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POWER FUNCTIONS RELATING EXCRETION

TO BODY BURDEN

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DATA RECORD DISTRIBUTION

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Abstract – Formulae necessary to relate the quantity

of radionuclides excreted to that assimilated in

exposures that are acute and those that are multiple

or continuous are derived from power function re-

lationships. Particular attention is given to

providing equations having variables for which the

bio-assayer can easily derive numerical values.

Introduction

Evidence accumulated since 1946 indicates that the decrease in body

retention of many bone-seeking elements, when administered as sol-

uble compounds readily accessible to circulating fluids, may be

better described in terms of a power rather than an exponential

function. In laboratory animals, this was found to be true for

calcium[1], strontium[2,5], and for radium[1,6,7,8,9]. In humans, Norris

et al[7] have shown that all available data from studies of radium

metabolism over a 25-year period following administration can be

better fitted by power functions. Stehney and Lucas[10] used the

power function to describe the increase in the radium burden of

human subjects at the natural level of dietary intake. Langham

et al presented data to show that retention and excretion of plutoni-

um by man over a 5-year period can also be described in this

manner[11,12,13].

A case in which soluble plutonium was accidently assimilated through

a wound by a normal healthy individual at the Savannah River Plant

has been followed for over four years[14]. Here the decrease in the

urinary excretion rate could also be best described in terms of a power function. Similarly, Bernard and Struxness[15] have found it applicable to the description of uranium retention by humans.

Admittedly, the application of the power function to the description of the metabolism of bone-seeking elements is based on experimental observations and is of little value in explaining the mechanisms involved. However, it is able to describe the fraction of certain radionuclides excreted per day in simple mathematical fashion over extended periods of time. For this reason the revised report of the Committee on Permissible Dose for Internal Radiation[16] includes an appendix on the calculation of the maximum permissible concentration in air and water based on a power function model.

Now bio-assayers are expected to interpret the dose to an individual due to radiation within the body from the quantity of radioactivity excreted. These interpretations should be based on a power function in cases where the retention of the assimilated radionuclide is known to be best represented by such a function. Formulae having the same parameters used in the appendix of the revised report[16] are derived in this report for use in making these interpretations.

Relation of Quantity Excreted to That

Assimilated in Acute Exposures

In experimental observations where the power function has been found to describe retention of bone-seeking elements, the basic equation of retention (neglecting radioactive decay) is expressed as the power function

$$R(t) = At^{-n} \qquad t \geq A^{1/n} \qquad 0 < n < 1 \qquad (1)$$

where $R(t)$ is the fraction of assimilated radioactive material retained t days postassimilation, A is a constant which is equal to the fraction of the assimilated dose retained when t is equal to one day, and $-n$ is a constant equal to the slope of the linear log-log transform of the retention function. It should be noted that since this slope is negative, the power function increases without bound as t approaches zero. Therefore it has no meaning between the time of assimilation ($t = 0$) and some minimum value of t ($t = x$) where the fraction retained equals unity. If $R(x) = 1$, then $Ax^{-n} = 1$ and $x = A^{1/n}$. From equation (1) the accompanying instantaneous rate of excretion is obtained by differentiating

$$\frac{dR(t)}{dt} = -Ant^{-(n + 1)} . \qquad (2)$$

Substituting equation (1), the basic equation of retention, in equation (2) gives

$$\frac{dR(t)}{dt} = - \frac{nR(t)}{t} . \qquad (3)$$

The expression is negative since the fraction retained must always decrease. To the rate of loss by excretion may be added the rate of loss by radioactive decay to obtain the rate at which the assimilated material is eliminated from the body should the half life be short. This gives

$$\frac{dR(t)}{dt} = -\frac{nR(t)}{t} - \lambda_r R(t) \tag{4}$$

where λ_r is the radiological decay constant. The fraction of the assimilated radioactive material retained, $R(t)$, when the radioactive decay is appreciable* can be obtained by integrating equation (4) giving

$$R(t) = At^{-n} \; e^{-\lambda_r(t - 1)} \text{, when } t > 1 . \tag{5}$$

$$\boxed{\text{See Appendix I}}$$

However, of interest to the bio-assayer is that fraction excreted. Thus the fractional excretion rate at time t, $Y(t)$, expressed as the fraction excreted per unit time, is the rate of elimination from the body at time t less the rate of loss by radioactive decay or

$$Y(t) = -\frac{dR(t)}{dt} - \lambda_r R(t) . \tag{6}$$

Substituting equation (4) and (5) in equation (6) gives

$$Y(t) = nAt^{-(n + 1)}e^{-\lambda_r(t - 1)} \tag{7}$$

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* If the radiological half life, $T_r \left(\frac{.693}{\lambda_r}\right)$, is $\geq 6.6t$, $0.9 < e^{-\lambda_r t} < 1.0$, and if $T_r \geq 66t$, $0.99 < e^{-\lambda_r t} < 1.00$.

