are written as u_i, and n_j represents the jth component of the outward unit normal to S. By mean of Gauss' theorem, $\dot{W}$ can be put in either of the forms

$$\dot{W} = \int_V \sigma_{ij}\, d_{ij}\, dV = \int_M \sigma_{ij}\, d_{ij}\, v\, dM \quad , \tag{4.3}$$

where the first integral is taken over the volume V and the second over the mass M, provided the mass is considered to be in static equilibrium so that $\sigma_{ij,j} = 0$. The stress tensor is assumed symmetric, and v denotes the specific volume. Thus, the stretching $\mathbf{D} = \left(d_{ij}\right)$ introduced in Eq. 3.3 arises naturally in energy considerations. If only mechanical effects are considered, the internal energy I per unit mass is given by

$$I = I_o + \int_o^t \boldsymbol{\sigma}_{ij}\, d_{ij}\, v\, dt \quad , \tag{4.4}$$

where I_o represents the initial specific internal energy. Then, Eq. 4.3 becomes

$$\dot{W} = \int_V \dot{I} \rho \, dV \quad , \tag{4.5}$$

where $\rho = 1 / v$.

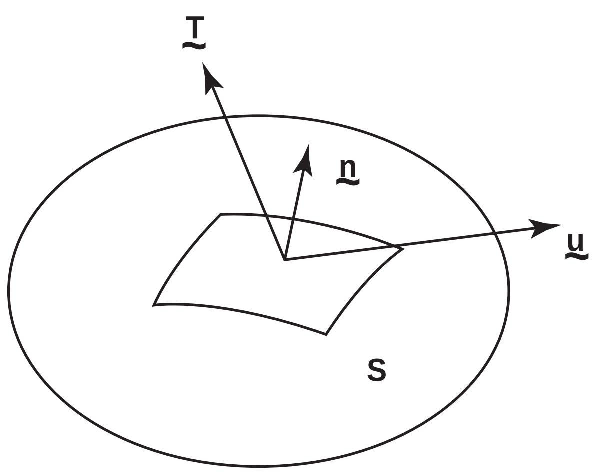

Fig. 4.1 Illustration of a mass M bounded by the surface S. In index notation, the local normal is given by n_i; the traction, $T_j = \sigma_{ij} n_i$; and the local velocity, u_i.

A constitutive law represents elastic behavior if the strain energy per unit mass depends only on the current strain $\bar{\boldsymbol{\varepsilon}}$, and is, consequently, independent of strain history, i.e.,

$$I - I_o = \int_0^t v \, tr \left(\bar{\boldsymbol{\sigma}} \bar{\mathbf{D}} \right) dt = f(\bar{\boldsymbol{\varepsilon}}) \quad , \qquad 4.6$$

where *tr* denotes the trace of the matrix product. The function $f(\bar{\boldsymbol{\varepsilon}})$ may be an anisotropic function of the strain $\bar{\boldsymbol{\varepsilon}}$. The integrand is written in terms of the polar quantities $\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbf{D}}$, rather than in terms of the spatial quantities $\boldsymbol{\sigma}$ and $\mathbf{D}$, in order to account for material anisotropy in the subsequent analysis. It is straightforward to show that this is valid, since

$$tr \, \bar{\boldsymbol{\sigma}} \bar{\mathbf{D}} = tr \, \boldsymbol{\sigma} \mathbf{D} \quad , \qquad 4.7$$

using Eq. 3.14 and the definition of trace given as Eq. 3.30. In order to determine a criterion for the internal energy I to be history independent, it is useful to define a quasi-stress

$$\bar{\boldsymbol{\tau}} = \frac{v}{v_o} \bar{\mathbf{V}}^{-1} \, \bar{\boldsymbol{\sigma}} \, \bar{\mathbf{V}}^{-1} \quad . \qquad 4.8$$

Then, using Eq. 3.19,

$$v \, tr \, \bar{\boldsymbol{\sigma}} \bar{\mathbf{D}} = v_o \, tr \bar{\boldsymbol{\tau}} \dot{\bar{\boldsymbol{\varepsilon}}} \quad , \qquad 4.9$$

and the integral of Eq. 4.6 becomes

$$I - I_o = v_o \int_o^{\varepsilon} tr \left(\bar{\boldsymbol{\tau}} \, d\bar{\boldsymbol{\varepsilon}} \right) \quad . \qquad 4.10$$

This integral is history-independent if

$$\bar{\tau}_{ij} = \rho_o \frac{\partial f}{\partial \bar{\varepsilon}_{ij}} \quad . \qquad 4.11$$

Consequently, from Eq. 4.8, the stress must be of the form

$$\bar{\boldsymbol{\sigma}} = \frac{\rho}{\rho_o} \bar{\mathbf{V}} \, \bar{\boldsymbol{\Sigma}} \, \bar{\mathbf{V}} \qquad 4.12$$

to ensure elastic behavior, where

$$\bar{\Sigma}_{ij} = \rho_o \frac{\partial f}{\partial \bar{\varepsilon}_{ij}} \qquad 4.13$$

and *f* denotes an elastic potential. The relation of this formula to similar formulas given by Rivlin and Ericson (1955) and discussed by Truesdell (1952) is discussed in Appendix D.

Isotropic Material

If a material is isotropic, the elastic potential f depends on the strain only through its invariants. An equivalent but more convenient formulation of isotropy is to let it depend on the invariants of stretch. It is straightforward to show that the invariants of $\mathbf{B}$ and $\bar{\mathbf{B}}$ are the same by means of Eq. 3.10. The invariants can be selected in numerous ways, but it is particularly convenient to select

$$I_1 = tr\ \overline{\mathbf{B}} = tr\ \mathbf{B} = \overline{B}_{ii} = B_{ii} \quad , \qquad 4.14$$

$$I_2' = \frac{1}{2} tr\ \overline{\mathbf{B}}' \overline{\mathbf{B}}' = \frac{1}{2} tr\ \mathbf{B}' \mathbf{B}' \quad , \qquad 4.15$$

$$I_3 = \left|\overline{\mathbf{B}}\right| = |\mathbf{B}| \quad , \qquad 4.16$$

where

$$\overline{\mathbf{B}}' = \overline{\mathbf{B}} - \frac{1}{3} I_1 \mathbf{I} \qquad 4.17$$

is the deviator of $\overline{\mathbf{B}}$. We can write the potential in terms of these invariants as

$$\mathrm{f}(\overline{\boldsymbol{\varepsilon}}) = \mathrm{g}\left(\mathrm{I}_1, \mathrm{I}_2', \mathrm{I}_3\right) \quad . \qquad 4.18$$

Now, it is straightforward to show that

$$\frac{\partial I_1}{\partial \overline{B}_{ij}} = \delta_{ij} \quad , \qquad 4.19$$

$$\frac{\partial \mathrm{I}_2'}{\partial \overline{\mathrm{B}}_{\mathrm{ij}}} = \overline{\mathrm{B}}'_{\mathrm{ij}} \quad , \qquad 4.20$$

and

$$\frac{\partial I_3}{\partial \overline{B}_{ij}} = \tilde{B}_{ij} \quad , \qquad 4.21$$

where $\tilde{B}_{ij}$ is the cofactor of $\overline{B}_{ij}$ in the determinant of $\overline{\mathbf{B}}$ and, consequently,

$$\tilde{\mathbf{B}} \overline{\mathbf{B}} = I_3 \mathbf{I} \quad , \qquad 4.22$$

as indicated in Eq. A.21. Denoting the matrix of cofactors of $\overline{\mathbf{V}}$ by $\tilde{\mathbf{V}}$, so that

$$\tilde{\mathbf{V}} \overline{\mathbf{V}} = \sqrt{I_3}\ \mathbf{I} \quad , \qquad 4.23$$

then Eqs. 4.12 and 4.13 reduce to

$$\overline{\boldsymbol{\sigma}} = 2\rho \left[\frac{\partial g}{\partial I_1} \overline{\mathbf{B}} + \frac{\partial g}{\partial I_2} \left(\overline{\mathbf{B}}^2 - \frac{1}{3} I_1 \overline{\mathbf{B}} \right) + \frac{\partial g}{\partial I_3} I_3 \mathbf{I} \right] \quad . \qquad 4.24$$

By means of the rotation operation indicated in Eq. 3.10, this can be put into the spatial form

$$\boldsymbol{\sigma} = 2\rho \left[\frac{\partial g}{\partial I_1} \mathbf{B} + \frac{\partial g}{\partial I_2} \left(\mathbf{B}^2 - \frac{1}{3} I_1 \mathbf{B} \right) + \frac{\partial g}{\partial I_3} I_3 \mathbf{I} \right] \quad . \qquad 4.25$$

The density is related to I_3 through

$$\left(\rho_o / \rho\right)^2 = I_3 \quad . \qquad 4.26$$

The validity of Eq. 4.25 can be verified in two steps. The first involves Eq. 3.14, from which the relations

$$2B_{ij}d_{ij} = tr\,\hat{\mathbf{B}} = \dot{I}_1 \quad , \qquad 4.27$$

$$2tr\left[\left(\mathbf{B}^2 - \frac{1}{3}I_1\mathbf{B}\right)\mathbf{D}\right] = \dot{I}_2' \quad , \qquad 4.28$$

and

$$2I_3 d_{ii} = |\mathbf{B}| d_{ii} = \dot{I}_3 \qquad 4.29$$

follow readily. Proof of Eq. 4.29 requires the use of Euler's identity, given in Appendix A as Eq. A.23. In the second step, the relations 4.25 and 4.27–4.29 are combined to show that

$$v\sigma_{ij}d_{ij} = v\,tr(\boldsymbol{\sigma}\mathbf{D}) = \dot{g} \quad . \qquad 4.30$$

This verifies that the rate of change of potential for an isotropic elastic material is equal to the rate at which work is done by the applied stresses.

When the deformation vanishes so that **B** = **I**, the stress should vanish (unless the material is in a prestress condition). In order for this condition to hold, it is necessary that

$$\frac{\partial g}{\partial I_1} + \frac{\partial g}{\partial I_3} = 0 \quad . \qquad 4.31$$

Since it follows from Eq. 4.14 that $I_1 = 3$, from Eq. 4.15 that $I_2' = 0$, and from Eq. 4.16 that $I_3 = 1$, then Eq. 4.31 is sufficient to make the stress vanish when **B** = **I**.

Anisotropic Material

We now return to the more general case in which the elastic material is not isotropic. If the expression for material stress given as Eqs. 4.12 and 4.13 is converted to spatial (Cauchy) stress by rotation, then

$$\boldsymbol{\sigma} = \mathbf{R}\bar{\boldsymbol{\sigma}}\mathbf{R}^T = \frac{\rho}{\rho_o}\mathbf{F}\,\bar{\bar{\boldsymbol{\Sigma}}}\,\mathbf{F}^T \quad . \qquad 4.32$$

This expression is given by Truesdell (1952) as Cosserat's form for an elastic material. (The dependence on **F** is something of an accident which occurs because the polar stress is rotated into spatial axes. As discussed elsewhere, the constitutive equation for an isotropic material should not involve the rotation **R**, and, hence, should not involve **F**.) In linear elasticity, the elastic potential of Eq. 4.13 is quadratic in the strains, as discussed by Landau and Lifshitz (1986). This requires 36 elastic constants, which reduce to 21 as a result of energy considerations.

It is interesting that $\bar{\tau}_{ij}$, defined by Eq. 4.8, is the Piola-Kirchoff stress, usually written as

$$\bar{\boldsymbol{\tau}} = \frac{\rho_o}{\rho}\mathbf{F}^{-1}\boldsymbol{\sigma}\,\mathbf{F}^{-T} \quad . \qquad 4.33$$

This expression can be obtained by combining Eqs. 4.32, 4.11, and 4.13. Thus, Eq. 4.11 shows that the Piola-Kirchoff stress is the gradient of an elastic potential with respect to the material strain.

It is possible to define a spatial Piola-Kirchoff tensor by rotation of $\bar{\boldsymbol{\tau}}$,

$$\boldsymbol{\tau} = \mathbf{R}\bar{\boldsymbol{\tau}}\mathbf{R}^T \quad , \qquad 4.34$$

using the transformation of Eq. 3.10. This presumes that the Piola-Kirchoff stress is a polar stress (a point not emphasized in the continuum mechanics literature). This is natural since the polar strain used in obtaining Eq. 4.11 is regarded as a material strain, as is indicated by the use of bars in Eq. 3.17. The rate of change of internal energy per unit volume can be written

$$\rho\dot{I} = tr\left(\boldsymbol{\sigma}\mathbf{D}\right) = \frac{\rho}{\rho_o} tr\left(\mathbf{F}\bar{\boldsymbol{\tau}}\mathbf{F}^T\mathbf{D}\right) \quad , \qquad 4.35$$

in view of Eq. 4.33. Then, using the polar decomposition expressed in Eqs. 2.2, 4.34, and 3.15, the rate of change of energy per unit mass can be expressed in terms of $\boldsymbol{\tau}$ as

$$\dot{I} = v_o tr\left(\boldsymbol{\tau}\hat{\boldsymbol{\varepsilon}}\right) \quad . \qquad 4.36$$

This pair of conjugate variables $\boldsymbol{\tau}$ and $\hat{\boldsymbol{\varepsilon}}$ is not standard in continuum mechanics, but it arises naturally when considering elastic behavior of anisotropic materials in spatial coordinates. The spatial Piola-Kirchoff stress for an isotropic elastic material has the form

$$\boldsymbol{\tau} = 2\rho\left[\frac{\partial g}{\partial I_1}\mathbf{I} + \frac{\partial g}{\partial I_2}\left(\mathbf{B} - \frac{1}{3}I_1\mathbf{I}\right) + \frac{\partial g}{\partial I_3}I_3\mathbf{B}^{-1}\right] \quad , \qquad 4.37$$

as can be deduced from Eqs. 4.34, 4.33 with 2.2, and 4.25.

5. SIMPLE SHEAR

In simple shear, material is displaced parallel to a fixed plane. The coordinates of displaced material can be written in the general case as

$$x_i = X_i + U_i\left(X_j, t\right) \quad , \qquad 5.1$$

where U_i is the displacement. When the x_1 axis is selected as the displacement direction, the only coordinate that varies is

$$x_1 = X_1 + 2\varepsilon X_2 \quad , \qquad 5.2$$

where ε is a scalar measure of shear. The deformation gradient takes the form

$$\mathbf{F} = \begin{pmatrix} 1 & 2\varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \qquad 5.3$$

In general, ε can depend on time in an arbitrary way. The velocity gradient, given by Eq. 3.1, reduces to

$$\mathbf{G} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \begin{pmatrix} 0 & 2\dot{\varepsilon} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \qquad 5.4$$

and the stretching and vorticity are, from Eq. 3.3,

$$\mathbf{D} = \begin{pmatrix} 0 & \dot{\varepsilon} & 0 \\ \dot{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \quad \mathbf{W} = \begin{pmatrix} 0 & \dot{\varepsilon} & 0 \\ -\dot{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \qquad 5.5$$

It will prove convenient to define an angle of shear, β, by

$$\varepsilon = \tan\beta \qquad 5.6$$

and to abbreviate the form of various matrices that follow by writing

$$s = \sin\beta \,, \quad c = \cos\beta \,. \qquad 5.7$$

It is straightforward to calculate that the spatial deformation tensor is

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{T} = \begin{pmatrix} 1+4\varepsilon^2 & 2\varepsilon & 0 \\ 2\varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \qquad 5.8$$

The matrix of eigenvalues of **B** can be readily found by solving the characteristic equation, A.42 of Appendix A, with the result

$$\mathbf{\Lambda} = \begin{pmatrix} (1+s)^2 / c^2 & 0 & 0 \\ 0 & (1-s)^2 / c^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \qquad 5.9$$

The corresponding matrix of eigenvectors can be found by solving the appropriate simultaneous equations and normalizing, leading to

$$\mathbf{T} = \begin{pmatrix} \sqrt{(1+s)/2} & -\sqrt{(1-s)/2} & 0 \\ \sqrt{(1-s)/2} & \sqrt{(1+s)/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \qquad 5.10$$

It follows that the spatial (left) stretch is, using Eq. 2.4,

$$\mathbf{V} = \begin{pmatrix} (1+s^2)/c & s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \qquad 5.11$$

and the Signorini strain of Eq. 3.7 is

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 2\varepsilon^2 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \qquad 5.12$$

Construction of Mohr's circle, or direct calculation of the eigenvalues of $\boldsymbol{\varepsilon}$, shows that the principal strains are $\varepsilon^2 \pm \sqrt{1+\varepsilon^2}\ \varepsilon$ and 0, and that the mean strain is $2\varepsilon^2/3$, which is tensile. Thus, shearing produces a second-order effect that tends to open up voids and cracks with certain orientations and, hence, this effect is termed dilatancy. The compressive principal strain $\left(\varepsilon^2 - \sqrt{1+\varepsilon^2}\ \varepsilon\right)$ decreases monotonically from $-\varepsilon$ to –1/2 with increasing deformation, whereas the tensile principal strain increases without bound. Though the density does not change in simple shear, the tensile character of ε_{11} suggests that cracks whose normals lie in a shearing plane will open, while those that are perpendicular to the shearing plane will remain closed, as they are in compression. This is a nonlinear effect not accounted for in ordinary elasticity. If cracks are present, they will cause the elastic constants to depend on the sign of the stress. This subject is outside the scope of the current treatment, which is essentially kinematic, but is an essential part of Statistical Crack Mechanics.

The rotation matrix $\mathbf{R} = \mathbf{V}^{-1}\mathbf{F}$ is, from Eq. 2.2,

$$\mathbf{R} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \qquad 5.13$$

Differentiation of this result shows that the rate of polar rotation is

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^{-1} = \begin{pmatrix} 0 & \dot{\beta} & 0 \\ -\dot{\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \qquad 5.14$$

For small deformations, the vorticity $\mathbf{W}$, given by Eq. 5.5, and $\boldsymbol{\Omega}$ are nearly equal, but for large deformations, $\boldsymbol{\Omega}$ is smaller by a factor $\cos^2\beta$, in view of Eq. 5.6. This is consistent with the observation that the amount of polar rotation is limited to $\pi/2$, so its rate must go to zero as β approaches $\pi/2$.

It is interesting to verify the general relation between the polar rate of rotation and vorticity given by Eq. 3.35. This can be done in two steps. The first is to compute the deformation commutator of Eq. 3.25,

$$\mathbf{Z} = \mathbf{DV} - \mathbf{VD} = \frac{2\dot{\varepsilon}\sin^2\beta}{\cos\beta} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad 5.15$$

and its vector counterpart, defined by Eq. 3.28:

$$\mathbf{z} = \frac{2\dot{\varepsilon}\sin^2\beta}{\cos\beta}\begin{pmatrix}0\\0\\1\end{pmatrix} \ . \tag{5.16}$$

Also,

$$\left(\mathbf{I}\ tr\mathbf{V} - \mathbf{V}\right)^{-1} = \frac{1}{4}\frac{c^2}{1+c}\begin{pmatrix}4/c^2+2/c-2 & 2\varepsilon & 0\\ 2\varepsilon & 2(1+1/c) & 0\\ 0 & 0 & 2(1+1/c)\end{pmatrix} \ . \tag{5.17}$$

By combining this result with Eqs. 5.5 and 5.15, we find

$$\mathbf{w} + \left(\mathbf{I}\ tr\mathbf{V} - \mathbf{V}\right)^{-1}\mathbf{z} = \begin{pmatrix}0\\0\\-\dot{\beta}\end{pmatrix} \ , \tag{5.18}$$

which is indeed equal to $\boldsymbol{\omega}$, as computed directly from 5.14.