The excretion rate at time t, $E(t)$, is equal to the product of the fractional excretion rate at time t and the quantity assimilated $q(o)$, or

$$E(t) = q(o) \, Y(t) \, . \tag{8}$$

Substituting equation (7) in equation (8) and solving for $q(o)$ gives

$$q(o) = \frac{E(t) \, t^{n+1}}{nAe^{-\lambda_r(t-1)}} \, . \tag{9}$$

It should be remembered that $\frac{dR(t)}{dt}$ in equation (6) is the instantaneous rate of elimination, and as such, should be integrated to determine the amount eliminated per day. However, this is unnecessary except at values of t close to the time of assimilation or when the unit of time considered is several weeks instead of one day, since the decrement in retention shortly becomes very small. When t is close to the time of assimilation or the unit of time over which the excreta is collected is long, the fraction of material which will be eliminated from the body during the period of time from t_1 to t_2 will be $R(t_1) - R(t_2)$. The fraction of the material which will be found in the urine and feces during this same period, Y, will be the fraction eliminated less that which has decayed due to radioactivity

$$Y = R(t_1) - R(t_2) - R(t_1)(1 - e^{-\lambda_r(t_2 - t_1)})$$

or

$$Y = R(t_1) \, e^{-\lambda_r(t_2 - t_1)} - R(t_2) \, . \tag{6'}$$

Substituting equation (5) in (6') gives

$$Y = At_1^{-n} \, e^{-\lambda_r(t_1-1)} e^{-\lambda_r(t_2 - t_1)} - At_2^{-n} \, e^{-\lambda_r(t_2 - 1)}$$

or

$$Y = A(t_1^{-n} - t_2^{-n}) \, e^{-\lambda_r(t_2 - 1)} \tag{7'}$$

Substituting equation (7') in equation (8) and solving for q(o) gives

$$q(o) = \frac{E}{A(t_1^{-n} - t_2^{-n})\, e^{-\lambda_r(t_2 - 1)}} \qquad (9')$$

where E is the quantity excreted during this period.

Should the quantity of material in the body at the time a sample was taken, q(t), be desired, it can be calculated by substituting equation (5) and equation (9) in the simple relationship

$$q(t) = q(o)\, R(t) \qquad (10)$$

giving

$$q(t) = \frac{E(t)t}{n} . \qquad (11)$$

Frequently an acute assimilation is discovered as the result of routine sampling, and later confirmed by a resample. The time of assimilation is usually not known and must be calculated before the quantity assimilated, q(o), can be determined. Here equation (9) for the excretion rate of the first sample is

$$E(t_1) = \frac{q(o)\, nAe^{-\lambda_r(t_1 - 1)}}{t_1^{\,n + 1}} \qquad (12)$$

and for the second sample is

$$E(t_2) = \frac{q(o)\, nAe^{-\lambda_r(t_2 - 1)}}{(t_1 + \Delta t)^{n + 1}} \qquad (13)$$

where Δt is $t_2 - t_1$. Assuming that the exponential functions in these two equations are approximately equal, they can be solved simultaneously for t_1 giving $\boxed{\text{See Appendix II}}$

$$t = \frac{\Delta t}{\left(\dfrac{E(t_1)}{E(t_2)}\right)^{\frac{1}{n + 1}} - 1} . \qquad (14)$$

Substituting this in equation (9) gives

$$q(o) = \frac{E(t_1)}{nAe^{-\lambda_r(t_1 - 1)}} \left[\frac{\Delta t}{\left(\frac{E(t_1)}{E(t_2)}\right)^{\frac{1}{n + 1}} - 1} \right]^{n + 1} . \qquad (15)$$

It should be noted here that the exponential term for radioactive decay, $e^{-\lambda_r t}$, in the above equation is essentially unity for plutonium-239 and natural uranium. With radium-226 the value of the exponential term becomes less than unity when t becomes very large; however, it can never be less than 0.98 (the value when t is equal to an average occupational exposure of 50 years), and thus can also be considered unity. With strontium-90, $e^{-\lambda_r t}$ becomes appreciably less than unity for large values of t. For values of t less than four years, however, it may be assumed to be unity without introducing an error greater than ten percent.

Table 1 gives the values which have been recommended for the constants used in the calculation of MPC values for soluble compounds and are applicable in the above equations.

In the above equation the term E(t) refers to the total amount excreted per unit time. This includes material eliminated via the urine, $E_u(t)$, and via the feces, $E_f(t)$. Consequently E(t) must be calculated from either one or both of the above terms before it can be used in these calculations.

Table 1. Values Recommended for Power Function Constants

Nuclide	A	n	x*
$_{38}Sr^{90}$	0.65**	0.35**	0.29
	.95**†	.254**†	.81
$_{88}Ra^{226}$	.54**††	.52**††	.31
$_{94}Pu^{239}$	.99**	.01**	.37
Natural Uranium	.72**	.80**	.66
	0.60‡	0.50‡	0.36

* $x = A^{1/n}$.