In polar axes, the deformation is characterized by the polar stretching (using Eq. 3.10)

$$\overline{\mathbf{D}} = \dot{\varepsilon}\begin{pmatrix}-\sin 2\beta & \cos 2\beta & 0\\ \cos 2\beta & \sin 2\beta & 0\\ 0 & 0 & 0\end{pmatrix} \ , \tag{5.19}$$

the stretch

$$\overline{\mathbf{V}} = \begin{pmatrix}c & s & 0\\ s & \left(1+s^2\right)/c & 0\\ 0 & 0 & 1\end{pmatrix} \ , \tag{5.20}$$

and the strain

$$\bar{\boldsymbol{\varepsilon}} = \begin{pmatrix}0 & \varepsilon & 0\\ \varepsilon & 2\varepsilon^2 & 0\\ 0 & 0 & 1\end{pmatrix} \ . \tag{5.21}$$

Then, the relation between polar strain rate $\dot{\bar{\boldsymbol{\varepsilon}}}$ and polar stretching $\overline{\mathbf{D}}$ given by Eq. 3.19 reduces, in this example, to

$$\dot{\bar{\boldsymbol{\varepsilon}}} = \overline{\mathbf{V}}\overline{\mathbf{D}}\overline{\mathbf{V}} = \begin{pmatrix}0 & \dot{\varepsilon} & 0\\ \dot{\varepsilon} & 4\varepsilon\dot{\varepsilon} & 0\\ 0 & 0 & 0\end{pmatrix} \ , \tag{5.22}$$

as can be verified by direct calculation. The corresponding relation in spatial axes, given by Eq. 3.8, becomes, for simple shear,

$$\hat{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{\varepsilon} = \mathbf{VDV} = \dot{\beta}\begin{pmatrix} 2\varepsilon + 4\varepsilon^3 & 1+3\varepsilon^2 & 0 \\ 1+3\varepsilon^2 & 2\varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad 5.23$$

which can also be verified by direct calculation. It is straightforward to show that the invariants of $\dot{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\varepsilon}}$ are the same.

The relation of material response in spatial and polar axes is illustrated in Fig. 5.1, which illustrates the stretch ellipsoid and the three important systems in which it can be represented—the spatial, polar, and principal axes. The stretch can be represented by the ellipsoid

$$B_{ij}x_i x_j = \overline{B}_{ij}\overline{x}_i\overline{x}_j = \Lambda_{ij}\tilde{x}_i\tilde{x}_j = 1 \qquad 5.24$$

in any of several ways which are equivalent. The $\overline{x}_i$ are defined by Eq. 3.9. The ellipsoid is analogous to the strain quadric discussed by Sokolnikoff and Specht (1946), but is simpler to think about than the strain quadric because its character does not change (the strain quadric may be hyperbolic or any second-order surface). The stretch is always positive, but is less than one when compressive and exceeds one when tensile. The stretch quadric is always an ellipsoid. (We refer to either quadric, $V_{ij}x_i x_j$ or $B_{ij}x_i x_j$, as a stretch ellipsoid—they are the same except for the magnitude of the principle stretches.) The ellipsoid is related to the Signorini strain by Eq. 3.7. The spatial axes are fixed, but as the shear increases, the principal and polar axes rotate. The principal axes, initially bisecting the spatial axes, rotate clockwise until those two systems coincide. The polar axes also rotate clockwise from the initial orientation, in which they coincide with the spatial axes, until they have rotated 90 degrees, when the shear becomes infinite. The stretch ellipsoid is initially a circle (actually a cylinder), but at small strains its principal axes essentially bisect the spatial axes, as mentioned above. A principal stretch is the square root of the inverse of the length of the principal axis of the stretch ellipsoid. The product of the principal stretches, given by Eq. 5.9, is unity, indicating that the specific volume does not change for simple shear. The Λ matrix, by definition, is diagonal, and the $\tilde{x}_i$ denotes coordinates in principal axes. Thus,

$$x_i = T_{ik}\tilde{x}_k \quad . \qquad 5.25$$

The matrix of eigenvectors given by Eq. 5.10 can be written in the alternative form

$$\mathbf{T} = \begin{pmatrix} \cos\upsilon & -\sin\upsilon & 0 \\ \sin\upsilon & \cos\upsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad 5.26$$

involving the angle from the principal to the spatial axes

$$\upsilon = \pi/4 - \beta/2 \quad . \qquad 5.27$$

The relation of the x_i and the $\overline{x}_i$ is given by Eq. 3.9, which, in analogy with the notation of Eq. 5.25, can be written

$$\overline{x}_i = R_{ji}x_j \quad . \qquad 5.28$$

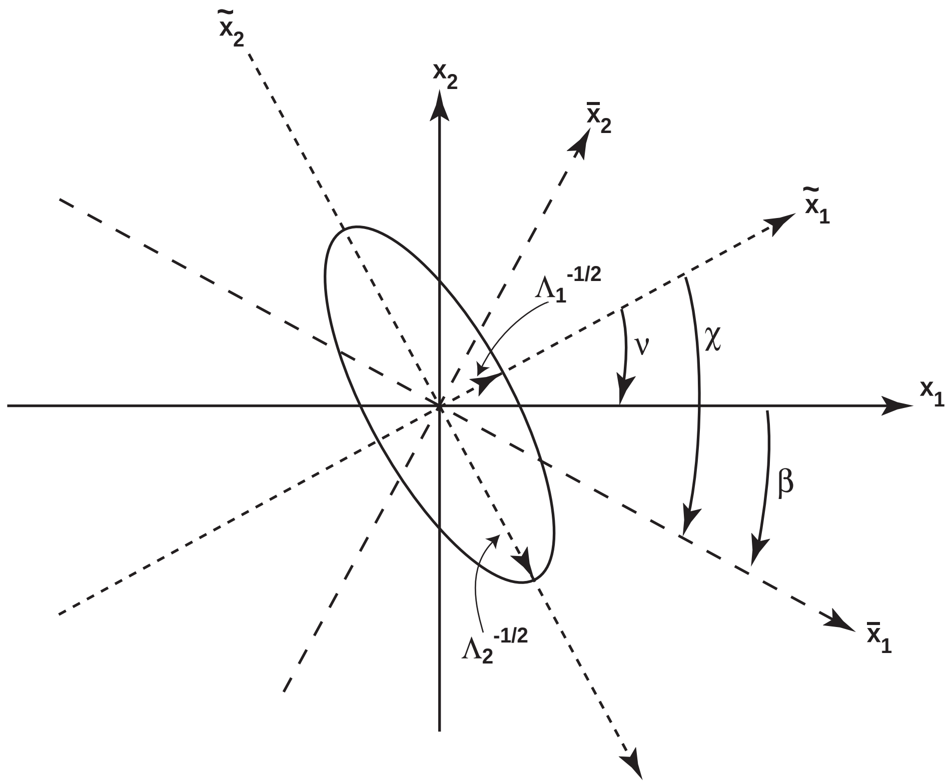

Fig. 5.1. The stretch ellipsoid for simple shear. The ellipsoid is the same whether referred to spatial, polar, or principal axes. It is initially a circle of unit radius. At early times, it becomes an ellipse with principal axes roughly bisecting the spatial axes. At late times, it becomes very thin and elongated, with the major axis approaching the x_2 axis. Its size is not the stretch, but is related to it. For example, the length of the major axis is $1/\sqrt{\Lambda_2} = \cos\beta/(1-\sin\beta)$, and the length of the minor axis is $1/\sqrt{\Lambda_1} = \cos\beta/(1+\sin\beta)$, with $\tan\beta$ equal to the scalar strain ε. Both the polar and principal axes rotate clockwise with increasing deformation. The Signorini strain is related to the difference between the stretch ellipsoid and the reference circle in spatial axes, while the Green-St. Venant strain is related to the difference in polar axes.

The transformation from polar to principle axes is given by $\bar{\boldsymbol{x}} = \boldsymbol{S}^{\boldsymbol{T}}\tilde{\boldsymbol{x}}$, with $\boldsymbol{S}$ given by Eq. 2.12. For simple shear, this reduces to

$$\mathbf{S} = \begin{pmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \chi = \frac{\pi}{4} + \frac{\beta}{2} \quad . \qquad 5.29$$

The strain measures $\boldsymbol{\varepsilon}$ and $\bar{\boldsymbol{\varepsilon}}$ discussed above measure the deviation of the ellipsoid from a circle in the various systems. *These strains should not, therefore, be considered as competitive, but as alternate representations of the same physical entity.* A broader discussion of measures of strain is given in Appendix C.

To compute the strain energy for isotropic elastic deformation, it is necessary to determine the invariants of stretch, which are, of course, the same in the various systems discussed:

$$I_1 = B_{ii} = 3 + 4\varepsilon^2 \quad , \tag{5.30}$$

$$I_2 = \frac{1}{2} B'_{ij} B'_{ij} = 4\varepsilon^2 + \frac{16}{3}\varepsilon^4 \quad , \tag{5.31}$$

$$I_3 = 1 \ . \tag{5.32}$$

In application, it is necessary to relate g to these invariants by means of experimental data or, possibly, by atomic or microstructural models. Still, it is possible to make Eq. 4.25 for the stress explicit, by use of Eq. 5.8 in the example of simple shear:

$$\sigma_{11} = 2\rho_o \left[\left(g_1 + \frac{5}{3} g_2 \right) 4\varepsilon^2 + \frac{32}{3} g_2 \ \varepsilon^4 \right] + \zeta \quad , \tag{5.33}$$

$$\sigma_{22} = \zeta + \frac{16}{3} \rho_o g_2 \ \varepsilon^2 \quad , \tag{5.34}$$

$$\sigma_{33} = \zeta - \frac{8}{3} \rho_o g_2 \ \varepsilon^2 \quad , \tag{5.35}$$

$$\sigma_{12} = 2\mu\varepsilon + \frac{32}{3} \rho_o g_2 \ \varepsilon^3 \quad , \tag{5.36}$$

where ρ has been set to ρ_o, since $I_3 = 1$, and the abbreviation

$$g_n = \frac{\partial g}{\partial I_n} \tag{5.37}$$

has been introduced. In addition, it is convenient to set

$$2\rho_o (g_1 + g_2) = \mu \tag{5.38}$$

and

$$2\rho_o (g_1 + g_3) = \zeta \quad . \tag{5.39}$$

The term $g_1 + g_3$ in Eq. 5.39 vanishes for $\varepsilon = 0$ in view of Eq. 4.31, but it may not vanish for large ε, so the general form is retained. The fact that σ_{12} is linear in ε up to a cubic term is consistent with a comment by Mooney (1940) that for rubber, linearity of the shear stress as a function of strain holds to very large strains. If it is linear to 200% strain, as he finds, then g_2 must be very small in addition to the quadratic term being absent.

6. RIGID-BODY MOTION

In the case of rigid-body rotation, the deformation is trivial, but it is interesting to see how the general results are specialized. Such a deformation can be represented by

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \tag{6.1}$$

where the z-axis is taken as the axis of rotation and

$$\theta = \omega\, t + \phi \qquad 6.2$$

and r and ϕ depend on the initial values X and Y of x and y through

$$r = \sqrt{X^2 + Y^2} \quad , \quad \tan\phi = Y / X \quad . \qquad 6.3$$

Then, applying Eq. 2.1,

$$\mathbf{F} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \qquad 6.4$$

It can be readily shown that

$$\mathbf{B} = \mathbf{V} = \mathbf{I} \quad , \quad \mathbf{R} = \mathbf{F} \quad . \qquad 6.5$$

It follows that the velocity gradient is

$$\mathbf{G} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \qquad 6.6$$

and $\mathbf{D} = \mathbf{0}$, $\mathbf{Z} = \mathbf{0}$. It is straightforward to verify that

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T = \mathbf{W} \quad , \qquad 6.7$$

demonstrating that the vorticity and rate of material rotation are equal in this case.

7. TORSION

When a uniform circular cylinder is twisted so that each section normal to its axis rotates without strain and there is no elongation, the deformation may be described as simple torsion. Let the material points be defined by their initial coordinates X and Y, Z, and take the Z axis as the axis of the cylinder. Then, the angular orientation θ can be written

$$\theta = \theta_o(Z,t) + \phi(X,Y) \quad , \qquad 7.1$$

where ϕ denotes the initial orientation of a point with the initial coordinates X, Y, and θ_o the amount of twist. The Cartesian coordinates of material position at time *t* are given by

$$x = r\cos\theta \;,\; y = r\sin\theta \quad , \quad z = Z \quad , \qquad 7.2$$

with *r* remaining constant during the deformation. Then, substitution of these expressions into Eq. 2.1 for the deformation gradient leads to

$$\mathbf{F} = \begin{pmatrix} \cos\theta_o & -\sin\theta_o & a \\ \sin\theta_o & \cos\theta_o & b \\ 0 & 0 & 1 \end{pmatrix} \quad , \qquad 7.3$$

where

$$a = -r\theta_o' \sin\theta \quad , \quad b = r\theta_o' \cos\theta \quad , \quad \theta_o' = \frac{\partial \theta_o}{\partial Z} \quad , \qquad 7.4$$

and θ_0' is constant in view of the uniformity of deformation of the cylinder. The corresponding Cauchy-Green tensor is

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \begin{pmatrix} 1+a^2 & ab & a \\ ab & 1+b^2 & b \\ a & b & 1 \end{pmatrix} , \qquad 7.5$$

which is independent of ϕ. It proves convenient to define

$$\tan\alpha = 2 / r\theta_o' \quad , \qquad 7.6$$

which is closely related to the helical angle of twist, as discussed below and illustrated in Fig. 7.1.

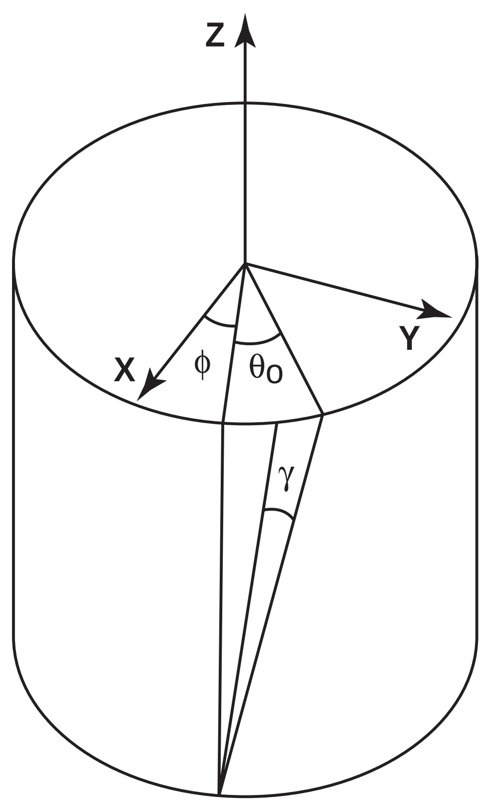

Fig. 7.1. Cylinder in torsion. The angle $\gamma = tan^{-1}(r\theta_0 / 2Z)$ proves to be a convenient measure of deformation.

The eigenvalues of **B** are given by

$$\mathbf{\Lambda} = \begin{pmatrix} \cot^2\frac{\alpha}{2} & 0 & 0 \\ 0 & \tan^2\frac{\alpha}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad 7.7$$

The determinant of $\mathbf{\Lambda}$ is unity, indicating that the deformation proceeds without change in volume. Calculation of the normalized eigenvectors of **B** leads to the expression

$$\mathbf{T} = \begin{pmatrix} -\cos\frac{\alpha}{2}\sin\theta & \sin\frac{\alpha}{2}\sin\theta & \cos\theta \\ \cos\frac{\alpha}{2}\cos\theta & -\sin\frac{\alpha}{2}\cos\theta & \sin\theta \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} & 0 \end{pmatrix} . \qquad 7.8$$

In subsequent calculations, it proves convenient to define the complement of α

$$\gamma = \pi / 2 - \alpha \ , \qquad 7.9$$

which has the convenient property that it vanishes when the twist vanishes, since

$$2\tan\gamma = r\theta_o' = A = 2\cot\alpha = \sqrt{a^2 + b^2} \ . \qquad 7.10$$

The angle of twist is $\tan^{-1} r\theta_o'$. Thus, γ is closely related to the angle of twist, but is algebraically more convenient. This equation also defines A, a quantity useful in shortening some lengthy expressions. Then, using Eq. 2.4, the stretch is given by

$$\mathbf{V} = \begin{pmatrix} 1 + Q\sin^2\theta & -Q\sin\theta\cos\theta & -\sin\theta\sin\gamma \\ -Q\sin\theta\cos\theta & 1 + Q\cos^2\theta & \cos\theta\sin\gamma \\ -\sin\theta\sin\gamma & \cos\theta\sin\gamma & \cos\gamma \end{pmatrix} , \qquad 7.11$$

where

$$Q = \frac{2}{\cos\gamma} - \cos\gamma - 1 \ . \qquad 7.12$$

It can be verified that $|\mathbf{V}| = 1$ by direct calculations, demonstrating that simple torsion causes no change in volume, even in the strongly nonlinear regime. When the twist is zero, $\gamma = 0$, Q = 0, and **V** = **I**. To calculate $\mathbf{V}^{-1}$, one can either determine the matrix of cofactors of Eq. 7.11 or use the spectral relation

$$\mathbf{V}^{-1} = \mathbf{T}\Lambda^{-1/2}\mathbf{T}^{\mathrm{T}} \ , \qquad 7.13$$

with the result

$$\mathbf{V}^{-1}=\begin{pmatrix}\cos\gamma\sin^2\theta+\cos^2\theta & \sin\theta\cos\theta(1-\cos\gamma) & \sin\theta\sin\gamma\\ \sin\theta\cos\theta(1-\cos\gamma) & \sin^2\theta+\cos^2\theta\cos\gamma & -\cos\theta\sin\gamma\\ \sin\theta\sin\gamma & -\cos\theta\sin\gamma & \dfrac{2}{\cos\gamma}-\cos\gamma\end{pmatrix}. \qquad 7.14$$

Then, the rotation matrix is

$$\mathbf{R}=\mathbf{V}^{-1}\mathbf{F}=\begin{pmatrix}\sin\theta\sin\phi\cos\gamma+\cos\theta\cos\phi & -\sin\theta\cos\phi\cos\gamma+\cos\theta\sin\phi & -\sin\theta\sin\gamma\\ \sin\theta\cos\phi-\cos\theta\sin\phi\cos\gamma & \sin\theta\sin\phi+\cos\theta\cos\phi\cos\gamma & \cos\theta\sin\gamma\\ \sin\phi\sin\gamma & -\cos\phi\sin\gamma & \cos\gamma\end{pmatrix}. \qquad 7.15$$

Further calculations are greatly simplified by considering the case $\phi=0$, which is sufficiently general because the deformation is the same for all generators of a cylinder in torsion. Let $\mathbf{R}_o$ denote the value of $\mathbf{R}$ on $\phi=0$, given by

$$\mathbf{R}_o=\begin{pmatrix}\cos\theta_o & -\sin\theta_o\cos\gamma & -\sin\theta_o\sin\gamma\\ \sin\theta_o & \cos\theta_o\cos\gamma & \cos\theta_o\sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{pmatrix}. \qquad 7.16$$

When there is no twist, $\theta_o=0$. Consequently, $\theta'_o=0$, and it follows from Eq. 7.10 that $\gamma=0$ and the rotation reduces to $\mathbf{R}_o=\mathbf{I}$. The rate of polar rotation is given by

$$\mathbf{\Omega}_o=\dot{\mathbf{R}}_o\mathbf{R}_o^{-1}=\begin{pmatrix}0 & 0 & -\sin\theta_o\\ 0 & 0 & \cos\theta_o\\ \sin\theta_o & -\cos\theta_o & 0\end{pmatrix}\dot{\gamma}+\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\dot{\theta}_o\ , \qquad 7.17$$

where

$$\dot{\gamma}=\frac{1}{2}\dot{A}\cos^2\gamma=\frac{2z}{r}\frac{\dot{\theta}_o}{\theta_o^2}\sin^2\gamma\ . \qquad 7.18$$

Though $\dot{\gamma}$ and $\dot{\theta}_o$ are related, they are different, since γ is related to the helical twist angle, while θ_o is the angle of rotation in the plane normal to the Z axis, as shown in Fig. 7.1.