** ICRP Sub-Committee on Permissible Dose for
   Internal Radiation .

† Marinelli[21].

†† Norris[7].

‡ Bernard[15].

Generally, data on the rate of urinary elimination are more available
and the total amount excreted is calculated from it. (This may be
done even when data on fecal elimination are available due to the
difficulty in establishing the period of time represented by a
single defecation and the fluctuations in the quantity of radioac-
tivity in successive samples.) The total excretion rate for plutonium
compounds readily accessible to circulating body fluids at time t
can be calculated from the urinary excretion rate alone using the
formula

$$E(t) = 4E_u(t) \, t^{-0.2} , \quad \text{when} \quad t < 1024 . \qquad (16)$$

This can be calculated from data provided by Langham[12] Bernard[15] found that the excretion of uranium takes place largely in the urine. The fraction of the daily excretion from humans injected with soluble compounds which appeared in the feces ranged from 1.1×10^{-3} to 6.6 percent but seldom exceeded one percent. Thus all of the uranium accessible to the circulating body fluids may be considered to be excreted in the urine. In contrast to uranium, the excretion of radium takes place largely in the feces. The average value for the radium excreted in the urine probably lies between 1.5 and 5 percent[17]. There also appears to be no trend of this ratio as a function of time after radium assimilation as in the case of plutonium. The ratio for a single day's sampling may vary several fold from individual to individual and in the same individual. Therefore, inferences about the daily excretion rate based on a single 24-hour urine sample will be, at best, approximations.

Very little systematic study of strontium metabolism in man is available. Clinical investigation by Harrison[18] using nonradioactive strontium and by Laszlo, Spencer, *et al*[19,20] using strontium-85 have resulted in some data which can be used to relate urinary to fecal excretion. For the first five days following intravenous injection of soluble strontium, the main avenue of excretion is via the urine. However, as the urinary excretion rapidly drops, the fecal excretion increases. For times longer than five days, roughly 60 percent of the daily excretion is urinary. About 80 percent of the ingested soluble strontium remains unabsorbed and is eliminated in the feces over a six-day period. After this time roughly 80 percent of the daily excretion is urinary.

It might be explained that the power function formulae used for plutonium excretion rates in the past[11],[12],[13] cannot be used to relate retention to excretion, since their negative exponents are less than unity and the integral of such a function is divergent and has no value. However, using the above constants, the negative differential of the retention equation for plutonium,

$$R(t) = 0.99t^{-0.01} \tag{17}$$

is

$$-\frac{dR}{dt} = 0.0099t^{-1.01} = Y(t) . \tag{18}$$

This is not too different from the equation previously used by Langham[11],

$$Y_{u + f} = 0.0079t^{-0.94} . \tag{19}$$

These two equations may be further compared by taking their ratio

$$\frac{Y(t)}{Y_{u + f}} = \frac{0.0099t^{-1.01}}{0.0079t^{-0.94}} = 1.25t^{-0.07} . \tag{20}$$

The numerical values of this ratio with increasing time are given in table 2.

Table 2. Change in Ratio of Two Plutonium Retention Equations With Time

t, days	$Y(t)/Y_{u + f}$
1	1.25
10	1.07
100	0.91
1000	0.77

Calculation of Integral Absorbed Dose

in an Acute Assimilation

Having estimated the quantity assimilated into the body as the result of a single acute exposure, $q(o)$, it is usually desirable to assess the relative dose the body will receive from the ionizing radiation. It should be noted that the values given in the literature[22,23] for q*, are calculated on the premise that the accumulation of the burden in the critical body organ is due to the exclusive use of air or water contaminated with a fixed concentration of the nuclide for a time that is much longer than the effective half life of the nuclide in the critical body organ, but not greater than 50 or 70 years. These values are not comparable in the case of a single acute assimilation. In assessing these cases, the average dose rate in rem per unit of time over a specified interval of time, *viz.* rem per week for the first week, must be calculated from $q(o)$.

The RBE dose in rem can be defined as the quotient of the integral absorbed RBE dose divided by the mass of either the critical body organ or the whole body, depending on the parameters used. The unit of integral absorbed RBE dose is the gram-rem, which is the product of the integral absorbed dose in gram-rad[24] and an agreed conventional value of the RBE with respect to a particular form of radiation effect. Since radiation dosage accumulates as the product of

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* The maximum permissible body burden of a radioactive nuclide in a

standard man.

concentration of the radioelement and time, the integrated dosage*

over t days, D(t), is

$$D(t) = q(o) \ K \ f(R) \qquad (21)$$

where q(o) is the quantity assimilated in disintegrations per minute.