The vorticity is found by application of Eqs. 3.1 and 3.3. As above, only the value $\mathbf{W}_o$ on the generator $\phi=0$ is required:

$$\mathbf{W}_o=\begin{pmatrix}0 & 0 & -\sin\theta_o\\ 0 & 0 & \cos\theta_o\\ \sin\theta_o & -\cos\theta_o & 0\end{pmatrix}\frac{\dot{A}}{2}+\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\dot{\theta}_o\ . \qquad 7.19$$

The first term is the same as the first term of $\mathbf{\Omega}_o$, except for a factor of $\cos^2\gamma$, just as in simple shear. The second term is the same as the second term of $\mathbf{\Omega}_o$. For θ_o small, the first term is a rotation essentially about the x_1 axis, and is like simple shear except for the difference in axes and calling the shear angle γ instead of β. The second term represents rotation about the axis of symmetry, the Z axis. In view of Eqs. 7.10 and 7.18,

$$\dot{\theta}_o = \frac{2z}{r\cos^2\gamma}\dot{\gamma} \quad , \quad \dot{A} = \frac{2}{\cos^2\gamma}\dot{\gamma} \quad . \qquad 7.20$$

It is shown in Appendix A that the rotation matrix can be expressed in the form

$$\mathbf{R} = e^{\alpha\,\mathbf{Q}} \quad , \qquad 7.21$$

where **Q** is an antisymmetric matrix associated with the unit vector **q** characterizing the axis of rotation, and α is the angle of rotation about **q**, not the α of Eq. 7.6. It is straightforward to show that

$$q_1 = -\nu\sin\gamma\sin\theta_o \,/\, \sqrt{1-\cos\theta_o} \quad , \qquad 7.22$$

$$q_2 = -\nu\sin\gamma\sqrt{1-\cos\theta_o} \quad , \qquad 7.23$$

$$q_3 = \nu(1+\cos\gamma)\sqrt{1-\cos\theta_o} \quad , \qquad 7.24$$

where

$$\nu = \left\{(1+\cos\gamma)\left[3-\cos\theta_o-(1+\cos\theta_o)\cos\gamma\right]\right\}^{-1/2} \quad . \qquad 7.25$$

When r is very small, so that γ is small, q_3 is very near to 1, so that the rotation is clockwise about the x_3 axis, and the deformation is essentially the same as in simple shear. However, as r increases without bound, γ approaches $\pi/2$, and the rotation vector becomes

$$q_1 = -\left(\frac{1+\cos\theta_o}{3-\cos\theta_o}\right)^{\frac{1}{2}} \quad , \quad q_2 = -\left(\frac{1-\cos\theta_o}{3-\cos\theta_o}\right)^{\frac{1}{2}} \quad , \quad q_3 = \left(\frac{1-\cos\theta_o}{3-\cos\theta_o}\right)^{\frac{1}{2}} \quad . \qquad 7.26$$

This direction varies in a complex fashion, but for θ_o small, it approaches the x_1 axis. The angle of rotation α of Eq. 7.21 is given by

$$\cos\alpha = \frac{1}{2}\left(\cos\theta_o+\cos\theta_o\cos\gamma+\cos\gamma-1\right) \quad . \qquad 7.27$$

When r is zero, γ is zero, and this reduces to $\alpha = \theta_o$. In general, however, α and θ_o are quite different. For example, if r and z are equal and $\theta_o = 45°$, then $\alpha = 49.6°$.

The stretching is given by

$$\mathbf{D} = -\frac{\dot{A}}{2}\begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & -\cos\theta \\ \sin\theta & -\cos\theta & 0 \end{pmatrix} \quad . \qquad 7.28$$

For θ small, this is the same as for simple shear, with γ playing the same role as β in simple shear. There is a difference in the labeling of axes, with X_2 in the torsion analysis playing the role of X_1 in the shear analysis when $\phi = 0$, and X_3 in the torsion analysis playing the role of X_2 in the shear analysis.

As in the analysis of shear, it is possible to verify the relation Eq. 3.35 between $\boldsymbol{\omega}$ and $\mathbf{w}$. On $\phi = 0$,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\mathbf{o} = \begin{pmatrix} -\dot{\gamma}\cos\theta_o \\ -\dot{\gamma}\sin\theta_o \\ \dot{\theta}_o \end{pmatrix} \quad , \tag{7.29}$$

$$\mathbf{w} = \mathbf{w}_o = \begin{pmatrix} -\frac{\dot{A}}{2}\cos\theta_o \\ -\frac{\dot{A}}{2}\sin\theta_o \\ \dot{\theta}_o \end{pmatrix} \quad , \tag{7.30}$$

and

$$\mathbf{z} = \dot{A}\frac{\sin^2\gamma}{\cos\gamma}\begin{pmatrix} \cos\theta_o \\ \sin\theta_o \\ 0 \end{pmatrix} \quad . \tag{7.31}$$

It is straightforward, then, to verify that

$$\mathbf{z}_o = \left(\mathbf{I}tr\mathbf{V}_o - \mathbf{V}_o\right)\left(\boldsymbol{\omega}_o - \mathbf{w}_o\right) \quad . \tag{7.32}$$

8. SIMPLE VORTEX

To illustrate the difference between $\boldsymbol{\Omega}$ and $\mathbf{W}$**,** it is useful to consider a simple vortex. For such an irrotational motion, **W** is zero everywhere (except the origin), whereas $\boldsymbol{\Omega}$ increases monotonically with time. (If it seems paradoxical that the motion in a simple vortex is *irrotational*, this is a consequence of some naïve statements in books on hydrodynamics—zero vorticity does not imply zero rotation, as shown by Eq. 3.31.) The components of velocity are given by the expressions

$$u = -K\frac{\sin\alpha}{r} \quad , \quad v = K\frac{\cos\alpha}{r} \quad , \tag{8.1}$$

where K is the strength of the vortex and

$$\alpha = \omega t + \phi \quad . \tag{8.2}$$

The position vector is given by

$$x = r\cos\alpha \quad , \quad y = r\sin\alpha \quad , \tag{8.3}$$

so that the phase ϕ can be determined by the initial coordinates through

$$\tan\phi = Y / X \quad . \tag{8.4}$$

The angular velocity is

$$\omega = K / r^2 \quad . \tag{8.5}$$

A straightforward calculation leads to the expression

$$\mathbf{F} = \begin{pmatrix} \cos\beta + 2\beta \sin\alpha \cos\phi & -\sin\beta + 2\beta \sin\alpha \sin\phi \\ \sin\beta - 2\beta \cos\alpha \cos\phi & \cos\beta - 2\beta \cos\alpha \sin\phi \end{pmatrix} \tag{8.6}$$

for the deformation gradient, where

$$\beta = \omega t \quad . \tag{8.7}$$

The left Cauchy-Green tensor is

$$\mathbf{B} = \begin{pmatrix} 1 + 2\beta \sin 2\alpha + 4\beta^2 \sin^2\alpha & -2\beta \cos 2\alpha - 2\beta^2 \sin 2\alpha \\ -2\beta \cos 2\alpha - 2\beta^2 \sin 2\alpha & 1 - 2\beta \sin 2\alpha + 4\beta^2 \cos^2\alpha \end{pmatrix} \quad . \tag{8.8}$$

In previous sections, **V** has been determined by computing its eigenvectors and eigenvalues, but here we use the relations

$$\mathbf{V} = \frac{\mathbf{B} + \sqrt{J_2}\,\mathbf{I}}{\sqrt{J_1 + 2\sqrt{J_2}}} \quad , \quad J_2 = |\mathbf{B}| \quad , \quad J_1 = tr\,\mathbf{B} \tag{8.9}$$

that follow from the Cayley-Hamilton theorem (see Appendix A) for 2 x 2 matrices. Then,

$$\mathbf{V} = \left(1+\beta^2\right)^{-\frac{1}{2}} \begin{pmatrix} 1 + \beta \sin 2\alpha + 2\beta^2 \sin^2\alpha & -\beta \cos 2\alpha - \beta^2 \sin 2\alpha \\ -\beta \cos 2\alpha - \beta^2 \sin 2\alpha & 1 - \beta \sin 2\alpha + 2\beta^2 \cos^2\alpha \end{pmatrix} \quad . \tag{8.10}$$

A fairly lengthy calculation leads to the simple result

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \omega \begin{pmatrix} \sin 2\alpha & -\cos 2\alpha \\ -\cos 2\alpha & -\sin 2\alpha \end{pmatrix} \quad , \tag{8.11}$$

which, in view of Eq. 3.3, leads to the conclusion that **W**=**0** and $\mathbf{D} = \dot{\mathbf{F}}\mathbf{F}^{-1}$.

Consider now the rate of polar rotation, $\boldsymbol{\omega}$, which can be determined algebraically by use of Eq. 3.35. To begin, one computes

$$\mathbf{DV} = \frac{\omega}{\sqrt{1+\beta^2}} \begin{pmatrix} \beta + \left(1+\beta^2\right)\sin 2\alpha & -\left(1+\beta^2\right)\cos 2\alpha - \beta^2 \\ -\left(1+\beta^2\right)\cos 2\alpha + \beta^2 & \beta - \left(1+\beta^2\right)\sin 2\alpha \end{pmatrix} \quad . \tag{8.12}$$

Then, Eqs. 3.21 and 3.24 lead to

$$\mathbf{z} = \left(0,\ 0,\ 2\beta^2\omega\left(1+\beta^2\right)^{-1/2}\right) \quad . \tag{8.13}$$

The computation of $\boldsymbol{\omega}$ can be completed using the relation

$$\mathbf{h} = (\mathbf{I}tr\mathbf{V} - \mathbf{V})^{-1}\mathbf{z} = \left(0,\ 0,\ z_3 / \left(V_{11} + V_{22}\right)\right), \tag{8.14}$$

which holds for plane flows. Completion of the calculation using Eq. 3.31 then leads to

$$\boldsymbol{\omega} = \left(0,\, 0,\, \omega\beta^2 / \left(1+\beta^2\right)\right) \quad . \qquad 8.15$$

Thus, the polar rotation has an angular velocity that is initially zero, but approaches a constant value of ω at late times, when β is large, i.e., the fluid rotates though the vorticity is zero.

9. STRESS RATE IN CONSTITUTIVE RELATIONS

Given an isotropic constitutive law having the rate form

$$\dot{\boldsymbol{\sigma}} = \psi(\boldsymbol{\sigma}\,,\,\mathbf{D}) \qquad 9.1$$

in the absence of rotation, it is often necessary to construct a general form that allows for rotation, especially when formulating computer algorithms. In particular, this situation arises in connection with elastic-plastic behavior, kinematic hardening, and Maxwell solids. In this section, it will be shown that the appropriate generalization of Eq. 9.1 is

$$\hat{\boldsymbol{\sigma}} = \psi(\boldsymbol{\sigma}\,,\,\mathbf{D}) \quad , \qquad 9.2$$

where the polar rate operator $(\hat{\ })$ is defined by Eq. 3.11.

When a material is rotating, Eq. 9.1 must apply in the rotating, or polar, axes. The stress in polar axes is

$$\bar{\boldsymbol{\sigma}} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R} \quad , \qquad 9.3$$

as indicated by Eq. 3.10. By differentiating, we find that

$$\dot{\bar{\boldsymbol{\sigma}}} = \dot{\mathbf{R}}^T\boldsymbol{\sigma}\mathbf{R} + \mathbf{R}^T\dot{\boldsymbol{\sigma}}\mathbf{R} + \mathbf{R}^T\boldsymbol{\sigma}\dot{\mathbf{R}} \quad . \qquad 9.4$$

Now, for isotropic materials,

$$\psi\left(\bar{\boldsymbol{\sigma}}\,,\,\bar{\mathbf{D}}\right) = \mathbf{R}^{\mathrm{T}}\psi\left(\mathbf{R}\bar{\boldsymbol{\sigma}}\mathbf{R}^{\mathrm{T}}\ ,\ \mathbf{R}\bar{\mathbf{D}}\mathbf{R}^{\mathrm{T}}\right)\mathbf{R} \qquad 9.5$$

since, by the Cayley-Hamilton theorem (Appendix A), ψ can always be expressed as a polynomial in $\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbf{D}}$, and the property indicated by Eq. 9.5 holds for polynomials. Then, since Eq. 9.1 must hold in rotating axes, neglecting dynamical effects so that

$$\dot{\bar{\boldsymbol{\sigma}}} = \psi\left(\bar{\boldsymbol{\sigma}}\,,\,\bar{\mathbf{D}}\right) \quad , \qquad 9.6$$

premultiplication by $\mathbf{R}$ and postmultiplication by $\mathbf{R}^T$ leads to

$$\hat{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \mathbf{R}\dot{\mathbf{R}}^T\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\mathbf{R}}\mathbf{R}^T = \psi(\boldsymbol{\sigma}\,,\,\mathbf{D}) \quad , \qquad 9.7$$

which is equivalent to Eq. 9.2, and had to be proved.

This stress rate is different from the ZJN stress rate of Zaremba (1903), Jaumann (1911), and Noll (1955),

$$\overset{\vee}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W} \quad , \qquad 9.8$$

though it is similar in form. At any time, one can choose the reference system so that **F** = **I**, forcing **V** = **I** and consequently, in view of Eqs. 3.21 and 3.31, $\mathbf{\Omega} = \mathbf{W}$. However, this cannot hold for all time, since the reference system can only be selected once. This point has led to great confusion in the literature. Note that here we have derived Eq. 9.7, not merely conjectured the form as was done by Zaremba and Jaumann. This analysis is similar to Noll's, but his final step of setting Ω = **W** is not generally valid.

To illustrate the difference in using the exact (polar) stress rate of Eq. 9.7 and the approximate ZJN stress rate of Eq. 9.8, consider the hypo-elastic material defined by

$$\hat{\boldsymbol{\sigma}} = \psi(\boldsymbol{\sigma}, \mathbf{D}) = \lambda \mathbf{I} tr \mathbf{D} + 2\mu \mathbf{D} \quad , \qquad 9.9$$

where λ and μ are the Lamé constants, as it undergoes simple shear. By virtue of 5.5, the trace of **D** is zero. Then, applying the stress rate of Eq. 9.7, we find

$$\hat{\sigma}_{11} = \dot{\sigma}_{11} - 2\Omega_{12}\sigma_{21} = 2\mu D_{11} = 0 \quad , \qquad 9.10$$

$$\hat{\sigma}_{12} = \dot{\sigma}_{12} - \Omega_{12}\sigma_{22} + \sigma_{11}\Omega_{12} = 2\mu D_{12} \quad , \qquad 9.11$$

$$\hat{\sigma}_{22} = \dot{\sigma}_{22} + 2\Omega_{12}\sigma_{12} = 2\mu D_{22} = 0 \quad . \qquad 9.12$$

These equations can be combined to obtain the ordinary differential equation

$$\frac{d^2\sigma_{11}}{d\beta^2} + 4\sigma_{11} = \frac{4\mu}{\cos^2\beta} \qquad 9.13$$

whose general solution is

$$\sigma_{11} = 4\mu\left(\cos 2\beta \ln\cos\beta + \beta \sin 2\beta - \sin^2\beta\right) + A\cos 2\beta + B\sin 2\beta \quad . \qquad 9.14$$

If we assume that all the stresses are initially zero, then

$$\sigma_{11} = 4\mu\left(\cos 2\beta \ln\cos\beta + \beta \sin 2\beta - \sin^2\beta\right) \quad , \qquad 9.15$$

$$\sigma_{12} = 2\mu \cos 2\beta\left(2\beta - 2(\tan 2\beta)(\ln\cos\beta) - \tan\beta\right) \quad , \qquad 9.16$$

$$\sigma_{22} = -\sigma_{11} \quad . \qquad 9.17$$

Note that there was an error in Dienes (1979) in which tan 2β was printed as $\tan^2\beta$. Also note that $\varepsilon = \tan\beta$ here (Eq. 5.6), where $e = 2\varepsilon = 2\tan\beta$. Thus, $D_{12} = \dot{\varepsilon}$ here while $D_{12} = \dot{e}/2$ in Dienes.

As the strain approaches infinity at late time, β approaches $\pi/2$, as indicated by Eq. 5.6, and the stress approaches infinity logarithmically. The stress-strain relation is given graphically in Fig. 9.1.

If the ZJN stress rate is used for this process, the governing equations are

$$\dot{\sigma}_{11} - a\sigma_{21} = 0 \quad , \qquad 9.18$$

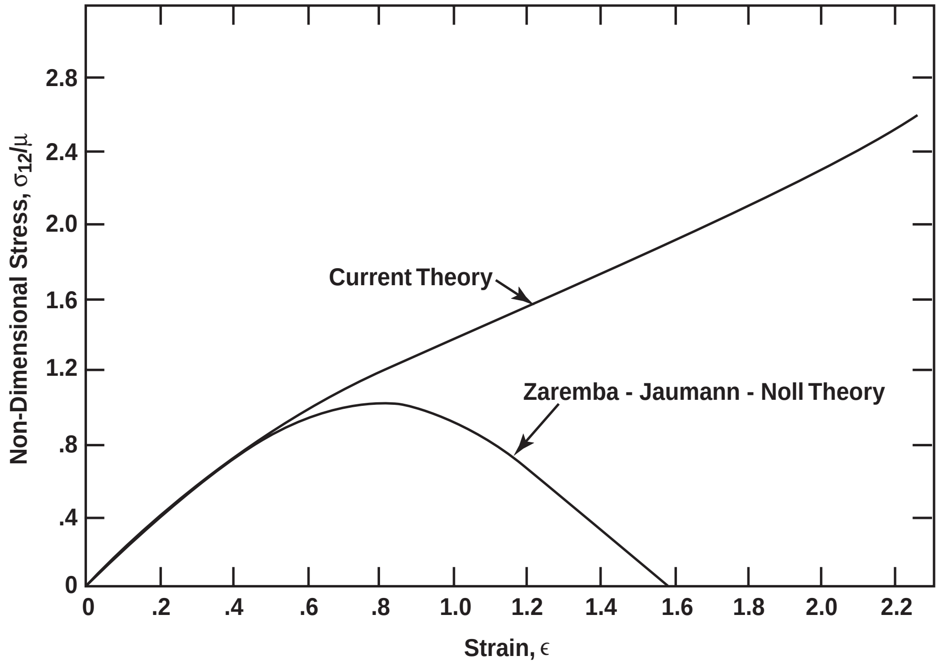

Fig. 9.1 Comparison of the stress-strain relation for simple shear, using the traditional method of Zaremba-Jaumann-Noll and the current finite-deformation theory based on polar decomposition. Note that a negative slope would lead to instability.

$$\dot{\sigma}_{12} - \frac{a}{2}\left(\sigma_{22} - \sigma_{11}\right) = a\mu \quad , \tag{9.19}$$

$$\dot{\sigma}_{22} + a\sigma_{21} = 0 \quad , \tag{9.20}$$

where

$$a = \frac{\partial \dot{x}_1}{\partial X_2} = 2\dot{\varepsilon} \tag{9.21}$$

is considered constant in this example. It is straightforward to show that, if the stresses vanish initially,

$$\sigma_{12} = \mu \sin at \quad , \tag{9.22}$$

$$\sigma_{11} = \mu(1 - \cos at) \quad , \tag{9.23}$$

$$\sigma_{22} = -\mu(1 - \cos at) \quad . \tag{9.24}$$

This behavior is compared with the exact solution in Fig. 9.1. Clearly, the periodic behavior of this solution is not physically realistic, though the solution agrees with the exact solution for small strains. The difference in these behaviors can be traced to the difference between **Ω** and **W**. The rate of polar rotation Ω in simple shear gradually drops from its initial value to zero as deformation proceeds, as given by Eqs. 5.14 and 5.6. The vorticity **W**, however, is constant, as shown by Eq. 5.5, when $\dot{\varepsilon}$ is constant. Thus, vorticity does not measure the polar rotation rate, except at the initial (reference) time.