The quantity f(R) is in every case the appropriate retention func-

tion integrated with respect to t over the period in question.   K

is the radiation dosage rate per day per disintegration per minute.

$$K = \frac{(1440)(1.6 \times 10^{-6}) \ \Sigma E(RBE)N}{100} = 2.3 \times 10^{-5} \ \Sigma E(RBE)N \qquad (22)$$

where

$$1440 = min/day$$

$$1.6 \times 10^{-6} = ergs/Mev$$

$$\Sigma E = \text{effective energy per disintegration in Mev}$$

$$RBE = \text{relative biological effectiveness in rem/rad}$$

$$N = \text{relative damage factor}$$

$$100 = ergs/gram\text{-}rad$$

Substituting equation (5) for the quantity, f(R) in equation (21)

gives

$$D(t) = q(o) \ KA \int_{x}^{t} \tau^{-n} \ e^{-\lambda_r(\tau - 1)} \ d\tau \qquad (23)$$

where x is that fraction of a day when the fraction retained is one

and is numerically equal to $A^{1/n}$. This is the Laplace transform of

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* The integrated dosage should not be confused with the integral

  absorbed dose which is related to the dose in rems as follows:

$$\text{dose in rems} = \frac{\text{integral dose}}{\text{weight of tissue}}$$

the function τ^{-n} and can be solved by converting it to the form

$$D(t) = \frac{q(o) \, KAe^{\lambda_r}}{\lambda_r^{(1-n)}} \left[\rceil(u,p) - \rceil(u',p) \right] \Gamma(1-n) . \quad (24)$$

See Appendix III

Where the arguments to the function $\rceil(u,p)$ are $p = -n, u = \dfrac{\lambda_r t}{\sqrt{1-n}}$,

and $u' = \dfrac{\lambda_r x}{\sqrt{1-n}}$. The numerical values for the function $\rceil(u,p)$ are

given in tables 2 and 3 of "Tables of the Incomplete Γ - Function"[25].

The numerical values for the function $\Gamma(1-n)$ can be computed from

any table of the Γ - function using the relationship

$$\Gamma(1-n) = \frac{\Gamma(2-n)}{1-n} . \quad (25)$$

Should tables of the incomplete Γ - function not be available, $D(t)$

can be calculated from the infinite series

$$D(t) = \frac{q(o) \, KAe^{\lambda_r}}{\lambda_r^{1-n}} \sum_{j=1}^{\infty} \frac{(\lambda_r t)^{j-n} \, e^{-\lambda_r t} - (\lambda_r x)^{j-n} \, e^{-\lambda_r x}}{j-n} . \quad (26)$$

See Appendix III

Where the radiological half life is very long and λ_r is insignificant

$$D(t) = \frac{q(o) \, KA}{1-n} \, (t^{1-n} - x^{1-n}) \quad (26')$$

and can be obtained from either equation (23) or equation (26).

However, should this equation be used to calculate the integral ab-

sorbed dose from strontium-90 over the average occupational exposure

of 50 years, $D(50y)$ would be 175 percent of the true value.

Relation of Quantity Excreted to Rate of Assimilation
in Multiple or Continuous Exposures

In the case of multiple assimilations of dissimilar quantities of a radioactive nuclide or of similar quantities assimilated at long or dissimilar intervals, estimates of retained amounts must be derived by summation of individual items over the appropriate time intervals. However, this may be impossible if, as is often the case, information on the time of the assimilations or the quantity previously assimilated is lacking. Here it is necessary to assume a chronic invariant assimilation. This is a hypothetical case in which an individual assimilates a radioactive nuclide continuously at a fixed rate, q, over an interval of time T. There are instances, however, in which such hypothetical cases are approached, in that radioactive nuclides are made available to the body from its own metabolic pool (*e.g.* the lungs), at very slow rates. However, when repeated assimilations of similar magnitude occur at reasonably short and constant intervals, the resulting picture can be closely approximated by the chronic invariant case.

If the excretion rate resulting from each increment assimilated is described by equation (9),

$$E(t) = q(o) \ nAt^{-(n + 1)} \ e^{-\lambda_r(t - 1)} \ ,$$

the total excretion on day t resulting from a continuous assimilation, $E_c(t)$, can be considered as the sum of the excretion rates from each of the incremental steps or

$$E_c(t) = q \ nA \int_x^T (t - \tau + x)^{-(n + 1)} \ e^{-\lambda_r(t - \tau + x - 1)} \ d\tau \quad (27)$$

where the time following assimilation in equation (9) is given by the difference between the time of excretion t and τ, the time of each incremental assimilation. The constant x, which is the time when the fraction retained is unity, is added here since, according to the previous model, no excretion occurs during times less than x. Thus the values of the time following assimilation will never be less than x, and will approach t - τ asymptotically as t increases. This equation cannot be solved using the present tables of incomplete Γ - functions, but can be expressed as the infinite series

$$E_c(t) = - q \; nAe^{-\lambda_r(t - 1)} \sum_{j = 0}^{\infty} \frac{\lambda_r^j(t - T + x)^{j - n} - t^{j - n}}{(1 - n)(2 - n) \cdots \cdots (j - n)} . \quad (28)$$

$$\boxed{\text{See Appendix IV}}$$

Where the radiological half life is very long and λ_r is insignificant

$$E_c(t) = q \; A \; [(t - T + x)^{-n} - t^{-n}] \quad\quad (29)$$

can be obtained from either equation (27) or equation (28).