10. SUPERPOSITION OF STRAIN RATES

Though superposition of strain rates dates back to Reuss (1931), who applied it in 1928 to characterize elastic-plastic behavior, no physical motivation was given, nor adumbrated, nor was it suggested that the principle could be extended to more complex situations in mechanics where processes other than plastic flow are involved. In this section, a motivation for the superposition principle is described and the extension to fragmentation and the behavior of porous materials is also considered. Extension to cover polycrystalline and composite materials is probably straightforward. Though the title of the section contains the phrase “strain rates,” the details of the analysis involve primarily the stretching **D** rather than the strain rate $\hat{\varepsilon}$. However, in view of Eq. 3.15, if

$$\mathbf{D} = \sum_{\alpha} \mathbf{D}^{\alpha} \quad , \tag{10.1}$$

with $\mathbf{D}^{\alpha}$ denoting the stretching due to the αth process, then

$$\hat{\varepsilon} = \mathbf{V}\left(\sum_{\alpha} \boldsymbol{D}^{\alpha}\right)\mathbf{V} = \sum_{\alpha} \hat{\varepsilon}^{\alpha} , \tag{10.2}$$

where $\hat{\boldsymbol{\varepsilon}}^{\alpha}$ is the Signorini strain rate due to the αth process. Thus, superposition of stretchings and superposition of Signorini strain rates are equivalent. Of course, **V** denotes the total stretch, and cannot be separated into parts in a corresponding way.

The premise of this section is that deformation is often the consequence of numerous independent physical processes which can be superimposed to obtain an overall deformation. This is illustrated in Fig. 10.1. The difference in velocity between two points P_1 and P_2 can be written

$$\Delta u_i = \sum \Delta u_i{}^{c} + \sum \Delta u_i{}^{d} \quad , \tag{10.3}$$

where the first sum is taken over the continuous elastic regions between defects, and the second sum is taken over the defects between elastic regions. The continuous regions can, in fact, be considered to have a more general character, such as thermoelastic behavior characterized by rather general thermodynamic laws. They can be anisotropic and may involve plastic flow or various phases. The flaws are considered to be either open or closed. (In real materials, flaws may be partially open and partially closed, and dislocations may move out of their plane, but the treatment of such flaws is beyond the current treatment.) The rules for elastic behavior were discussed in Section 4. More complex thermodynamic behavior is deferred to a subsequent volume. The success of the superposition method depends on the adequacy of the constitutive laws characterizing the various kinds of defects and their statistics. The independence of the behavior of defects is crucial to the method. This, too, will be treated in a separate volume.

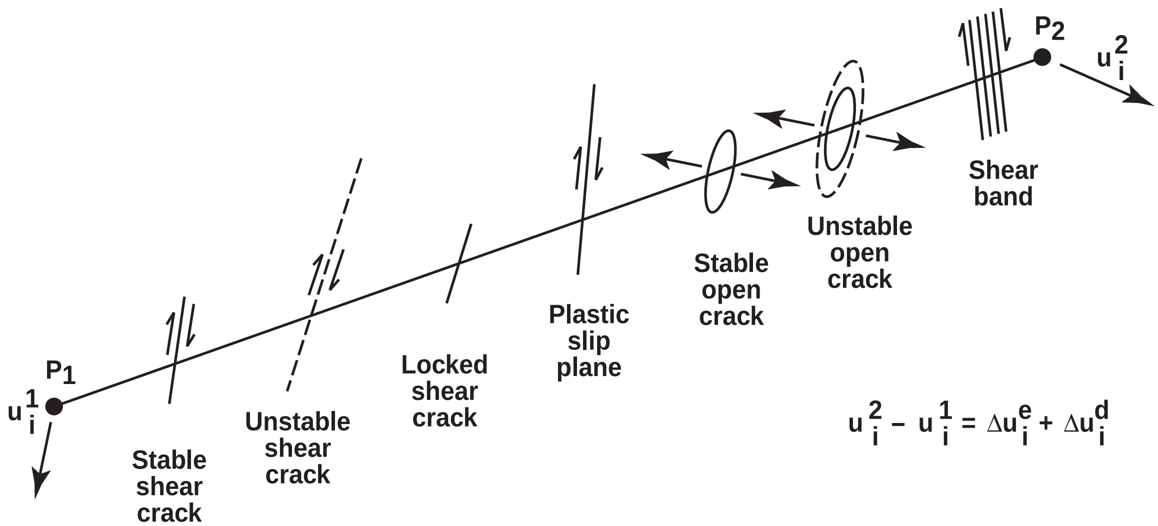

Fig. 10.1 Superposition of strain rates. An illustration of how the effects of various defects can combine to influence the velocity increment between two points P_1 and P_2.

The number of defects per unit length is denoted by $\tilde{L} = 1/\lambda$, where λ is the average separation *in the reference state* for a particular kind of flaw. Use of $\tilde{L}$ makes it possible to replace the summation over individual defects in Eq. 10.3 by a sum over sets of defects involving $\tilde{L}$, so that

$$\Delta v_i{}^d = \Delta S \sum \tilde{L}^\alpha \ \delta v_i{}^\alpha \qquad 10.4$$

represents the contribution to the velocity difference of the defects intersecting a segment between points P_1 and P_2 of length ΔS, as illustrated in Fig. 10.2. It is assumed that $\tilde{L}$ is not time dependent. The time-dependent case is addressed in Section 11.

It is very convenient to take the flaws as circular, and quite possibly a good approximation in many situations (Zhurkov 1975). This may be adduced with an analogy. In the development of the kinetic theory of gases, molecules such as O_2 were considered spherical for the purpose of computing collisions, though that is clearly far from precise. Nevertheless, the resulting theory was very successful in explaining the behavior of gases; for example, diffusion, conduction, and viscosity coefficients were computed to an accuracy typically on the order of 1%, according to Jeans (1940).

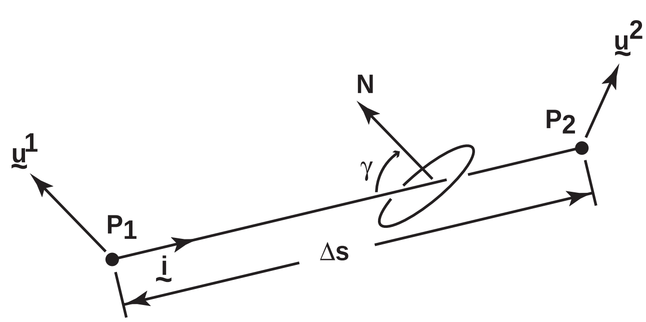

Fig. 10.2 The contribution of a defect to the difference in velocity between P_1 and P_2.

In the theory of solid defects, approximating their shape as circular greatly simplifies the statistical arguments (which are much more complex than for gases) as well as the behavior of individual defects. (Recall, in this connection, that the theory of circular cracks was not developed until the 1950s when Sack, Segedin, Sneddon, and Keer were able to solve the governing boundary value problems.) A more explicit result is stated by Kachanov (1994): "Replacement of an elliptic crack by the equivalent circular crack is adequate with very good accuracy."

The number of defects of each type per unit length is equal to the projected area per unit volume, as illustrated in Fig. 10.3. A set of circular defects with radii in the range $(c\ , c+\Delta c)$ and orientations roughly in the direction $\mathbf{\Omega}$ occupying a spherical cap of size $\Delta\Omega$ on the unit sphere, as illustrated in Fig. 10.4, is said to be a defect set. (Cracks have symmetry such that a reversal of 180° leaves them unchanged. Thus, half the unit sphere is sufficient to characterize defect orientation.) A useful subdivision of half the unit sphere is illustrated in Fig. 10.5. With these restrictions, the number of defects per unit length in each crack set can be written

$$\tilde{L} = -\pi c^2 L_{c\phi\theta} \Delta c \ \Delta\Omega \cos\gamma \quad , \qquad 10.5$$

where L is the number of cracks per unit volume exceeding c in radius, and the subscripts denote differentiation. The angles ϕ and θ are the usual polar coordinates (having nothing to do with polar axes), so that

$$\Delta\Omega = \Delta\theta \ \Delta\phi \sin\theta \quad . \qquad 10.6$$

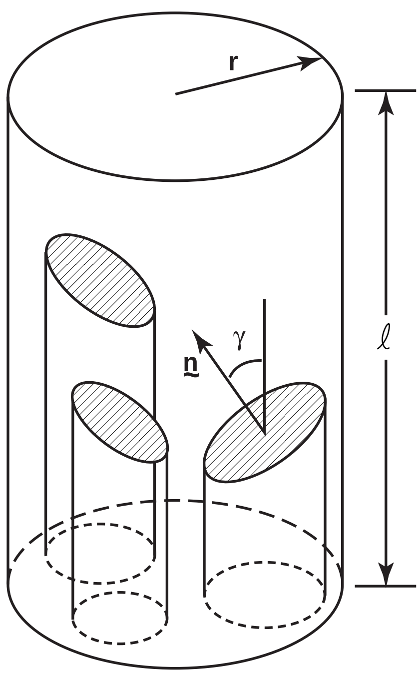

Fig. 10.3 Illustration of a control volume of radius r and length l containing a random assortment of circular defects of area "a" used in estimating the number of defects per unit length $\tilde{L} = \pi La$, when the defects have a number density "L" per 2π, per unit volume.

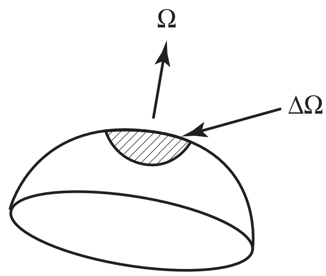

Fig. 10.4 Hemisphere defining crack orientation. The cap of area $\Delta\Omega$ and center Ω defines one crack set.

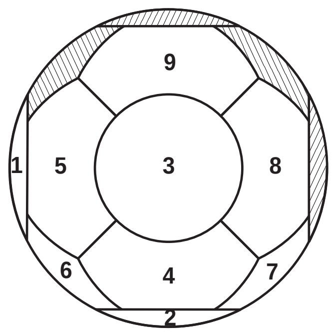

Fig. 10.5 Subdivision of a unit hemisphere into elements defining flaw orientation.

The angle between $\overline{P_1 P_2}$ and the crack normal **N** is denoted by γ. Now, observe that

$$\Delta S\, i_k = \Delta X_k \tag{10.7a}$$

and

$$\cos\gamma = N_k\, i_k \quad , \tag{10.7b}$$

where ΔX_k is the projection of the element of arc ΔS in a reference direction. Then, upon dividing Eq. 10.4 by Δx_j and considering the mean (in the sense of an ensemble average) deformation to be continuous, we may write

$$u^d_{i,j} = -F^{-1}_{kj} \sum \pi c^2 L_{c\phi\theta}\, \delta u_i\, N_k\, \Delta c\, \Delta\Omega \quad , \tag{10.8}$$

where F^{-1}_{kj} denotes an element of $\mathbf{F}^{-1}$. This relation can be put into spatial axes by means of Eq. A.66, so that

$$u^d_{i,j} = -\sum \nu n_j \pi c^2 L_{c\phi\theta}\, \delta u_i\, \Delta c\, \Delta\Omega \quad , \tag{10.9}$$

where

$$\nu = 1 / \Gamma = \left(\mathbf{N}\overline{\mathbf{V}}^{-2}\mathbf{N}\right)^{\frac{1}{2}} \quad . \tag{10.10}$$

The summation is taken over defect orientation, size, and type.

The treatment of plastic deformation has a somewhat different character. We consider plastic flow to consist of the motion of dislocation loops in slip planes. This is, of course, a great idealization, since loops may vanish, move out of their own plane, lose their circular character, coalesce, become pinned, or be nucleated by thermal or athermal mechanisms.

Equation 10.3 can be written

$$\Delta u_i = \sum_\alpha \Delta u_i^\alpha \quad , \qquad 10.11$$

where the sum is taken over various types of defects and over the continuum regions between defects, which are assumed to have the same strain rate. Dividing by Δx_j, we find, on taking the limit,

$$u_{i,j} = \sum_\alpha u_{i,j}^\alpha \quad . \qquad 10.12$$

These gradients may be separated into symmetric and antisymmetric parts, as indicated by Eq. 3.3. Then,

$$d_{ij} = \sum_\alpha d_{ij}^\alpha \qquad 10.13$$

and

$$w_{ij} = \sum_\alpha w_{ij}^\alpha \quad . \qquad 10.14$$

Thus, the proposed superposition of velocity gradients implies the superposition of stretchings, which, in turn, implies the superposition of strain rates as indicated in Eq. 10.2. It also implies superposition of spins.

This principle also implies superposition of energy rates. This conclusion is derived by operating on Eq. 10.13 with σ_{ij}, and writing

$$\dot{e}^\alpha = \sigma_{ij} d_{ij}^\alpha \quad . \qquad 10.15$$

The energy associated with each deformation mechanism is e^α. This energy may go into elastic deformation, and thereby represents a reversible process; into heating, and thereby represents an irreversible process; into formation of new surfaces; and into other microscopic processes such as local kinetic energy. The energy balance is written

$$\dot{e} = \sum_\alpha \dot{e}^\alpha \quad . \qquad 10.16$$

The rate of energy deposition into each type of mechanism can be written

$$\dot{e}^\alpha = a^\alpha \dot{e} \quad , \qquad 10.17$$

with a^α a fraction less than one and

$$\sum_\alpha a^\alpha = 1 \quad . \qquad 10.18$$

Each deformation mechanism involves a stretching which can be expressed with a constitutive law of the form

$$\mathbf{D}^{\alpha} = f^{\alpha}\left(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}, \mathbf{V}, \mathbf{D}\right) \quad , \tag{10.19}$$

though it may possibly involve other variables. This determines the a^{α}. However, this implies that

$$\mathbf{W}^{\alpha} = a^{\alpha}\mathbf{W} \quad , \tag{10.20}$$

for only if this relation holds can the constitutive laws embodied in Eq. 10.13 be made consistent with Eq. 10.12. This equation is not a necessary ingredient for calculating deformation, but once the deformation is known, it can be used to interpret the motion and determine the amount of spin associated with each type of deformation. If the stretching and spinning associated with each mode of deformation are applied to Eq. 3.31, we find

$$\boldsymbol{\omega}^{\alpha} = \mathbf{w}^{\alpha} + \left(\mathbf{I}\, tr\mathbf{V} - \mathbf{V}\right)^{-1}\mathbf{z}^{\alpha} \quad , \tag{10.21}$$

where $\mathbf{z}^{\alpha}$ is the vector associated with the matrix

$$\mathbf{Z}^{\alpha} = \mathbf{D}^{\alpha}\mathbf{V} - \mathbf{V}\mathbf{D}^{\alpha} \quad , \tag{10.22}$$

generalizing Eq. 3.21. Then, Eq. 3.5 can be resolved into

$$\dot{\mathbf{R}}^{\alpha} = \boldsymbol{\Omega}^{\alpha}\mathbf{R} \quad . \tag{10.23}$$

In addition, if **D** is represented by a superposition, this suggests that Eq. 3.14 can be resolved into

$$\hat{\mathbf{B}}^{\alpha} = 2\mathbf{V}\mathbf{D}^{\alpha}\mathbf{V} \quad , \tag{10.24}$$

and Eq. 3.18 can be resolved into

$$\dot{\overline{\mathbf{B}}}^{\alpha} = 2\overline{\mathbf{V}}\overline{\mathbf{D}}^{\alpha}\overline{\mathbf{V}} \quad . \tag{10.25}$$

Thus, one can determine by integration $\mathbf{B}^{\alpha}$, $\overline{\mathbf{B}}^{\alpha}$,

$$\mathbf{B} = \sum \mathbf{B}^{\alpha} \tag{10.26}$$

and

$$\overline{\mathbf{B}} = \sum \overline{\mathbf{B}}^{\alpha} \quad . \tag{10.27}$$

Similarly, the Green and Signorini strain rates of Eqs. 3.15 and 3.19 can be integrated with the results

$$\bar{\boldsymbol{\varepsilon}} = \sum \bar{\boldsymbol{\varepsilon}}^{\alpha} \tag{10.28}$$

and

$$\boldsymbol{\varepsilon} = \sum \boldsymbol{\varepsilon}^{\alpha} \quad . \tag{10.29}$$

We can also define

$$\dot{\mathbf{F}}^{\alpha} = \mathbf{G}^{\alpha}\mathbf{F} \qquad 10.30$$

and, thereby, define $\mathbf{F}^{\alpha}$ as the integral of $\dot{\mathbf{F}}^{\alpha}$. Then, a rotation can be defined by

$$\boldsymbol{R}^{\boldsymbol{\alpha}} = \boldsymbol{F}^{\boldsymbol{\alpha}}\boldsymbol{V}^{-1} \quad . \qquad 10.31$$

These $\boldsymbol{R}^{\boldsymbol{\alpha}}$ are not the same as those obtained by integrating Eq. 10.23. They are not orthogonal, nor are they the matrices that would be obtained by polar decomposition of F^{α}. This is not surprising, for rotations do not add. The superposition works only for rates.

The separation of **F** and other quantities into parts, each associated with a particular kind of defect, leads to new insights. In particular, the energy may be separated into parts associated with each defect, and this lends some insight into the dominant features of the overall motion in specific cases. More research into types of deformation and the conditions that allow one to dominate, as in brittle-ductile transition, would be of great interest.