Calculation of Quantity Assimilated in
Multiple or Continuous Exposures

Generally, with chronic assimilations the values for q, t, and T are not known. However, if two measurements are made of the quantity of material excreted at time t_1 and t_2, where $t_2 = t_1 + \Delta t$, the total quantity assimilated can be calculated by solving equation (29) for t. In order to do this the term $(t - T + x)^{-n}$ must be expanded. This can be done by considering it as a binomial, $[t - (T - x)]^{-n}$, and expanding it using the binomial series. Since the exponent is not a positive integer this will result in an infinite series which

is approximated here by using only the first two terms. Thus,

$$(t - T + x)^{-n} \cong t^{-n} + n(T - x)\, t^{-(n + 1)} \, . \qquad (30)$$

From the above relationship it can be seen that as T approaches t the approximation becomes poorer. Therefore a minimum value for the ratio of t - T + x to t below which this approximation will not be valid must be established for each element. Table 3 gives the values for the ratio of t - T + x to t above which the approximation of the binomial will be better than 90 percent.

Table 3. Values for the Ratio t - T + x to t Above Which the Approximation of the Binomial is Better Than 90%

Nuclide	n	$\dfrac{t - T + x}{t}$
$_{94}Pu^{239}$	0.01	0.0001
$_{38}Sr^{90}$	.35	.45
Natural Uranium	.50	.55
$_{88}Ra^{226}$	0.52	0.55

These ratios were calculated by relating the approximate formula to the binomial as a percent or

$$\% = \frac{[t^{-n} + n(T - x)t^{-(n + 1)}]\,100}{(t - T + x)^{-n}} \, . \qquad (31)$$

Setting T - x = 1, the values for t which would give 90 percent were found by trial and error. These values of t were then substituted in

$$\frac{t - 1}{t}$$

to give the above ratios. It can be seen from table 3 that the approximation may be used to describe the excretion of plutonium at

any time t; however, some discretion must be used with the other elements. Substituting this approximation [equation (30)] in equation (29) gives

$$E_c(t) = q\ A\ [t^{-n} + n(T-x)\ t^{-(n+1)} - t^{-n}]$$

or

$$E_c(t) = q\ An(T - x)\ t^{-(n + 1)} \tag{32}$$

Substituting t_1 and t_2 in the preceding equation and solving the two equations simultaneously for $q(T - x)$ similar to the derivation of equation (15) gives

$$q(T - x) = \frac{E_c(t_1)}{nA}\left[\frac{\Delta t}{\left(\frac{E_c(t_1)}{E_c(t_2)}\right)^{\frac{1}{n + 1}} - 1}\right]^{n + 1}. \tag{33}$$

This formula represents the quantity of material assimilated due to a chronic invariant exposure since it is the product of the quantity assimilated daily and the period of the exposure in days. It is particularly significant here that, using this formula, the total quantity assimilated can be calculated when only the quantity excreted on two different days and the time interval between the excretions are known.

Calculation of Integral Absorbed Dose
in Multiple or Continuous Exposures

For the sake of simplifying the derivations, only the hypothetical chronic invariant assimilation discussed earlier will be considered. In this case if the dose resulting from each increment assimilated is described by equation (26')

$$D(t) = \frac{q(o)\ KA}{1 - n}\ (t^{1 - n} - x^{1 - n})\ ,$$

where $K = 2.3 \times 10^{-5} \Sigma E(RBE)N$, the total dose can be considered as the sum of the doses from each of the incremental steps. In setting up this equation, the time since assimilation in equation (26') must be taken as the difference between the time of observation t and τ, the time of assimilation, or $(t - \tau + x)$. The reason for the constant x has been discussed earlier. Integrating equation (26') over the period of exposure, T,

$$D_c(t) = \frac{q\,KA}{1-n} \int_x^T [(t - \tau + x)^{1-n} - x^{1-n}]\,d\tau \qquad (34)$$

gives

$$D_c(t) = \frac{q\,KA}{n^2 - 3n + 2}\left[t^{2-n} - (t - T + x)^{2-n}\right] - \frac{q\,KAx^{1-n}}{1-n}(T - x). \quad (35)$$