11. SUPERPOSITION WITH TIME-DEPENDENT STATISTICS

If the statistics are time dependent as, for example, when cracks grow, the equations of Section 10 need to be generalized. One approach is to recognize that Eq. 10.3 for velocity increment has an analogue for displacement increment

$$\Delta v_i = \sum \Delta v_i^c + \Delta v_i^d \quad , \qquad 11.1$$

where Δv_i denotes the displacement in the x_i direction, Δv_i^c is the contribution from the continuum, and Δv_i^d denotes the contribution from defects. The summation is taken over all the individual defects. If now it is assumed that the cracks have a statistical distribution $\tilde{L}^{\alpha}$, then the contribution from defects can be expressed as

$$\Delta v_i{}^d = \Delta S \sum \tilde{L}^{\alpha} \; \delta v_i{}^{\alpha} \quad , \qquad 11.2$$

where, as before, $\tilde{L}^{\alpha}$ denotes the number of defects of type α per unit length, and the displacement across a defect of type α (α may characterize the radius) is written δv_i^{α}. The summation involves distributions of defects, which may change in the course of deformation. Thus, $\tilde{L}^{\alpha}$ may be time dependent. As before, the defects are taken to be circular, not a bad assumption in view of the statement by Kachanov quoted in the paragraph following Fig. 10.2. The relation of Eq. 10.5 continues to hold, but it should be recalled that L is time dependent here, and is considered to evolve according to an evolution law such as that of Eq. 3.2 of Vol. 2. Using Eq. 10.7a,b, we obtain

$$\Delta v_i^d = -\Delta X_k \sum \pi c^2 \Delta c \, \Delta\Omega \, L_{c\phi\theta} \, \delta v_i \, N_k \quad . \qquad 11.3$$

Here, it is noted that the summation is over crack orientation Ω as well as size, and could include type as well. We can find the displacement rate, the velocity due to defects, by differentiating Eq. 11.3 to obtain

$$\Delta \dot{v}_i^d = \Delta u_i^d = -\Delta X_k \sum \pi c^2 \Delta c \, \Delta\Omega (\dot{L}_{c\phi\theta} \, \delta v_i + L_{c\phi\theta} \, \delta \dot{v}_i) N_k \quad . \qquad 11.4$$

Dividing by Δx_j , taking the ensemble average of deformations with the same statistics, and letting ΔS go to zero, we have

$$G_{ij}^{d} = -F_{kj}^{-1} \sum \pi c^{2} \Delta c \, \Delta\Omega(\dot{L}_{c\phi\theta} \delta v_i + L_{c\phi\theta} \delta \dot{v}_i) N_k \quad . \qquad 11.5$$

Noting, as before, Eqs. A.66 and 10.10, we get the part of the velocity gradient due to defects

$$G_{ij}^{d} = -\sum \pi c^{2} \Delta c \, \Delta\Omega(\dot{L}_{c\phi\theta} \delta v_i + L_{c\phi\theta} \delta \dot{v}_i) n_j \,/\, \Gamma \quad . \qquad 11.6$$

The first term in parentheses is a contribution from the unsteady statistics that does not occur in the previous derivation. The second term is the same as Eq. 10.9. However, we use $1/\Gamma$ rather than υ of Eq.10.10, since it is difficult to distinguish between italic v (v) and Greek υ.

The deformation of foams provides a nice example of the GSSR, for the total deformation of a solid filled with spherical voids can be shown to involve two parts that can be added, one due to the matrix material and the other involving collapse of the voids. This is discussed by Dienes and Solem (1999).

12. CONCLUSIONS AND CLOSURE

This report began as a collection of notes concerning the feasibility of a novel theory of brittle materials in which the defects in an ensemble grow and coalesce, increasing the permeability, reducing strength, and forming hot spots, while the material experiences large strain and rotation. Such a theory was needed to understand the anisotropic response of oil shale and other geologic materials to blasting, the dynamics of gross structural failure, and the sensitivity of explosives and propellants to impact. Many other applications are extant, such as collapse of buildings, formation of the moon, and stimulation of reservoirs.

When this work began in the 1970s, continuum theories shed little light on the evolution of porosity and permeability or on the formation of hot spots in propellants. The work on NAGFRAG by Seaman and others at SRI (Seaman et al. 1972) appeared provocative but did not account for shear bands or shear cracks and made no reference to classical fracture theory. The work on associated flow laws and, later, nonassociated flow laws,[6] promised no insight or accounting for what happened on the mesoscopic or microscopic scales, and could not account for permeability or hot spots. For these reasons a new direction was taken. The goal was to have an approach that accounted for both microstructural behavior and large deformations. This report contains the framework for such a kinematic theory, though not in the complete form that would be expected of a formal treatise—it is still a collection of notes, but it should provide a useful reference for those interested in Statistical Crack Mechanics and its implementation in the SCRAM algorithm for material response.

It is shown that polar decomposition provides a useful and unique basis for a theory of large deformation, allowing us to place many of the standard measures of stress, strain, stress rate, strain

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[6] The associated flow law was based on the idea that the strain rate is parallel to the normal to the yield surface (in six-dimensional space), as discussed by Hill (1950). However, the use of this normality rule to account for dilatancy, as proposed by Drucker and Prager (1952), predicted far too much dilatancy (pore volume) due to shear, as discussed by Sandler and Baron (1985); so, various modifications were made to force the calculations to conform to experimental data. Experiments of Spitzig and Richmond (1984) demonstrated that there is no dilatancy at all in metals though the yield stress depends strongly on pressure, pretty much invalidating the whole idea of normality (and related concepts) in the view of this writer.

rate, rotation, and vorticity into a coherent framework. It allows us to show that the polar rate of rotation is different from the vorticity, that they are related by an algebraic formula, and that the polar rate is the appropriate choice for computing stress rate. It follows that vorticity does not provide a suitable measure of rotation for solids. In particular, it is shown that irrotational flows (zero vorticity) normally involve material rotation. Thus, the standard formulas for stress rate are not valid at large deformations, they exhibit unphysical periodic behavior under monotonic loading. The alternative, based on the polar stress rate, avoids this difficulty. It also turns out that polar decomposition provides a useful relation between stretching and Signorini strain rate, which, surprisingly, improves the stability of finite-element calculations. It has not been previously emphasized that stretching (the symmetric part of the velocity gradient) is not associated with any unique strain, casting doubt on its utility as a measure of strain rate. On the other hand, the Signorini strain rate is the polar rate of Signorini strain. Stretching and strain rate are related in a simple way (Eq. 3.6) useful in numerical calculations. The utility of the results from polar decomposition is shown in a variety of examples.

The main goal of this effort was to generalize the concept of superposition of strain rates. The traditional use was in superimposing elastic and plastic strain rates, an idea due to Reuss, but it was commonly thought that this concept is useful only for small deformations. Its utility lay in smoothing the transition between elastic and plastic states. In the current work, it is shown that the superposition principle is valid for large deformations as well, and that it applies when the Signorini strain rate is used as a measure of strain rate rather than stretching. It is considered especially useful to extend the superposition principle to a GSSR that allows us to combine the effects of various physical mechanisms by adding their strain rates. Thus, we can add the strain rates due to elasticity; plasticity; opening, shear, growth and coalescence of microcracks; and high-pressure equation of state. This allows for a very general formulation of constitutive laws, accounting for the failure of materials, the compaction of foams, and a myriad of other physical processes. This is an alternative to the popular concept of product decomposition, which takes the deformation gradient as a product of elastic and plastic contributions.[7] That concept has the drawback of leading to very complex algebra, even in the case of two contributions, elastic and plastic deformation. Were one to try to include fracture, high-pressure response, and void formation, the algebra would be quite intractable.

GSSR is based on the observation that the effects of defects can be added so long as they are geometrically separate and statistically independent. Large separation is not necessary, as illustrated by Kachanov (1984) in calculations of crack intersections. One effect of this rule is to allow for greatly reduced modulus of elasticity due to microcracking, and to allow for plasticity in the presence of this reduced elasticity.

To summarize GSSR, it extends the standard use of superposition in three ways. First, it is derived from a more fundamental basis, rather than presented as an ad hoc concept (a sort of *deus*

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[7] Product decomposition takes the deformation gradient as the product of elastic and plastic parts, $\mathbf{F} = \mathbf{F}^{\mathbf{e}}\mathbf{F}^{\mathbf{p}}$, claiming that the plastic part is physically superposed on the elastic part. It seems equally plausible, however, to argue that the plastic part dominates, so one should write $\mathbf{F} = \mathbf{F}^{\mathbf{p}}\mathbf{F}^{\mathbf{e}}$. This approach to finite deformation, which is often considered exact, leads to great algebraic complications. These are avoided by many researchers by arguing that for small strains, the rates of stretching due to elasticity and plasticity can be added, i.e., $\mathbf{D} = \mathbf{D}^{\mathbf{e}} + \mathbf{D}^{\mathbf{p}}$. On the other hand, in GSSR, this latter relation is considered exact, a logical consequence of defect statistics and polar decomposition.

ex machina). Second, the derivation shows that GSSR is valid for arbitrarily large overall deformation. Third, superposition is valid for combining any number of physical processes. These generalizations may be taken with a grain of salt, since counterexamples can surely be found involving strong defect interactions or unusual materials. Still, GSSR is believed to be a useful extension of the more traditional concept.

Appendices discuss a number of issues that would detract from the flow of the main document. Matrix concepts are reviewed in Appendix A, emphasizing simplicity and relevance to kinematics. It starts with the simplest concepts and ends with some new results. Appendix B compares the results of the main text with more traditional views on strain. The difficult subject of allowable formulations of constitutive laws is addressed in Appendices C and D, in which we attempt to delimit the use of tensor concepts in mechanics. In Appendix E, a very specialized calculation is undertaken to reconcile two apparently different points of view relating polar rate of rotation and vorticity (Hill 1978, Dienes 1978a), and it is shown that they are actually consistent. [This is in contrast to the implication by Nemat-Nasser (1983) that they are different.]

In fact, this work can be considered an effort to reconcile various points of view on many aspects of material behavior. It began with the effort to reconcile the rotation $\mathbf{R}$ obtained by polar decomposition and the vorticity, ω. The most recent effort concerns the formation of macroscopic cracks from microscopic defects, and reconciliation of continuum mechanics with the behavior of an ensemble of defects that exhibit discontinuities (Dienes 1996).

APPENDIX A
MATRIX THEORY

Summary of Matrix Notation and Concepts

A matrix is an array of numbers. It can be expressed in a variety of ways, such as

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \left(A_{ij}\right) = \mathbf{A} \quad . \qquad \text{A.1}$$

In general, the numbers A_{ij} can be complex, and the arrays need not be square, but in the theory of deformation the matrices of interest are usually 3 x 3, as in Eq. A.1, and the numbers are real. The transpose of **A** is obtained by diagonal reflection, viz.,

$$\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \left(A_{ji}\right) = \mathbf{A}^T \quad . \qquad \text{A.2}$$

Symmetric matrices have the property

$$\mathbf{A}^T = \mathbf{A} \quad or \quad A_{ij} = A_{ji} \quad , \qquad \text{A.3}$$

whereas, for antisymmetric matrices,

$$\mathbf{A}^T = -\mathbf{A} \quad or \quad A_{ji} = -A_{ij} \quad . \qquad \text{A.4}$$

Any matrix can be written as the sum of symmetric and antisymmetric parts, viz.,

$$\mathbf{A} = \mathbf{A}_s + \mathbf{A}_a \quad , \qquad \text{A.5}$$

where

$$\mathbf{A}_s = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) \qquad \text{A.6}$$

and

$$\mathbf{A}_a = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) \quad . \qquad \text{A.7}$$

In the case of a 3 x 3 matrix, the symmetric matrix contains 6 independent elements and the antisymmetric matrix contains 3 independent elements.

The sum of two matrices is given by

$$\mathbf{A} + \mathbf{B} = \left(A_{ij}\right) + \left(B_{ij}\right) = \left(A_{ij} + B_{ij}\right) \quad . \qquad \text{A.8}$$

A linear combination of matrices is given by

$$a\mathbf{A} + b\mathbf{B} = \left(aA_{ij} + bB_{ij}\right) \quad . \qquad \text{A.9}$$

The product of two matrices can be written

$$\mathbf{C}=\mathbf{AB} \qquad \text{A.10}$$

or

$$C_{ij}=\sum_k A_{ik}B_{kj}=A_{ik}B_{kj} \quad . \qquad \text{A.11}$$

It is often convenient to drop the sum over repeated indices, as indicated in the second part of Eq. A.11. The summation convention indicated above for matrices is more rudimentary than the convention in tensor analysis, which brings in a variety of physical and geometrical concepts. The transpose of a product is the product of the transposed matrices in reverse order,

$$\mathbf{C}^T=\mathbf{B}^T\mathbf{A}^T \qquad \text{A.12}$$

or

$$C_{ji}=\sum_k B_{ki}A_{jk}=A_{jk}B_{ki} \quad . \qquad \text{A.13}$$

Note that $\mathbf{AB}\neq\mathbf{BA}$ in general and that symmetry of **A** or **B** does not imply any symmetry properties for the product.

Inverse Matrices

The inverse of a matrix is written $\mathbf{A}^{-1}$. It has the property that when either premultiplied or postmultiplied by **A**, the product is the identity matrix

$$\mathbf{I}=\left(\delta_{ij}\right)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} , \qquad \text{A.14}$$

where δ_{ij} is the Kronecker delta. Thus,

$$\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I} \quad . \qquad \text{A.15}$$

The transpose of the inverse is the inverse of the transpose, allowing us to define $\mathbf{A}^{-T}$ unambiguously:

$$\left(\mathbf{A}^{-1}\right)^T=\left(\mathbf{A}^T\right)^{-1}=\mathbf{A}^{-T} \quad . \qquad \text{A.16}$$

The proof is given here to illustrate the power of matrix methods. Applying Eqs. A.12 to A.15 we have

$$\left(\mathrm{A}^{-1}\right)^T\mathrm{A}^T=\mathrm{I}^T=\mathrm{I} \quad .$$

Then, postmultiplying by $\left(\mathbf{A}^T\right)^{-1}$, one finds A.16. It is very awkward to show this with index notation.

Determinants

In the preceding paragraph, the existence and properties of inverse matrices were treated in outline form. Consider now the calculation of inverse matrices. For this, it is necessary to make use of determinants, written as

$$\left|A_{ij}\right| = |\mathbf{A}| = A \quad . \qquad \text{A.17}$$

For a 2 x 2 matrix

$$\left|A_{ij}\right| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21} \quad . \qquad \text{A.18}$$

For a 3 x 3 matrix

$$\left|A_{ij}\right| = e_{ijk}A_{i1}A_{j2}A_{k3} \quad , \qquad \text{A.19}$$

where e_{ijk} is the classic permutation symbol: if any 2 of ijk are the same, it is zero. Otherwise, it is 1 if ijk can be obtained from 123 by interchanging pairs of indices an even number of times, and –1 if obtained by an odd number of interchanges.

A cofactor a_{ij} is formed by striking out the ith row and jth column of the matrix **A** and forming the determinant of the remaining terms, with appropriate sign $(-1)^{i+j}$. Thus, for example,

$$a_{23} = (-1)^{2+3} \begin{vmatrix} A_{11} & A_{12} & \mathbf{A_{13}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{A_{23}} \\ A_{31} & A_{32} & \mathbf{A_{33}} \end{vmatrix} = -\left(A_{11}A_{32} - A_{12}A_{31}\right) \quad ,$$

where the terms to be struck out are indicated in copperplate bold. Then, the inverse of a matrix is given by

$$A_{ij}^{-1} = a_{ji} / A \quad . \qquad \text{A.20}$$

This is sometimes called the reduced cofactor. Also,

$$a_{ij}A_{kj} = \delta_{ik}A$$

or

$$\boldsymbol{a}\mathbf{A}^{T} = A\mathbf{I} \qquad \text{A.21a}$$

is called the Laplace expansion of A.

If the A_{ij} are considered functions of a parameter *s*, then the derivative of the determinant A is given by

$$\frac{dA}{ds} = a_{ij}\frac{d}{ds}A_{ij} \qquad \text{A.21b}$$

or

$$\frac{d|\mathbf{A}|}{ds} = tr\left(\boldsymbol{a}\frac{d\mathbf{A}^T}{ds}\right) \quad , \tag{A.22}$$

where a_{ij} is the cofactor of A_{ij}. An important application of this formula is the proof of Euler's identity concerning conservation of mass

$$\dot{J} = Ju_{i,i} \quad , \tag{A.23}$$

where the superimposed dot denotes the time derivative, and

$$J = v / v_o = \rho_o / \rho = |\mathbf{F}| \tag{A.24}$$

denotes the relative volume of an element of mass of specific volume v and density ρ. The zero subscripts denote initial values. The second term on the right of Eq. A.23 denotes the divergence of the velocity field, and **F** denotes the deformation gradient. To prove Eq. A.23, write Eq. 3.1 as

$$\dot{\mathbf{F}} = \mathbf{GF} \quad . \tag{A.25}$$

Differentiating Eq. A.24, applying Eq. A.21 and Eq. A.22, and denoting the cofactor of F_{ij} by f_{ij},

$$\dot{J} = \frac{d}{dt}|\mathbf{F}| = f_{ij}\dot{F}_{ij} = f_{ij}G_{ik}F_{kj} = |\mathbf{F}|\, G_{ii} = |\mathbf{F}|\, u_{i,i} \quad ,$$

verifying Eq. A.23. Alternatively, in matrix notation,

$$\dot{J} = \frac{d}{dt}|\mathbf{F}| = tr\left(f\,\dot{\mathbf{F}}^T\right) = tr\left(f\,\mathbf{F}^T\mathbf{G}^T\right) = |\mathbf{F}|\, tr\,\mathbf{G} \quad .$$

A useful formula given by Flugge (1972) is

$$\begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = e_{ijk}e_{lmn} \quad . \tag{A.26}$$

This formula is, in fact, only a special case of a formula that nicely characterizes the behavior of determinants, viz.,

$$\begin{vmatrix} A_{or} & A_{os} & A_{ot} \\ A_{pr} & A_{ps} & A_{pt} \\ A_{qr} & A_{qs} & A_{qt} \end{vmatrix} = Ae_{opq}e_{rst} \quad , \tag{A.27}$$

where A is the value of the determinant

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A \quad . \tag{A.28}$$

Expansion of the determinant in Eq. A.26 leads to

$$e_{ijk}e_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} \quad . \tag{A.29}$$

Contraction leads to the following useful relations:

$$e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad , \qquad \text{A.30}$$

$$e_{ijk}e_{ijn} = 2\delta_{kn} \quad , \qquad \text{A.31}$$

$$e_{ijk}e_{ijk} = 6 \quad . \qquad \text{A.32}$$

Eigenvectors and Eigenvalues

Matrices appear in the solution of simultaneous equations, which can be written

$$A_{ij}x_j = y_i \quad ,$$

or

$$\mathbf{Ax} = \mathbf{y} \quad .$$

The solution can be written formally as

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

provided that the determinant of **A** is not zero. An important variation on this theme occurs when the vectors x and y are parallel, so that

$$\mathbf{Ax} = \lambda\mathbf{x} = \lambda\mathbf{Ix} \quad . \qquad \text{A.33}$$

Here, λ is described as an eigenvalue of **A**. Since Eq. A.33 is homogeneous in **x**, it has a nonzero solution only if the determinant of the matrix $\mathbf{A} - \lambda\mathbf{I}$ is zero. If **A** is an n x n matrix, then there are n values of λ, denoted by λ^α, which are eigenvalues; each has its own eigenvector, $\mathbf{x}^\alpha$, for which Eq. A.33 is satisfied. Expanding the determinant results in a polynomial in λ. The solutions satisfy the n relations

$$\mathbf{Ax}^\alpha = \lambda^\alpha\mathbf{x}^\alpha \qquad \text{A.34}$$

not summed on α. In some applications, there may be double roots, so that the number of distinct eigenvalues is less than n, but this does not present an important aspect of the theory of deformation. It is generally useful to deal with normalized eigenvectors having the property

$$\sum_i x_i^\alpha x_i^\alpha = 1 \quad .$$

This relation is subsumed in the more general relation

$$x_i^\alpha x_i^\beta = \delta^{\alpha\beta} \qquad \text{A.35}$$

expressing orthogonality as well as normalization; the summation on i is now understood. When **A** is symmetric, the λ^α are real. This is often the case in the theory of deformation. However, when dealing with rotations, **A** is orthogonal and, consequently, not symmetric. Then, it can be shown that one of the λ^α is real (assuming 3 dimensions) and the others are complex. This subject is discussed in detail by Goldstein (1959).