$\boxed{\text{See Appendix V}}$

Where the radionuclide is Pu-239 and n is a small number, equation (35) can be approximated by

$$D_c(t) \cong \frac{q\,K\,A}{2}\left[t^2 - (t-T+x)^2\right] - q\,K\,A\,x\,(T-x)$$

or

$$D_c(t) \cong q(T - x)\,KA\left(\frac{2t - T - x}{2}\right). \qquad (36)$$

$\boxed{\text{See Appendix VI}}$

Although a value for $q(T - x)$ may be obtained from equation (33), this formula, as well as equation (35), will be of little value unless some information on t and T is available. Therefore, a better appraisal of the magnitude of the damage resulting from a multiple or continuous assimilation of these nuclides, particularly if t and T are large, may be made by comparing the quantity assimilated, calculated using equation (33), with the values of the maximum permissible total body burden for a 70-kg man found in the literature[22,23].

Summary

If a power function formula is used to describe the retention of radionuclides in the body, the relationship between the quantity assimilated in an acute exposure $q(o)$ and that excreted at time t days postassimilation, $E(t)$, is given by

$$q(o) = \frac{E(t)\ t^{n+1}}{nAe^{-\lambda_r(t-1)}}$$

where A is a constant which is equal to the fraction of the assimilated material retained when t is equal to one day, n is a constant equal to the negative slope of the linear log-log transform of the retention function, and λ_r is the radiological decay constant. When the values of t are close to the time of assimilation or when the unit of time, $t_1 - t_2$, over which the excreta is collected is long, the relationship will be better represented by

$$q(o) = \frac{E}{A(t_1^{-n} - t_2^{-n})\ e^{-\lambda_r(t_2-1)}}\ .$$

Whereas in the past, power function equations have been given for urinary or fecal excretion alone, the quantity expressed here by $E(t)$ or E represents the total excretion. Therefore, if only urinary excretion data are available, the value of $E(t)$ or E must be extrapolated based on experimental data.

If multiple assimilations are of similar magnitude and occur at reasonably short and constant intervals, and if the radiological decay constant is approximately zero, the relationship between the quantity assimilated per unit time, q, over a period of T days and

$E_c(t)$, the quantity excreted at time t days after the commencement of the exposure can be given by

$$E_c(t) = q \, A \, [(t - T + x)^{-n} - t^{-n}]$$

where x is that fraction of a day when the fraction retained is unity.

Frequently, the fact that a radionuclide has been assimilated is discovered as the result of routine sampling and later confirmed by a resample. Here the quantity assimilated as the result of an acute exposure can be calculated using the relationship

$$q(o) = \frac{E(t)}{nAe^{-\lambda_r(t_1 - 1)}} \left[\frac{\Delta t}{\left(\dfrac{E(t_1)}{E(t_2)}\right)^{\frac{1}{n + 1}} - 1} \right]^{n + 1}$$

where $E(t_1)$ and $E(t_2)$ are the quantities excreted on the two different occasions Δt days apart. Within the limits shown in table 3, the relationship between the quantity of a radionuclide assimilated over a period of T - x days, $q(T - x)$, and that excreted on two different occasions can be given by

$$q(T - x) = \frac{E_c(t_1)}{nA} \left[\frac{\Delta t}{\left(\dfrac{E_c(t_1)}{E_c(t_2)}\right)^{\frac{1}{n + 1}} - 1} \right]^{n + 1}.$$

The calculation of the integral absorbed dose due to a chronic invariant assimilation requires information which in most cases is not available and must be approximated. Therefore, the magnitude of the damage resulting from such an exposure may be better appraised by comparing the quantity assimilated with the values of the maximum permissible total body burden for a 70-kg man found in the

literature[22,23]. The integral absorbed dose in gram-rem over a period of time t, which is due to an acute exposure, may be calculated from the equation

$$D(t) = \frac{q(o)\ KA}{1 - n} (t^{1 - n} - x^{1 - n})$$

where $K = 2.3 \times 10^{-5}\ \Sigma E(RBE)N$. When the radiological decay constant is appreciable the equation

$$D(t) = \frac{q(o)\ KA}{\lambda_r(1 - n)} \left[I(u,p) - I(u',p) \right]\ \Gamma(1 - n)$$

must be used. Here the arguments to the function $I(u,p)$ are $p = -n$, $u = \frac{\lambda_r t}{\sqrt{1 - n}}$, and $u' = \frac{\lambda_r x}{\sqrt{1 - n}}$. The numerical values for the function $I(u,p)$ are given in tables 2 and 3 of "Tables of the Incomplete Γ - Function"[25], and the values for the function $\Gamma(1 - n)$ can be computed from any table of Γ - functions using the relationship

$$\Gamma(1 - n) = \frac{\Gamma(2 - n)}{1 - n}.$$

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Appendix I

Integration of equation (4).