To show that the λ^α are real when **A** is symmetric, suppose for a moment that they are complex, and let an overbar denote the complex conjugate. Then, conjugation of Eq. A.34 leads to

$$\boldsymbol{A}\bar{\boldsymbol{x}}^{\boldsymbol{\alpha}} = \bar{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\bar{\boldsymbol{x}}^{\boldsymbol{\alpha}} \quad .$$

Switching to index notation, and then operating with x_i^α, we find

$$x_i^{\boldsymbol{\alpha}} A_{ij}\bar{x}_j^{\boldsymbol{\alpha}} = \bar{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} x_i^{\boldsymbol{\alpha}}\bar{x}_i^{\boldsymbol{\alpha}}$$

Similarly, operating on Eq. A.34 with $\bar{x}_i^\alpha$ leads to

$$\bar{x}_i^{\boldsymbol{\alpha}} A_{ij} x_j^{\boldsymbol{\alpha}} = \boldsymbol{\lambda}^{\boldsymbol{\alpha}} x_i^{\boldsymbol{\alpha}}\bar{x}_i^{\boldsymbol{\alpha}} \quad .$$

But if $A_{ij} = A_{ji}$, the left sides of these two equations are equal. Then, the λ^α are equal to their conjugates, implying that they are real, which was to be shown. Similar arguments lead to Eq. A.35.

A symmetric matrix can be expanded into the form

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^T \quad , \qquad \text{A.36}$$

where X denotes the matrix of normalized eigenvectors,

$$\mathbf{X} = \left(x_i^\alpha\right) \quad . \qquad \text{A.37}$$

In this notation, A.35 can be written

$$\mathbf{X}\mathbf{X}^T = \mathbf{I} \quad . \qquad \text{A.38}$$

Matrices with this property are said to be orthogonal. If a matrix is interpreted as defining a quadric surface

$$A_{ij} y_i y_j = 1 \quad ,$$

this has a simple geometric interpretation, for quadrics of this form have principal axes about which they are symmetric. The transformation **X** rotates the coordinate axes to the principal axes of the quadric. Orthogonality is, thus, the algebraic equivalent of a rotation in geometry.

The expansion of Eq. A.36 can be regarded as a reduction of **A** to diagonal form. It is useful in many contexts. For example, any integer power of a matrix can be written

$$\mathbf{A}^n = \mathbf{X}\boldsymbol{\Lambda}^n\mathbf{X}^T \quad , \qquad \text{A.39}$$

in view of Eq. A.38. It readily follows that Eq. A.39 holds for any rational or negative n. The extension to irrational powers is beyond the scope of this material.

The eigenvalues of Eq. A.33 can be found in practice in numerous ways, but one important way in analysis is to notice that Eq. A.33 can be written in the form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \quad , \qquad \text{A.40}$$

which is homogeneous in **x**. This equation has nontrivial solutions only if the determinant of the coefficients is zero,

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad . \tag{A.41}$$

This determinant can be expanded as a polynomial of degree n in λ,

$$\lambda^n - \sum_{m=1}^{n} a_{m-1}\lambda^{m-1} = f(\lambda) = 0 \quad , \tag{A.42}$$

where the a_m are "invariants" of **A**. This term expresses the fact that if **A** is transformed to different axes, the a_m are unchanged. The n roots of Eq. A.42 are the eigenvalues of **A**. Equation A.42 is often called the characteristic equation, and the roots the characteristic values, a term equivalent to eigenvalues.

These properties can be used to show that a matrix satisfies its own characteristic equation, the Cayley-Hamilton theorem. This is of considerable value in many aspects of the analysis of deformation. To show this, note that in view of Eq. A.39

$$f(\mathbf{A}) = \mathbf{X} f(\mathbf{\Lambda})\mathbf{X}^T \quad ,$$

where the polynomial function f is defined by Eq. A.42. Since $\mathbf{\Lambda}$ is diagonal, we have, for n = 3,

$$f(\mathbf{\Lambda}) = \begin{pmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & 0 \\ 0 & 0 & f(\lambda_3) \end{pmatrix} \quad ,$$

and each of the diagonal terms of this matrix is zero in view of Eq. A.42. The extension to general n is obvious. Thus,

$$f(\mathbf{A}) = 0 \quad , \tag{A.43}$$

and **A** satisfies its own characteristic equation.

If the λ^α are all positive, then the matrix is said to be positive definite. It is often convenient to refer to the quadric in **X**, $f(\mathbf{A}) = \mathbf{X} f(\mathbf{\Lambda})\mathbf{X}^T$. The quadric is an ellipsoid when the λ^α are all positive. There are many criteria concerning the conditions under which the λ^α are positive, but these are beyond the current scope.

Rotation

A matrix having the orthogonal property

$$\mathbf{R}^T = \mathbf{R}^{-1} \tag{A.44}$$

represents a rotation. To see that this algebraic property represents the geometric process that we call a rotation, consider the process in which an element of length ds is unchanged, though the solid experiencing this process is deformed. (We use "deformation" to include stretch and rotation). Then, if dx_i defines the element after rotation and dx_i' the element before rotation, we must have, using Eq. 2.1,

$$ds^2 = dx_i dx_i = F_{ij} dx_j' F_{ik} dx_k' = dx_j' dx_j' \quad .$$

This holds if

$$F_{ij} F_{ik} = \delta_{jk} \quad ,$$

that is,

$$\mathbf{F}^T \mathbf{F} = \mathbf{I} \quad .$$

Thus,

$$\mathbf{F}^T = \mathbf{F}^{-1} \quad ,$$

showing that the length of an element of material is unchanged if the transpose and inverse of the deformation matrix are equal, and this is what is meant by a rotation.

An orthogonal matrix can be expressed as the exponential

$$\mathbf{R} = e^{\phi \mathbf{Q}} \quad , \qquad \text{A.45}$$

where **Q** is a skew-symmetric matrix representing the axis of rotation and ϕ is the angle of rotation. To show this, let **r** be a vector that is unchanged by the rotation **R**, so that

$$\boldsymbol{rR} = \lambda \boldsymbol{r}, \quad \lambda = 1 \quad . \qquad \text{A.46}$$

That is, if one of the eigenvalues of **R** is unity, **R** represents a rotation about **r**. To prove this, first apply Eq. A.44 and then Eq. A.46 to the product given below on the left, with an overbar denoting the complex conjugate,

$$r_i R_{ij} \bar{r}_k R_{kj} = r_i \bar{r}_i = \lambda r_j \bar{\lambda} \bar{r}_j \quad .$$

Then, on division by $r_i \bar{r}_i$, one finds that $\lambda \bar{\lambda} = 1$. Now, there are 3 eigenvalues for a rotation in three dimensions which are the roots of the cubic equation given as Eq. A.41. Since at least one root must be real, it must be ± 1**.** But **R** = **I**, at the beginning of deformation, so $\lambda = 1$. Since rotations are continuous, $\lambda = 1$ must continue to be the real root as deformation proceeds. It is straightforward to show that there are no other real roots. This shows that Eq. 46 holds.

To continue the proof of Eq. 45, we relate the antisymmetric matrix **Q** to the unit vector **r** by

$$Q_{ik} = e_{ijk} r_j \quad , \quad \mathbf{Q} = \left(Q_{ik}\right) \quad . \qquad \text{A.47}$$

Now, expand the exponential of A.45 in a power series, so that

$$\mathbf{R} = \mathbf{I} + \phi \mathbf{Q} + \phi^2 \mathbf{Q}^2 / 2! + \ldots \quad . \qquad \text{A.48}$$

But it can be shown by means of Eq. A.30 that

$$\mathbf{Q}^2 = \left(r_i r_l - \delta_{il}\right) \quad , \qquad \text{A.49}$$

$$\mathbf{Q}^4 = -\mathbf{Q}^2 \quad , \tag{A.50}$$

$$\mathbf{Q}^{2n+2} = -\mathbf{Q}^{2n} \quad , \tag{A.51}$$

and

$$\mathbf{Q}^{2n+1} = -\mathbf{Q}^{2n-1} \quad . \tag{A.52}$$

Care must be expressed in manipulations involving **Q** because its determinant is zero and, consequently, it has no inverse. Thus, n in Eqs. A.51 and A.52 is restricted to be a positive integer. Using these relations, Eq.A.48 can be rearranged so that

$$\mathbf{R} = \mathbf{I} + \sin\phi\,\mathbf{Q} + (1-\cos\phi)\mathbf{Q}^2 \quad . \tag{A.53}$$

It readily follows from Eq. A.47 that **Rr**=**r**, proving that this representation of **R** generates a rotation about **r**.

By taking the trace of both sides of Eq. A.53, it can be shown that

$$\cos\phi = \frac{1}{2}(tr\mathbf{R} - 1) \quad . \tag{A.54}$$

The angle ϕ is the amount of rotation represented by **R** about the axis of rotation, **r**. This can be verified by considering a rotation **S** of coordinate axes such that

$$\mathbf{S}^T\mathbf{R}\mathbf{S} = \mathbf{P} = \mathbf{S}^{-1}\mathbf{R}\mathbf{S} \quad , \tag{A.55}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{A.56}$$

The similarity transformation indicated in Eq. A.55 is discussed by Bellman (1960), Gantmacher (1959), and other treatises on matrices. Since

$$S_{ij}S_{kj} = \delta_{ik} \quad ,$$

it can be immediately shown that

$$tr\mathbf{P} = tr\mathbf{R} \quad ,$$

which, in view of Eqs. A.53 and A.56, is tantamount to

$$1 + 2\cos\phi = 1 + 2\cos\theta \quad ,$$

or $\theta = \phi$. Thus, the rotation represented by **R** is equivalent to a rotation about an axis **r**, the real, normalized eigenvector of **R**, through an angle ϕ determined by the trace of **R**.

If $\mathbf{R}_1$ and $\mathbf{R}_2$ represent successive rotations, their order matters, for $\mathbf{R}_1\mathbf{R}_2$ is different from $\mathbf{R}_2\mathbf{R}_1$. However, any two rotations are equivalent to a single rotation about a suitable axis, for it is readily shown that

$$\left(\mathbf{R}_1\mathbf{R}_2\right)^T = \left(\mathbf{R}_1\mathbf{R}_2\right)^{-1} \ ,$$

so that the product has the orthogonal property. It can be shown that the angle of rotation is the same whatever the order of the rotations $\mathbf{R}_1$ and $\mathbf{R}_2$. To see this, let ϕ_1 and ϕ_2 denote the two rotation angles, and let

$$\cos\alpha = r_i^1 r_i^2 \qquad \text{A.57}$$

so that α is the angle between the two axes of rotation. Then, it can be shown that

$$tr\mathbf{R}_1\mathbf{R}_2 = 1 + 2(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2\cos\alpha) + \sin^2\alpha\left(\cos\phi_1 + \cos\phi_2 - 1 - \cos\phi_1\cos\phi_2\right) \text{A.58}$$

Since ϕ_1 and ϕ_2 can be interchanged in this formula, and $\cos\alpha$ is symmetric in $\mathbf{r}^1$ and $\mathbf{r}^2$, the order of rotations does not affect $tr\,\boldsymbol{R}_1\boldsymbol{R}_2$ and, hence, the angle of rotation. Of course, the axis depends on the order.

If the two rotations have the same axis, so that α=0, it follows from Eq. A.58 that the rotation angle is the sum of ϕ_1 and ϕ_2.

It can also be shown that when the rotation is time dependent,

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T = \dot{\phi}\,\mathbf{Q} + \sin\phi\,\dot{\mathbf{Q}} + \left(1 - \cos\phi\right)\left(\dot{\mathbf{Q}}\mathbf{Q} - \mathbf{Q}\dot{\mathbf{Q}}\right) \qquad \text{A.59}$$

and

$$\dot{\mathbf{Q}}\mathbf{Q} - \mathbf{Q}\dot{\mathbf{Q}} = \left(r_i\dot{r}_j - \dot{r}_i r_j\right) \ . \qquad \text{A.60}$$

The three terms on the right of Eq. A.59 represent rates about perpendicular axes. This can be shown by noting that the angle between two unit vectors is simply related to the inner product of the corresponding antisymmetric matrices, for, using Eq. A.31,

$$A_{ij}B_{ij} = e_{ikj}a_k e_{ilj}b_l = 2a_k b_k \ .$$

Thus, if the inner product is zero, the corresponding vectors are perpendicular. This result can be exploited to show that the three terms contributing to $\mathbf{\Omega}$ in Eq. A.59 represent rotations about three perpendicular axes.

Deformation of an Element of Area

In the course of deformation, perpendicular material fibers do not remain perpendicular, nor do perpendicular planes. A material fiber characterized by dX_i is transformed into $F_{ij}dX_j$ in the course of deformation. The change in element of area, however, requires a more elaborate calculation, as follows. The directed elements of area of a pair of fibers dx_i^1 and dx_i^2 are

$$ds_k = dx_i^1 dx_j^2 e_{ijk} = F_{il}F_{jm}dX_l^1 dX_m^2 e_{ijk} \ . \qquad \text{A.61}$$

Now, it can be shown that

$$F_{il}F_{jm}e_{ijk} = |\mathbf{F}|\, e_{lmn}H_{nk} \ , \qquad \text{A.62}$$

where

$$\mathbf{H} = \mathbf{F}^{-1} \quad . \tag{A.63}$$

(This can be obtained from the determinant expansion

$$a e_{lmn} = a_{il} a_{jm} a_{kn} e_{ijk} \quad ,$$

which is a slight generalization of Eq. A.19, by multiplying by the inverse A_{lp}). Then

$$dx_i^1 dx_j^2 e_{ijk} = |\boldsymbol{F}| H_{nk} e_{lmn} dX_l^1 dX_m^2 \quad . \tag{A.64}$$

Thus, a differential vector area $e_{lmn} dX_l^1 dX_m^2 = dS_n^o$ in reference axes is transformed into $ds_k = |\mathrm{F}| H_{nk} dS_n^o$ in current spatial axes, or, in vector notation,

$$d\mathbf{s} = |\mathbf{F}|\, d\mathbf{S}^o\, \mathbf{F}^{-1} = |\mathbf{F}|\, \mathbf{F}^{-T} d\mathbf{S}^o \quad . \tag{A.65}$$

The last term on the right is the same as Eq. 1.12.3 of Eringen (1967) or Eq. 1.32 of Hill (1978). In the absence of strain, this reduces to the simple rotation indicated by Eq. 3.13. The unit normal to the deformed element of area is given by normalizing d**S** and $d\mathbf{S}^o$

$$\mathrm{n} = \mathrm{F}^{-T} \mathrm{N} \left(\mathrm{N} \overline{\mathrm{V}}^{-2} \mathrm{N} \right)^{-1/2} = \mathrm{LN} \quad , \tag{A.66}$$

where **N** is the unit normal in reference axes and

$$\mathbf{L} = \Gamma\, \mathbf{F}^{-T} \quad , \tag{A.67}$$

with

$$\Gamma = \left(\mathbf{N} \overline{\mathbf{V}}^{-2} \mathbf{N} \right)^{-\frac{1}{2}} \quad . \tag{A.68}$$

But **L** is not a rotation matrix, as it depends on the normal **N**, nor does it satisfy Eq. A.44. The quantity Γ can be written in terms of spatial variables by rewriting Eq. A.66 as

$$n_i F_{ik} = \Gamma N_k \quad . \tag{A.69}$$

Since $N_k N_k = 1$, it readily follows that

$$\Gamma^2 = n_i B_{ij} n_j \quad . \tag{A.70}$$

Thus, Γ^2 can be regarded as the *normal component of position*. (Just as traction is a force per unit area related to stress by $T_i = \sigma_{ij} n_j$, so can position be related to stretch by $P_i = B_{ij} n_j$.)

The derivative of **L** is needed in some calculations, requiring the derivative of an inverse matrix. To find the needed relation, take the derivative of Eq. A.15. Then,

$$\dot{\mathbf{A}} \mathbf{A}^{-1} + \mathbf{A} \frac{d}{dt} \left(\mathbf{A}^{-1} \right) = 0 \quad ,$$

and it follows that

$$\frac{d}{dt}\left(\mathbf{A}^{-1}\right) = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1} \quad . \tag{A.71}$$

The derivative of Γ can be found by using the result

$$\dot{\mathbf{n}} = -\mathbf{G}^T\mathbf{n} + \left(\dot{\Gamma} / \Gamma\right)\mathbf{n} \quad , \tag{A.72}$$

which follows from differentiating Eq. A.66 and using Eqs. A.71 and 3.1. Then, since $n_i n_i = 1$, it follows that

$$n_i \dot{n}_i = 0 \quad . \tag{A.73}$$

Applying this to Eq. A.72, we find, using Eq. 3.3, that

$$\dot{\Gamma} / \Gamma = n_i D_{ij} n_j \quad , \tag{A.74}$$

since the antisymmetry of W_{ij} results in

$$n_i W_{ij} n_j = 0 \quad . \tag{A.75}$$

Alternatively, Eq. A.74 can be found by differentiating the relation

$$\Gamma^{-2} = \mathbf{N}\overline{\mathbf{B}}^{-1}\mathbf{N} \tag{A.76}$$

that follows from Eq. A.67, but the calculation is considerably longer.

Using Eq. A.75, Eq. A.72 can be written as

$$\dot{\mathbf{n}} = \mathbf{W}\,\mathbf{n} + \left(\mathbf{nDn}\;\mathbf{I} - \mathbf{D}\right)\mathbf{n} \quad . \tag{A.77}$$

If n is chosen so that it lies parallel to a principal direction of D, then the second term on the right of Eq. A.77 vanishes, leaving

$$\dot{\mathbf{n}}' = \mathbf{W}\mathbf{n}' \quad , \tag{A.78}$$

where the prime denotes a vector that is aligned with one of the principal directions of **D**. This result is erroneously described by Truesdell and Toupin (1960) and reproduced by Eringen (1967) as “the spin is the angular velocity of the principle axes of extension.” The correct statement is, “A material line element **m** of unit length that is aligned with one of the principle directions of **D** rotates with velocity $\dot{\mathbf{m}} = \mathbf{W}\mathbf{m}$.” Note that the skew character of **W** leads to $\mathbf{m} \bullet \dot{\mathbf{m}} = 0$.

It is no coincidence that **m** and **n** satisfy the same relations, since they represent dual triads. An **n** vector is the cross product of two material vectors, the pair defining the orientation of a surface element.

APPENDIX B

REMARKS ON THE MEASURE OF STRAIN

For small deformations, the analysis of strain presents no uniqueness problem, but for large deformations, numerous competing definitions have been proposed, of which many have been discussed by Truesdell and Toupin (1960). The choice of strain measure is of more than academic interest, for mechanical data are usually presented in the form of stress-strain curves. Unfortunately, the choice of strain measure is rarely discussed by experimentalists. It is, however, crucial to nonlinear material science that theory and experiment should refer to the same thing, and desirable that the choice should be a reasonably good one. In Appendix C, the Signorini strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{B} - \boldsymbol{I})$, as given by Eq. 3.7, is compared with the two most common strain measures, that of Green and St. Venant, $\boldsymbol{E} = \frac{1}{2}\left(\overline{\boldsymbol{B}} - \boldsymbol{I}\right)$, and that of Almansi and Hamel, $\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right)$, from the point of view of metric tensors. In this appendix, some general remarks are made about the ambiguities in finite-deformation strain.

In Truesdell and Toupin (1960), the problem of simple shear is discussed at length, and it is shown that second-order terms appear in the strain tensors described above with $E_{22} = 2\varepsilon^2$, $e_{22} = -2\varepsilon^2$, and $\varepsilon_{22} = 0$ in current notation. The fact that these alternatives are entirely different is not discussed! In fact, the measures **E, e,** and $\underline{\boldsymbol{\varepsilon}}$ are different representations of the same physical strain. The theory underlying this apparent discrepancy is resolved in Section 3 and an example is discussed in detail in Section 5. The relations between various theoretical measures of strain are discussed in Appendix C. In this section, we are more concerned with the actual measurement of strain. The approach taken by Hill is also discussed, and it is shown that his strain measures are fixed in the material. Thus, though his treatment is said to be general, the Signorini strain is not one of those included in Hill's general theory.

Strain gauges can be used to measure the Green-St. Venant strain, but do not provide any information about rotation. This may not matter for some purposes, but there is a risk that stress may be measured (or inferred) in one system and strain in another. This will not matter much in small deformations, but could lead to large errors when the deformation (stretch and rotation) is large. Strain gauge measurements are made in polar axes. They could be obtained in space axes by application of Eq. 3.10 if **R** were known, but **R** cannot be deduced without further measurements. Consequently, it appears that the spatial (Signorini) strain cannot be obtained with strain gauges, though the Green-St. Venant strain can be found in this way. Of course, strain gauges are not reliable at large strains anyway, so such approaches are not of interest for large deformations.

A more general approach to measurement of strain is to observe the deformation of a square grid of lines inscribed on the test sample. The deformation gradient **F** for the surface can be obtained, approximately, by the finite difference

$$F_{ij} \sim \frac{\Delta x_i}{\Delta X_j} \quad , \qquad \text{B.1}$$

where ΔX_j denotes the initial grid length and Δx_i the changing grid length. The stretch and rotation can be obtained, in two dimensions, by computing **B** with Eq. 2.3 and the **V** from Eq. 8.9. The rotation follows from Eq. 2.2. The spatial (Signorini) strain can be obtained from Eq. 3.7, and material (Green) strain follows from Eq. 3.17. This approach to measurement of strain is not used much (if at all), though it would appear to lead to more complete results than strain gauges for large deformation. Extension to three dimensions is possible, in principle, by use of internal markers and precise depth measurement, but is probably very difficult. Alternatively, thickness measurements of thin sheets might be used to obtain the full three-dimensional strain matrix.

A different approach to the analysis of strain has been taken by Hill (1978), who defines a general measure of strain as

$$\tilde{E}_{ij} = \sum_{\alpha} \tilde{e}_{\alpha}\, \mu^{L}_{i\alpha}\, \mu^{L}_{j\alpha} \quad , \tag{B.2}$$

where the $\mu^{L}_{i\alpha}$ are equivalent to **S** (Eq. 2.12), and

$$\tilde{e}_{\alpha} = \left(\lambda_{\alpha}^{2m} - 1\right) / \, 2m \quad . \tag{B.3}$$

For m = 1, this is the Green strain. For $m = \frac{1}{2}$, it reduces to engineering strain. For m = 0, it is the logarithmic strain, and for m = -1, it is the Almansi strain. In the current notation this is equivalent to

$$\tilde{\boldsymbol{E}} = \boldsymbol{S}^{T} \tilde{\boldsymbol{e}}\, \boldsymbol{S} \quad , \tag{B.4}$$

where $\tilde{\mathbf{e}}$ is the diagonal matrix of strains. Though Hill makes use of different notation and emphasizes vector triads more than matrices, his results are identical to those given here. Hill's relations and the corresponding matrix equations are listed in Table B-2. Unit vectors in the directions of the principal axes of the stretch ellipsoid measure in space axes (Hill's "Eulerian" frame) are denoted by $\boldsymbol{\mu}^{E}_{\alpha}$. The $\boldsymbol{\mu}^{L}_{\alpha}$ are the same unit vectors in polar axes (Hill's "Lagrangean" frame). The Signorini strain given in Eq. 3.7 would be

$$\boldsymbol{\varepsilon} = \frac{1}{2} \sum_{\boldsymbol{\alpha}} \left(\boldsymbol{\lambda}^{2}_{\boldsymbol{\alpha}} - 1\right) \mathrm{m}^{E}_{\boldsymbol{\alpha}} \times \mathrm{m}^{E}_{\boldsymbol{\alpha}} \tag{B.5}$$

in Hill's notation. It lies outside the realm of strains included in Eq. B.5.

It is common for authors and computer programmers to describe their work as "general" or "unified" when it combines several (or perhaps only two) ideas. Keynes's general theory, Einstein's general and unified theories, and Hill's (1978) "unified and definitive" article are sometimes considered by readers (or those who have not read, but have heard of, those works) to encompass everything that is relevant. This is a great danger indeed. *Caveat lector*. Though a magnificent contribution, Hill's article does not discuss $\dot{\mathbf{R}}\mathbf{R}^{T}$, $\hat{\boldsymbol{\sigma}}$, Signorini strain, or the topic of strain rate in a manner suitable for computation or measurement of large deformation. These subjects deserve more attention.

Table B-2. Comparison of Hill's and Current Notations

Current	Hill
$\mathbf{G}$	$\mathbf{\Gamma}$
$\mathbf{D}$	$\boldsymbol{\varepsilon}$
$\mathbf{W}$	$\mathbf{\Omega}$
$\mathbf{F}$	$\mathbf{A}$
$\mathbf{D} - \mathbf{W} = \mathbf{G}^T$	$\mathbf{\Gamma}'$
$\mathbf{F}^{-T}$	$\mathbf{B}$
$\mathbf{S}$	$\left[\boldsymbol{\mu}_\alpha^L\right] = \left(\mu_{i\alpha}^L\right)$ see note
$\mathbf{T}$	$\left[\boldsymbol{\mu}_\alpha^E\right] = \left(\mu_{i\alpha}^E\right)$ see note
$\overline{\mathbf{V}} = \mathbf{S}^T \boldsymbol{\lambda} \mathbf{S}$	$\mathbf{\Lambda} = \sum_\alpha \lambda_\alpha \, \boldsymbol{\mu}_\alpha^L \times \boldsymbol{\mu}_\alpha^L$
$\mathbf{R}=\mathbf{TS}$	$\mathbf{R} = \sum_\alpha \boldsymbol{\mu}_\alpha^E \times \boldsymbol{\mu}_\alpha^L$
$\mathbf{T} = \mathbf{RS}^T$	$\boldsymbol{\mu}_\alpha^E = \mathbf{R}\boldsymbol{\mu}_\alpha^L$

Note: The square bracket denotes a one-dimensional array. The one-dimensional array of vectors is a matrix.

APPENDIX C
REMARKS ON THE TENSOR CHARACTER OF STRAIN

In much of the literature of continuum mechanics, the quantities $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are said to be tensors on the grounds that they appear in quadratic forms representing the square of arc length, and they satisfy certain transformation laws. In this appendix, a more physical and restricted notation of tensor is considered which requires that the transformation involve coordinates that characterize current physical space and not just some mathematical reference space. Thus, the quantity represented by the components A^{ij} is considered a contravariant physical tensor only if it transforms according to

$$A'^{ij} = A^{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} \quad , \qquad \text{C.1}$$

and a covariant physical tensor only if it transforms according to

$$A'_{ij} = A_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \quad , \qquad \text{C.2}$$

where the x^i and x'^i are coordinates of the current physical space in which physical processes take place. Many authors call a tensor a quantity that transforms according to Eqs. C.1 or C.2 when the x^i are any system of coordinates. With the more limited notion of physical tensor used here, some quantities that are called strains in the literature are not tensor invariant and, thus, are not physical tensors.

According to this restricted definition, $\overline{\mathbf{B}}$ ($\mathbf{C}$ in Truesdell's 1952 notation), defined in Eq. 3.16, is not a physical tensor, for under a change of coordinates

$$\overline{B}'_{kl} = \sum_i \frac{\partial x'^i}{\partial X^k} \frac{\partial x'^i}{\partial X^l} = \frac{\partial x^m}{\partial X^k} \frac{\partial x^n}{\partial X^l} \sum_i \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^i}{\partial x^n} \quad , \qquad \text{C.3a,b}$$

which does not have the form of Eq. C.2. Though $\overline{\mathbf{B}}$ represents a mathematical tensor in the reference space with coordinates X^i**,** in view of Eq. C.3a, this does not have direct physical significance. Though often called a tensor in the literature of continuum mechanics, it is not a tensor in the sense commonly used in physics.

The deformation matrix $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, or

$$B^{ij} = \sum_k \frac{\partial x^i}{\partial X^k} \frac{\partial x^j}{\partial X^k} \quad , \qquad \text{C.4}$$

does transform as a physical tensor, for application of the chain rule shows that it transforms as indicated by Eq. C.1. It is no coincidence that **B** is said to be "frame indifferent" (Truesdell and Toupin 1960, Truesdell 1966) while $\overline{\mathbf{B}}$ (Truesdell's **C**) is not, for the same sorts of transformations are involved in specifying frame indifference as for showing tensor invariance in the sense used in physics.

Let us consider now the Cauchy deformation

$$c_{km} = \sum_K \frac{\partial X^K}{\partial x^k} \frac{\partial X^K}{\partial x^m} \quad . \tag{C.5}$$

This metric appears when the euclidian element of arc in reference space

$$dS^2 = \sum_K dX^k dX^k \tag{C.6}$$

is expressed in physical space as

$$dS^2 = c_{km} dx^k dx^m \quad . \tag{C.7}$$

Application of the chain rule shows that in the x'^m coordinate system

$$c'_{ml} = c_{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^l} \quad , \tag{C.8}$$

which is a transformation of the form indicated by Eq. C.2. Thus, **c** is a covariant physical tensor.

It appears to be useful in tensor analysis in many instances to adopt matrix notation. Let us introduce here the transformation matrix

$$\boldsymbol{\gamma} = \left(\gamma^i_{\bullet j}\right) = \left(\frac{\partial x^i}{\partial x'^j}\right) \quad . \tag{C.9}$$

Then, the contravariant tensor property indicated in Eq. C.1 can be written

$$\mathbf{A}' = \boldsymbol{\gamma}^{-1} \mathbf{A} \boldsymbol{\gamma}^{-T} \quad , \tag{C.10}$$

and the covariant tensor property indicated in Eq. C.2 can be written

$$\mathbf{A}' = \boldsymbol{\gamma}^T \mathbf{A} \boldsymbol{\gamma} \quad . \tag{C.11}$$

Then, if **A** is a tensor (contravariant or covariant), it follows that its inverse,

$$(\mathbf{A}')^{-1} = \boldsymbol{\gamma}^{-1} \mathbf{A}^{-1} \boldsymbol{\gamma}^{-T} \quad , \tag{C.12}$$

is also a tensor (covariant or contravariant). Now, in matrix notion, Eq. C.5 is

$$\mathbf{c} = \mathbf{F}^{-T} \mathbf{F}^{-1} = \mathbf{B}^{-1} \quad . \tag{C.13}$$

Thus, if **B** is a contravariant physical tensor, then **c** is a covariant physical tensor, as concluded above on the basis of a longer calculation.

These results can be summarized in tabular form as indicated in Table C-1, illustrating three ways to characterize the element of arc in continuum mechanics and the corresponding three measures of stretch and strain.

In the current view of the theory of deformation, quantities that transform like tensors only in the mathematical sense are called "metrics." The term "tensor" is then reserved for quantities that transform in this way in physical space, those called "physical tensors" above.

Table C-1. Comparison of Various Arc Meaures, Metrics, and Strains

Rotated reference coordinates (polar axes)	Euclidian physical space with reference coordinates	Euclidian space of reference coordinates
$d\sigma^2 = B_{kl} d\bar{X}^k d\bar{X}^l$	$ds^2 = dx^k dx^k = \bar{B}_{KM} dX^K dX^L$	$dS^2 = dX^I dX^I = c_{km} dx^k dx^m$
Signorini strain tensor	Green-St. Venant metric	Almansi strain tensor
$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{B} - \boldsymbol{I})$	$\bar{\boldsymbol{\varepsilon}} = \frac{1}{2}(\bar{\boldsymbol{B}} - \boldsymbol{I}) = \boldsymbol{E}$	$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{c})$
$\mathbf{d\bar{X}} = \mathbf{RdX}$	$\bar{\boldsymbol{\varepsilon}} = \mathbf{R}^T \boldsymbol{\varepsilon} \mathbf{R}$	

The first column of Table C-1 characterizes an element of arc in the material as it deforms, with $\mathbf{d\bar{X}}$ selected as the differential element to eliminate the rotational part of the deformation. Note that the rotation indicated in the last row of the first column rotates the *reference* axes so as to eliminate the rotation from the deformation. This is the inverse of the rotation indicated by Eq. 3.9. The metric in reference axes is noneuclidian. The arcs in such a space interpolate between the atoms characterizing the material. Integrating $d\sigma$ along the arc C_0 in the rotated reference axes results in the length of the deformed arc C. To show this, put $\mathbf{d\bar{X}} = \mathbf{R}\,\mathbf{dX} = \mathbf{V}^{-1}\mathbf{F}\,\mathbf{dX} = \mathbf{V}^{-1}\mathbf{dx}$ (using Eqs. 3.9 and 2.2), and note that **B** cancels the two occurrences of $\mathbf{V}^{-1}$in the expression for $d\sigma^2$. Thus, the length of the arc is that of the map of C_0 in the current Euclidean space. The metric **B** is a physical tensor related to the stretching **D** by Eq. 3.14. The Cauchy stress for an isotropic elastic material can be expressed in terms of **B,** as indicated by Eq. 4.25.

In the second column, the element of arc of euclidean physical space is defined, as well as its characterization in the reference space involving the initial coordinates. Note that $d\bar{x}_i\ d\bar{x}_i = dx_i\ dx_i$, as discussed following Eq. A44. The elements of arc ds and $d\sigma$ are actually the same, but the representation is different. The metric $\bar{\mathbf{B}}$ is not a physical tensor, as indicated in the discussion of Eq. C.5. The polar elastic stress for isotropic materials can be expressed in terms of $\bar{\mathbf{B}}$, as indicated by Eq. 4.24. For anisotropic materials, it is necessary to use the more basic formulation of Eq. 4.12.

The Euclidian element of arc in reference space is given in the third column. The metric **c** transforms like a physical tensor, but its rate is not related to the stretching in any simple way like Eq. 3.14. For small deformation, all three strains are equivalent. The three strains in the table are closely related, so that any one determines the other two, though in the case of the Green-St. Venant strain it is necessary to use the rotation **R** to make the connection, viz.,

$$\boldsymbol{E} = \bar{\boldsymbol{\varepsilon}} = \boldsymbol{R}^T \boldsymbol{\varepsilon} \boldsymbol{R} \quad . \qquad \text{C.14}$$

There is a fourth measure of arc, as pointed out to me by W. Cook, which seems to have not aroused any previous interest. It arises from the lack of symmetry in the three strains of Table C-1, which can conceptually be considered as part of a 2 x 2 array whose elements are spatial, and rotated axes and spatial or reference axes. We can define by analogy with the third column of the table

$$\bar{\mathbf{e}} = \mathbf{R}^T \mathbf{e} \mathbf{R} = \frac{1}{2}\left(\mathbf{I} - \bar{\mathbf{B}}^{-1}\right) \quad , \qquad \text{C.15}$$

with

$$\bar{\mathbf{B}}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T} = \mathbf{b} = \sum_j \left(\frac{\partial X_i}{\partial x_j} \frac{\partial X_k}{\partial x_j} \right) \quad . \qquad \text{C.16}$$

The associated element of arc is

$$d\,\Sigma^2 = b_{ik} dX_i dX_k \ , \qquad \text{C.17}$$

representing a Riemannian geometry with coordinates X_i and metric b_{ik}. However, **b** does not transform like a physical tensor.

Consider now the velocity gradient

$$\mathbf{G} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \left(\frac{\partial \dot{x}^i}{\partial X^j} \right) \left(\frac{\partial X^j}{\partial x^k} \right) \quad . \qquad \text{C.18}$$

Analysis of the metric properties of **G** requires considerations from tensor analysis, rather than simply the tensor algebra used in the analysis of strain. The covariant derivative of the velocity vector is

$$u^i_{;j} = \frac{\partial u^i}{\partial x^j} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} u^k \quad , \qquad \text{C.19}$$

where

$$\left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} = g^{il} [jk, l] \qquad \text{C.20}$$

is the Christoffel symbol of the second kind, and

$$[ik, l] = \frac{1}{2}\left(g_{il,k} + g_{kl,i} - g_{ik,l} \right) \qquad \text{C.21}$$

is the Christoffel symbol of the first kind. Here, the g_{ij} are the components of the metric tensor of the space in which the velocity is imbedded, and commas denote partial differentiation. Now, the X_i and t are independent variables, so one can write

$$\frac{\partial u^i}{\partial x^j} = \left(\frac{\partial}{\partial t} \frac{\partial x^i}{\partial X^k} \right) \frac{\partial X^k}{\partial x^j} \quad . \qquad \text{C.22}$$

Let us consider the effect of a coordinate transformation from coordinates x'^k, so that $x^j = f^j\left(x'^k\right)$ but there is not explicit dependence on t. Then, from

$$u^i = \frac{\partial x^i}{\partial x'^k} u'^k \quad ,$$

it follows that

$$\frac{\partial u^i}{\partial x^j} = \frac{\partial^2 x^i}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^j} u'^n + \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^j} \frac{\partial u'^m}{\partial x'^n} \quad . \qquad \text{C.23}$$

Now, it is shown in texts on tensor analysis, such as Eisenhart (1947), that

$$\frac{\partial^2 x^i}{\partial x'^m \partial x'^n} + \begin{Bmatrix} i \\ l\ k \end{Bmatrix} \frac{\partial x^l}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} = \begin{Bmatrix} k \\ m\ n \end{Bmatrix}' \frac{\partial x^i}{\partial x'^k} \quad , \qquad \text{C.24}$$

where $\begin{Bmatrix} k \\ m\ n \end{Bmatrix}'$ denotes the Christoffel symbol for the coordinate system x_i'. Then, direct calculation shows that

$$u^i_{;j} = \left(\begin{Bmatrix} n \\ m\ \ l \end{Bmatrix}' u'^l + \frac{\partial u'^n}{\partial x'^m} \right) \frac{\partial x^i}{\partial x'^n} \frac{\partial x'^m}{\partial x^j} \quad ,$$

which can be written

$$u^i_{;j} = u'^n_{;m} \frac{\partial x^i}{\partial x'^n} \frac{\partial x'^m}{\partial x^j} \quad . \qquad \text{C.25}$$

This demonstrates that $u^l_{;j}$ is a mixed tensor. It is useful to know that it is a tensor, but its mixed character precludes a separation into symmetric and antisymmetric parts. However, a covariant tensor

$$G_{mj} = g_{mi} u^i_{;j} \qquad \text{C.26}$$

can be formed from the mixed tensor. It can be readily shown that G_{mj} transforms like a covariant tensor, and it can be separated into symmetric and antisymmetric parts as indicated in Eq. 3.3. It follows that both **D** and **W** are covariant tensors.

The tensor character of strain, compelling it to satisfy a transformation of the type

$$E'_{ij} = E_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \quad ,$$

is also obligatory for stress. Tensor properties of geometric quantities such as strain can be examined by direct calculation, but the tensor character of stress must be stated as a physical principle. In special cases where the stress is given by a constitutive law involving strain, it should be possible to verify its tensor character by direct calculation. The tensor character of stress, however, arises from physical necessity rather than direct calculation. The underlying basis for this is that physical relations must hold in any coordinate system. For example, since the first law of thermodynamics,

$$\dot{I} = V \sigma^{ij} d_{ij} + \dot{Q} \quad ,$$

is a scalar equation, σ^{ij} and d_{ij} must both satisfy tensor transformations given above with σ^{ij} contravariant and d_{ij} covariant. Here, I denotes internal energy, V; specific volume; and Q, heat supplied to the system.