$$\frac{dR(t)}{dt} = -\frac{nR(t)}{t} - \lambda_r R(t) \tag{4}$$

$$\int \frac{dR(t)}{R(t)} = -\int \frac{ndt}{t} - \int \lambda_r dt$$

$$\ln R(t) = C - n \ln t - \lambda_r t$$

Taking $R(1)$ equal to $\underline{A}$, as defined in equation (1), then

$$\ln A = C - \lambda_r$$

$$C = \ln A + \lambda_r$$

$$\ln R(t) = \ln A - \lambda_r(t - 1)$$

$$R(t) = At^{-n} e^{-\lambda_r(t - 1)}. \tag{5}$$

Appendix II

Simultaneous solution of equation (12) and (13) for t_1.

$$E(t_1) = \frac{q(o) \; nAe^{-\lambda_r(t_1 - 1)}}{t_1^{\,n+1}} \qquad (12)$$

$$E(t_2) = \frac{q(o) \; nAe^{-\lambda_r(t_2 - 1)}}{(t_1 + \Delta t)^{n+1}} \qquad (13)$$

$$\frac{E(t_1)}{D(t_2)} = \left(\frac{t_1 + \Delta t}{t_1}\right)^{n+1}$$

$$\left(\frac{E(t_1)}{E(t_2)}\right)^{\frac{1}{n+1}} = \frac{t_1 + \Delta t}{t_1}$$

$$\left(\frac{E(t_1)}{E(t_2)}\right)^{\frac{1}{n+1}} - 1 = \frac{\Delta t}{t_1}$$

$$t_1 = \frac{\Delta t}{\left(\dfrac{E(t_1)}{E(t_2)}\right)^{\frac{1}{n+1}} - 1} \;. \qquad (14)$$

APPENDIX III

Conversion of equation (23) to a soluble form

$$D(t) = q(o) \, KA \int_{x}^{t} \tau^{-n} \, e^{-\lambda_r(\tau - 1)} \, d\tau \qquad (23)$$

If the function $\lambda_r \tau = v$, the differential $d\tau = \dfrac{dv}{\lambda_r}$, the lower limit $\tau = x$ becomes $v = \lambda_r x$, and the upper limit $\tau = t$ becomes $v = \lambda_r t$. Substituting in equation (23) gives

$$D(t) = \frac{q(o) \, KA}{\lambda_r} \int_{\lambda_r x}^{\lambda_r t} \frac{v^{-n}}{\lambda_r^{-n}} \, e^{-(v - \lambda_r)} \, dv$$

or

$$D(t) = \frac{q(o) \, KAe^{\lambda_r}}{\lambda_r \, 1 - n} \left[\int_{o}^{\lambda_r t} v^{-n} \, e^{-v} \, dv - \int_{o}^{\lambda_r x} v^{-n} \, e^{-v} \, dv \right] \quad .$$

The values given in the tables of incomplete Γ - functions[25] are for the arguments u and p in the function

$$I(u,p) = \frac{1}{\Gamma(p + 1)} \int_{o}^{u\sqrt{p + 1}} v^{p} e^{-v} \, dv \quad .$$

Therefore, the above equation can be written

$$D(t) = \frac{q(o) \, KAe^{\lambda_r}}{\lambda_r \, (1 - n)} \left[I(u,p) - I(u',p) \right] \Gamma(p + 1)$$

where $p = n$, $u = \dfrac{\lambda_r t}{\sqrt{1 - n}}$, and $u' = \dfrac{\lambda_r x}{\sqrt{1 - n}}$.

The equation

$$D(t) = \frac{q(o)\ KAe^{\lambda_r}}{\lambda_r^{1-n}} \int_{\lambda_r x}^{\lambda_r t} v^{-n}\ e^{-v}\ dv$$

can also be integrated using the fundemental theorem

$$\int x\,dy = xy - \int y\,dx$$

where x and y denote the following functions of v:

$$x = e^{-v} \qquad\qquad y = -\frac{v^{1-n}}{1-n}$$

$$dx = -e^{-v}\ dv \qquad dy = v^{-n}\ dv$$

giving

$$D(t) = \frac{q(o)\ KAe^{\lambda_r}}{\lambda_r^{1-n}}\left[\frac{v^{1-n}e^{-v}}{1-n} + \int v^{1-n}\ e^{-v}\ dv\right]\ .$$

Integrating the last term again using $y = \dfrac{v^{2-n}}{2-n}$ and $dy = v^{1-n}$

gives

$$D(t) = \frac{q(o)\ KAe^{\lambda_r}}{\lambda_r^{1-n}}\left[\frac{v^{1-n}\ e^{-v}}{1-n} + \frac{v^{2-n}\ e^{-v}}{2-n} + \int \frac{v^{2-n}}{2-n}\ e^{-v}\ dv\right]\ .$$