Formulating the constitutive laws with physical tensor quantities ensures physical validity and that the formulation is valid in various nonrotating coordinate systems. The laws would not be expected to hold in rotating systems since, then, centrifugal force would play a role. Note that energy conservation coupled with Galilean invariance ensures momentum conservation, as shown by Dienes (1978b). This illustrates the importance of invariance (which is not essentially different from Truesdell's "indifference"). On the other hand, it is useful to have constitutive laws fixed in the material, insofar as possible, so polar axes may be appropriate for certain purposes. This discussion has not included constitutive laws, but clearly they should be written with all quantities defined in the same coordinate system. This is assured if all quantities are physical tensors. It may be useful for computational purposes to express the constitutive relations in polar axes as an alternative, in order to eliminate rotation, but the constitutive laws should be valid in fixed axes.

APPENDIX D
REMARKS ON ELASTIC CONSTITUTIVE LAWS

Truesdell (1952) gives various expressions obtained by Boussinesq, Kelvin, Cosserat, Neumann, and Kirchhoff relating stress and deformation for an elastic material. Equation 4.12 is equivalent to those results. I argue here that $\overline{\mathbf{B}}$ (Truesdell's **C**) is not a physical tensor, though it is an adequate measure of stretch. Nor is $\overline{\mathbf{B}}$ frame indifferent. In numerical simulation, where the velocity and, hence, **D** and $\overline{\mathbf{D}}$, are given, Eq. 3.18

$$\dot{\overline{\mathbf{B}}} = 2\overline{\mathbf{V}}\overline{\mathbf{D}}\overline{\mathbf{V}}$$

(which does not involve the rate of rotation) is useful to update $\overline{\mathbf{B}}$, but then calculations must be made in polar axes. Alternatively, $\overline{\mathbf{B}}$ can be obtained by computing $\mathbf{FF}^T$ and using Eq. 3.10.

Another way of viewing an essential difficulty in some formulations is to consider the dangerous formulation " $\boldsymbol{\sigma}$ = f(**F**) " in the special (distortion free) case when **V** = **I**. Then " $\boldsymbol{\sigma}$ = f(**R**) ," which is clearly unphysical since the left side is the stress tensor and the right side is not a tensor. Of course, if **R** drops out because f involves the argument $\mathbf{FF}^T$, then the formulation works out physically, but this is an additional consideration.

In the expression of Neumann and Kirchoff involving the potential *f*, and in Eqs. 39.6 and 39.7 of Truesdell (1952), the stress is given by

$$T_i^{\alpha} = \frac{\partial f}{\partial x_{\alpha}^{i}} \quad ,$$

with the components x_i^a of **F** taken as the independent variables; but, since there are nine and the elastic potential involves only six strains. This seems confusing and unnatural. It is as though the potential depended on rotation! It is formally possible if the $\overline{\mathbf{B}}$ are written out in terms of the elements of **F** as indicated above, but this is not usually done. On the other hand, in Section 4, the potential is considered to depend only on the 6 components of strain.

If the stress of Eq. 4.12 is put into space axes by application of Eq. 3.14, then the spatial (Cauchy) stress, which can be expressed as

$$\boldsymbol{\sigma} = \frac{\rho}{\rho^o}\mathbf{F}\overline{\boldsymbol{\Sigma}}\mathbf{F}^T \quad , \quad \overline{\Sigma}_{ij} = \frac{\partial f}{\partial \bar{\varepsilon}_{ij}} \quad , \qquad \text{D.1}$$

is equivalent to the second expression given by Truesdell (1952) in his Eq. 39.2 as Boussinesq's form of the stress-strain relation. Rivlin and Ericson (1955) write an expression equivalent to Eq. 4.24, except that they use **C** to denote what is called **B** here (and by Truesdell); also, their I_2 (Eq. 1.2) is $-I_2'$, as given in Eq. 4.15. A factor of 1/3 seems to be omitted in their expression for I_2, when the expression for I_2 is compared with Eq. 4.15. The definition of $I_2{}'$ given here as Eq. 4.15 has the advantage that its gradient is the deviatoric metric $\overline{B}_{ij}'$ defined by Eq. 4.20,

$$\frac{\partial I_2'}{\partial \overline{B}_{ij}} = \overline{B}_{ij}' \quad , \qquad \text{D.2}$$

which is convenient to remember and arises often in theoretical calculations.

APPENDIX E
POLAR RATE OF ROTATION

An alternative expression for the difference **H** of the rate of polar rotation $\boldsymbol{\Omega}$ and the vorticity **W**, defined in Eq. 3.29, is

$$H_{jk} = \sum_{r,t} \frac{\lambda_t - \lambda_r}{\lambda_t - \lambda_r} \tilde{D}_{rt} T_{jr} T_{kt} \quad , \qquad \text{E.1}$$

where

$$\tilde{D}_{rt} = T_{qr} D_{qs} T_{st} \qquad \text{E.2}$$

is the stretching in the principal axes of stretch. This expression for **H** was derived by Nemat-Nasser (1983) (who denotes the left side of Eq. E.1 by ε_{jk}) using the formalism developed by Hill (1978). Here, **D** is the stretching defined by Eq. 3.3, and **T** is the matrix of eigenvectors of **V** given by Eq. 2.4. The purpose of this appendix is to demonstrate that the result given above as Eq. E.1 is equivalent to that given as Eq. 3.35, and originally derived by Dienes (1978). This is considered to be of some interest since Nemat-Nasser and others have been critical of the use of polar stress rate, given here as Eq. 9.2. In essence, their argument seems to be that the ZJN stress rate (see Section 9) is adequate, and (argues Nemat-Nasser) that I introduced the polar stress rate as an *ad hoc* device to eliminate certain oscillations observed by Nagtegaal and de Jong (1981) in kinematic-hardening calculations. Nemat-Nasser takes the position that the difference in stress rates $\boldsymbol{\sigma}\mathbf{H} - \mathbf{H}\boldsymbol{\sigma}$, with **H** expressed above as E. 1, can be added to the right-hand side of an arbitrary constitutive law of the form

$$\hat{\boldsymbol{\sigma}} = f(\boldsymbol{\sigma}, \mathbf{D}) \quad , \qquad \text{E.3}$$

which, thereby, becomes

$$\check{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W} = f'(\boldsymbol{\sigma}, \mathbf{D}) = f + \mathbf{H}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{H} \quad . \qquad \text{E.4}$$

Since the right side is said to be arbitrary, he maintains that it makes no difference whether one uses $\hat{\sigma}$ or $\check{\sigma}$. However, in the calculation of *specific* plastic and viscoelastic flows, the function $f(\boldsymbol{\sigma}, \mathbf{D})$ is presumed known. Changing the function changes the material. Thus, subtracting

$$\hat{\boldsymbol{\sigma}} - \check{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\mathbf{H} - \mathbf{H}\boldsymbol{\sigma} \qquad \text{E.5}$$

from $f(\boldsymbol{\sigma}, \mathbf{D})$ results in a different flow rule. Furthermore, the expression in E.5 has no physical meaning (in the sense of representing material response), though it influences the motion and, consequently, should not be arbitrarily added to E.3.

It seems that the ZJN stress rate has become so entrenched in the literature that its validity cannot be questioned without vigorous reaction. One should recall, however, that it is usually introduced in an ad hoc manner, having the property that it is invariant under a change of frame. It is not unique, however, in this respect! It is preferable to some other stress rates, since it leaves the second invariant of stress unchanged, as discussed by Prager (1961). However, the combination of frame indifference, together with this invariance argument, forms an unwieldy and unnatural foundation for the selection of stress rate. The result is not unique and, as shown in Section 3, it is only approximate.

Noll actually derives an expression for stress rate. A similar procedure is followed in Section 9, except that we do not assume $\mathbf{V} = \mathbf{I}$, as does Noll. The choice $\mathbf{V} = \mathbf{I}$ leads to the ZJN stress rate. It is not sensible, however, to assume that $\mathbf{V} = \mathbf{I}$ for all time! Hence, it is not true in general that $\Omega = \mathbf{W}$, nor that the stress rate is given by

$$\overset{\vee}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W} \quad .$$

Hill (1978) supports the use of ZJN stress rate (Jaumann flux). In deriving his Eq. 1.20, however, he assumes that $\mathbf{A} = \mathbf{B} = \mathbf{Q}$ and $\dot{\boldsymbol{Q}} = \boldsymbol{W}\boldsymbol{Q}$, with $\mathbf{Q}$ orthogonal. Thus, the ZJN stress rate is proven valid only when the axes are orthogonal. His discussion of objective rates of stress in his Section 3 follows the same line. He states, “It is straightforward to calculate the representation of T on a fixed background with which the Lagrangian triad is momentarily coincident.” Whereas his statements and views are quite correct, they are not what is wanted in calculations in which the initial conditions are given and it is necessary to follow large deformation to late times with numerical calculation. In such calculations, the reference axes are fixed, and it would not be useful to keep changing them. The difference in these stress rates (ZJN and polar) is not important in flows with small deformation. However, when deformation is large, as in the analysis of shear bands and other microstructural instabilities, and in rubber, many solid rocket propellants, other viscoelastic materials, impact problems, geological flows, and a variety of other large deformation problems, the difference can be important. In plastic flow with kinematic hardening, the difference arising from different stress rates can be large, and in materials undergoing fragmentation or shear banding, the material shear and rotation can be enormous.

To demonstrate that Eq. E.1 follows from Eq. 3.31, it is convenient to define a matrix

$$\mathbf{Y} = \mathbf{T}^T\mathbf{S}\mathbf{T} \quad , \qquad \text{E.6}$$

where

$$\mathbf{S} = \left(\mathbf{I} \;\; tr\mathbf{V} - \mathbf{V}\right)^{-1} \quad . \qquad \text{E.7}$$

In view of Eq. 2.4, $\mathbf{Y}$ is diagonal, and can be written explicitly

$$\mathbf{Y} = \begin{pmatrix} (\lambda_2+\lambda_3)^{-1} & 0 & 0 \\ 0 & (\lambda_1+\lambda_3)^{-1} & 0 \\ 0 & 0 & (\lambda_1+\lambda_2)^{-1} \end{pmatrix} \quad . \qquad \text{E.8}$$

It is also convenient to define

$$Q_{jkpq} = S_{il}e_{jik}e_{plq} \quad . \qquad \text{E.9}$$

Now, Eq. 3.29 can be written

$$\mathbf{h} = \mathbf{S}\mathbf{z} \quad . \qquad \text{E.10}$$

Then, using Eq. E.9, the difference **h** can be expressed in matrix form as

$$H_{ik} = \frac{1}{2} Q_{ikpq} Z_{pq} \quad . \qquad \text{E.11}$$

It is also convenient to define

$$B_{ijk} = T_{li} e_{jlk} \quad . \qquad \text{E.12}$$

Then, on combining Eqs. E.6, E.7, E.9, and E.12, we find that

$$Q_{jkpq} = \sum_{i,l} B_{ijk} B_{lpq} Y_{il} \quad . \qquad \text{E.13}$$

Now, using Eq. 2.4, **Z** can be written

$$Z_{pq} = \sum_{r,t} T_{pr} \tilde{D}_{rt} \left(\lambda_t - \lambda_r \right) T_{qt} \quad . \qquad \text{E.14}$$

When substituted into Eq. E.11, a lengthy expression for **H** is obtained containing a group of terms that simplifies to the permutation symbol, viz.,

$$B_{spq} T_{pr} T_{qt} = e_{plq} T_{pr} T_{ls} T_{qt} = e_{rst} \quad . \qquad \text{E.15}$$

This formula expresses the expansion of the determinant of **T** when r = 1, s = 2, t = 3, which is 1. (When the rows of the determinant are interchanged, its sign is changed, as indicated by the permutation symbol e_{rst}. If any two of r, s, t are the same, then the determinant has two rows the same, and is zero). With this result, Eq. E.11 reduces to

$$H_{jk} = \frac{1}{2} \sum_{i} B_{ijk} Y_i h_i \quad , \qquad \text{E.16}$$

where

$$h_i = \sum_{r,t} e_{irt} \left(\lambda_r - \lambda_t \right) \tilde{D}_{rt} \quad . \qquad \text{E.17}$$

In Eq. E.16, we express the diagonal terms of **Y** as Y_i, so that

$$\mathbf{Y} = \left(Y_{il} \right) = \left(Y_i \delta_{il} \right) \quad . \qquad \text{E.18}$$

The expression in E.16 can be brought into agreement with E.1 by noting that

$$T_{in} e_{ijk} = T_{jl} T_{km} e_{lmn} \quad . \qquad \text{E.19}$$

With this relation, it can be shown that

$$H_{jk} = \frac{1}{2} \sum_{r,t} \tilde{D}_{rt} \left(\lambda_t - \lambda_r \right) T_{jl} T_{km} \sum_{n} e_{lmn} Y_n e_{nrt} \quad . \qquad \text{E.20}$$

The sum over n can be simplified by means of Eq. A.29, whence

$$\sum \mathrm{Y_n e_{lmn} e_{nrt} = Y_{ii} \left(\delta_{lr} \delta_{mt} - \delta_{lt} \delta_{mr} \right) + \left(Y_t + Y_r \right) \left(\delta_{lt} \delta_{mr} - \delta_{lr} \delta_{mt} \right)} \quad . \qquad \text{E.21}$$

Substitution into Eq. E.20 leads to

$$H_{jk} = \frac{1}{2}\sum_{r,t}\left(\lambda_t - \lambda_r\right)\tilde{D}_{rt}\left(T_{jr}T_{kt} - T_{jt}T_{kr}\right)\left(Y_{ii} - Y_t - Y_r\right) \quad . \qquad \text{E.22}$$

It can be verified by detailed enumeration that, as a result of the definition of **Y** given as Eq. E.8, the last term on the right can be replaced by $1/\left(\lambda_r + \lambda_t\right)$, and that the two terms in the middle parentheses make equal contributions to the sum. This completes the demonstration that Nemat-Nasser's result, given as Eq. E.1, is equivalent to Dienes's Eq. 3.29, which is repeated in this appendix as Eq. E.10.

REFERENCES

Autonne, M.L., 1902, "Sur les groupes lineaires, reels, et orthogonaux," *30*, pp. 121–130, Bull. De la Soc. Math. de France.

Bellman, R., 1960, *Introduction to Matrix Analysis*, McGraw-Hill.

Cook, W., Personal communication, 1982.

Dienes, J.K., 1978a, "On the Analysis of Rotation and Stress Rate in Deforming Bodies," Acta Mechanica *32*, 217–232.

Dienes, J.K., 1978b, "The Method of Generalized Coordinates and an Application to Rayleigh-Taylor Instability," Phys. Fluids *21*, 5, pp. 736–744.

Dienes, J.K., 1989, "Theory of Deformation," Part II, Physical theory, LA-11063-MS, Vol. II, September 1989.

Dienes, J.K., 1996, "A Unified Theory of Flow, Hot Spots, and Fragmentation with an Application to Explosive Sensitvity," in *High-Pressure Shock Compression of Solids II*, Springer-Verlag New York, Inc., Lee Davison et al., eds.

Dienes, J.K. and J.C. Solem, 1999, "Nonlinear behavior of some hydrostatically stressed isotropic elastomeric foams," Acta Mechanica *138*, 155–162.

Drucker, D.C. and W. Prager, 1952, "Soil mechanics and plastic analysis or limit design," Quarterly of Applied Mechanics, *10*, 157–175.

Eringen, A.D., 1967, *Mechanics of Continua*, Wiley, Theorem 2, p. 81.

Eisenhart, L.P., 1947, *An Introduction to Differential Geometry*, Princeton University Press.

Flugge, W., 1972, *Tensor Analysis and Continuum Mechanics*, Springer-Verlag.

Gantmacher, F.R., 1959, *The Theory of Matrices*, Chelsea.

Goldstein, H., 1959, *Classical Mechanics*, Addison-Wesley.

Green, G., 1839, "On the Laws of Reflexion and Refraction of Light at the Common Surface of Two Non-Crystallized Media," Trans. Cambr. Phil. Soc. *7*, 245–269.

Hill, R., 1950, *The Mathematical Theory of Plasticity*, Oxford.

Hill, R., 1978, "Aspects of Invariance in Solid Mechanics," in *Advances in Solid Mechanics, 18*.

Jeans, J., 1940, *An Introduction to the Kinetic Theory of Gases*, Cambridge.

Jaumann, G., 1911, "Geschlossenes System physikalischer und chemischer Differentialgesetze," Sitzber. Akad. Wiss. Wien (IIA), *120*, 385–530.

Kachanov, M., 1984, "Elastic Solids with Many Cracks and Related Problems," in *Advances in Applied Mechanics*, Academic Press.

Landau, L., and E. Lifschitz, 1986, *Theory of Elasticity*, Third Edition.

Mooney, M., 1940, "A Theory of Large Elastic Deformation," J. Appl. Phys. *2*, 582.

Nagtegaal, J.C. and J.E. de Jong, 1981, "Some Aspects of Non-Isotropic Work Hardening in Finite Strain Plasticity," Proc. Workshop on Plasticity of Metals at Finite Strain, E.H. Lee, and R.L. Mallet, eds., Stanford Univ., July 29–Aug. 1.

Nemat-Nasser, S., 1983, "On Finite Plastic Flow of Crystalline Solids and Geomaterials," J. Appl. Mech. *50*, 1114–1126.

Noll, W., 1955, "On the Continuity of the Solid and Fluid States," J. Rat. Mech. Anal. *4*, 3–1.

Prager, W., 1961, "An Elementary Discussion of Definitions of Stress Rate," Quart. Appl. Math. *18*, 403–407.

Reuss, A., 1931, "Beruchsichtigung der elastischen Formanderung in der Plastizitatstheorie," Zeits. Ang. Math. Mech., 10, pp. 266–269.

Rivlin, R.S. and Ericson, J.L., 1955, "Stress-Deformation Relations for Isotropic Materials," J. Rat. Mech. Anal., *4*, 3, 323–425. Also reprinted in *Continuum Mechanics II, The Rational Mechanics of Materials*, C. Truesdell, ed., Gordon and Breach, 1965.

Sandler, I. and M. Baron, 1985, "Numerical Models for Dynamic Loading," in *Mechanics of Geomaterials*, Z.P. Bazant, ed., John Wiley and Sons.

Seaman, L., D.A. Shockey, and D.R. Curran, 1972, "The growth law for crack propagation under shock conditions," in *Dynamic crack propagation, Proceedings of an international conference*," G.C. Sih, ed., Lehigh University, Bethlehem, PA, July 10–12, 1972, pp. 629–647.

Signorini, A., 1931, "Sulle deformazioi termoelastische finite," *Third International Congress for Applied Mechanics*, *Proceedings*, Vol. 2, Pt. 2, pp. 80–89, August 24–29, 1930, Stockholm, Kungl, Boktryckeriet, P.A., Norstedt & Soener.

Sokolnikoff, I.S. and R.D. Specht, 1946, *Mathematical Theory of Elasticity*, McGraw-Hill.

Spitzig , W.A. and O. Richmond, 1984, "The Effect of Pressure on the Flow Stress of Metals," Acta Metall., 32, 3, pp. 457–463.

Truesdell, C., 1952, "The Mechanical Foundations of Elasticity and Fluid Dynamics," J. Rat. Mech. Anal., *1*, 125–300.

Truesdell, C., 1954, *The Kinematics of Vorticity*, Indiana University Press.

Truesdell, C. and Toupin, R.A., 1960, "The Classical Field Theories," in *Handbuch der Physik*, Vol. III.

Truesdell, C., 1966, *The Elements of Continuum Mechanics*, Springer-Verlag.

Zaremba, S., 1903, "Sur une forme perfectionnée de la théorie de la relaxation," Bull. Int. Acad. Sci. Cracovie No. 8, 594–616.

Zhurkov, S.N. and V.S. Kuksenko, 1975, "The Micromechanics of Polymer Fracture," Int. J. Fracture, *11*, No. 4.

Zuo, K., Personal communication, 2001.

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