This may be continued *ad infinitum* giving

$$D(t) = \frac{q(o)\ KAe^{\lambda_r}}{\lambda_r^{1-n}}\left[v^{1-n}\ e^{-v}\left(\frac{1}{1-n} + \frac{v}{2-n} + \frac{v^2}{3-n} + \cdots\right)\right]_{\lambda_r x}^{\lambda_r t}$$

or

Integrating using the fundamental theorem

$$\int x\,dy = xy - \int y\,dx$$

where x and y denote the following functions of v:

$$x = e^{-v} \qquad y = -\frac{v^{-n}}{n}$$

$$dx = -e^{-v}\,dv \qquad dy = v^{-(n+1)}\,dv$$

gives

$$E_c(t) = \frac{q\ nAe^{\lambda r}}{\lambda_r^{-n}}\left[\frac{v^{-n}\,e^{-v}}{n} + \int \frac{v^{-n}}{n}\,e^{-v}\,dv\right]\ .$$

Integrating the last term again using $y = \dfrac{v^{1-n}}{1-n}$ and $dy = v^{-n}\,dv$

gives

$$E_c(t) = \frac{q\ nAe^{\lambda r}}{\lambda_r^{-n}}\left[\frac{v^{-n}\,e^{-v}}{n} + \frac{v^{1-n}\,e^{-v}}{n(1-n)} + \int \frac{v^{1-n}\,e^{-v}}{n(1-n)}\,dv\right]\ .$$

Again integrating the last term gives

$$E_c(t) = q\ Ae^{-\lambda_r(t-1)}\left[(t-T+x)^{-n}\,e^{\lambda_r(T-x)} - t^{-n} + \right.$$

$$\frac{\lambda_r(t-T+x)^{1-n}\,e^{\lambda_r(T-x)} - \lambda_r t^{1-n}}{1-n} +$$

$$\left.\frac{\lambda_r^2(t-T+x)^{2-n}\,e^{\lambda_r(T-x)} - \lambda_r^2 t^{2-n}}{(1-n)(2-n)} + \ .\ .\ .\ .\ .\right]\ .$$

This may be written as an infinite series

$$E_c(t) = -q\ nAe^{-\lambda_r(t-1)}\sum_{j=0}^{\infty} \frac{\lambda_r^j(t-T+x)^{j-n} - t^{j-n}}{(1-n)(2-n)\ \cdots\cdots(j-n)}\ .\ (28)$$

APPENDIX V

The integration of equation (34).

$$D_c(t) = \frac{q\ KA}{1-n} \int_x^T [(t-\tau+x)^{1-n} - x^{1-n}]\, d\tau \qquad (34)$$

If the function $t - \tau + i = v$, the differential $d\tau = -dv$, the lower limit $\tau = x$ becomes $v = t$, and the upper limit $\tau = T$ becomes $v = t - T + x$.

Substituting in equation (34) gives

$$D_c(t) = -\frac{q\ KA}{1-n} \int_t^{t-T+x} (v^{1-n} - x^{1-n})\, dv$$

Integrating gives

$$D_c(t) = -\frac{q\ KA}{(1-n)(2-n)} v^{2-n} \Big]_t^{t-T+x} + \frac{q\ KA}{1-n} x^{1-n} v \Big]_t^{t-T+x}$$

or

$$D_c(t) = \frac{q\ KA}{n^2 - 3n + 2} \left[t^{2-n} - (t-T+x)^{2-n} \right] + \frac{q\ KAx^{1-n}}{1-n}(t-T+i-t)$$

or

$$D_c(t) = \frac{q\ KA}{n^2 - 3n + 2} \left[t^{2-n} - (t-T+x)^{2-n} \right] - \frac{q\ KAx^{1-n}}{1-n}(T-x). \quad (35)$$

APPENDIX VI

When n is a small number, equation (35) can be approximated by

$$D_c(t) = \frac{q\ KA}{2}\left[t^2 - (t - T + x)^2\right] - q\ KАx\ (t - x)$$

The expression $(t - T + x)^2$ can be expanded as follows:

$$(t - T + x)^2 = t^2 + T^2 + x^2 - 2tT - 2xT + 2xt$$

$$(t - T + x)^2 = t^2 + (T - x)^2 - 2t\ (T - x)$$

Substituting this in the above equation gives

$$D_c(t) = \frac{q\ KA}{2}\left[2t\ (T - x) - (T - x)^2\right] - q\ KАx\ (T - x)$$

or

$$D_c(t) = KA\ q\ t\ (T - x) - \frac{q\ KA}{2}\ (T - x)^2 - q\ KАx\ (T - x)$$

or

$$D_c(t) = KA\ q\ (T - x)\ (t - x) - \frac{q\ KA}{2}\ (T - x)^2$$

or

$$D_c(t) = KA\ q\ (T - x)\left[(t - x) - \frac{T - x}{2}\right]$$

or

$$D_c(t) = KA\ q\ (T - x)\left(\frac{2t - T - x}{2}\right)$$