Final Report
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Computational Semidefinite Programming

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Semidefinite programming (denoted SDP) is an extension of linear programming (LP), with vector variables replaced by matrix variables and componentwise nonnegativity replaced by positive semidefiniteness. SDP's are convex, but not polyhedral, optimization problems. SDP is well on its way to becoming an established paradigm in optimization, with many current and potential applications. Consequently, efficient methods and software for solving SDP's are of great importance.

During the award period, we have primarily focused our attention on three aspects of computational semidefinite programming.

I. General-purpose methods for semidefinite and quadratic cone programming.

During the first part of the grant period we put substantial effort into investigating efficient *general-purpose* interior-point methods for computational semidefinite programming, together with the related problem of quadratic cone programming. Interior-point methods came to be recognized as very powerful techniques for solving convex optimization problems, including LP, following Karmarkar's breakthrough work for LP in the 1980's. The interior point methods that we investigated exploit sparse linear algebra as far as possible as and are based on "higher-order" search directions. This was the subject of Publication 1 listed below, which appeared in *Optimization Methods and Software*. Higher-order directions are based on ideas which were originally suggested for LP, but they seem to hold a greater payoff for SDP. In brief, the idea is to get greater mileage out of the factorization of the Schur complement matrix defining the search direction for the algorithm, by performing corrector steps, each of which requires only a backsolve and not a new factorization.

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II. Specific applications:

During the second part of the grant period we turned our attention to more specific application classes. These included:

1. **LMI problems arising in control.** SDP has been enormously successful and influential in control, where it goes under the name of LMI: Linear Matrix Inequalities. However, many control engineers are unaware of the dramatic algorithmic improvements of the last few years. Publication 2 below was invited as a chapter of a new book on LMI's and SDP for control engineers. Using notation which is standard in that community (and quite different from that used by the optimization community) our goal in writing this chapter was to make our SDPpack code accessible and convenient for engineers wishing to solve LMI's with semidefinite and quadratic constraints. The book was published by SIAM in 2000 and has already proved quite influential in the control engineering community.

2. **Minimizing a sum of Euclidean norms.** The problem of minimizing a sum of Euclidean norms dates from the 17th century and may be the earliest example of duality in the mathematical programming literature. This nonsmooth optimization problem arises in many different kinds of modern scientific applications, including robotics, VLSI design, image reconstruction, planetary "fly-by's" and plastic collapse analysis. Together with K.D. Andersen, E. Christiansen and A.R. Conn, we derived a specialized primal-dual interior-point algorithm for this problem class, which is substantially simpler than for general quadratic cone or semidefinite programs. The algorithm has been used to solve applied problems arising in plastic collapse analysis with hundreds of thousands of variables and millions of nonzeros in the constraint matrix. The algorithm typically finds accurate solutions in less than 50 iterations and determines physically meaningful solutions that were considered unobtainable until now. This work (Publication 3 below) was published in *SIAM J. Scient. Comp.* We chose this journal because we think the readership is particularly broad and likely to include interested scientists who would not ordinarily read the optimization literature.

3. **A quantum mechanics application of SDP.** The possibility of using the one-body and two-body reduced density matrices, rather than the many-body wavefunction, as the fundamental object of study for electronic structure calculations was actively explored in the 1960's and 1970's but interest has waned since. In this approach the calculation of ground-state properties is reduced to a linear optimization problem subject to the representability conditions for the density matrices,

which are a mixture of linear equalities and bounds on eigenvalues. This leads to SDP's whose solutions approximate the desired energy states. These problems can be very large. With our new interior-point method described above, we were able to solve small (chemists would say tiny) problems quite satisfactorily. These have 1000-2000 independent variables, meaning that in each interior-point iteration we must factor a dense matrix of this size. However, it is clear that for the problem size of interest to chemists, interior-point methods are out of the question. We were therefore led to bundle-subgradient methods of the type developed by C. Lemarechal in France. Lemarechal's former student F. Oustry spent the calendar year 1998 in New York collaborating extensively with our group on this project, and this was a particularly productive year. This work was joint with Bastiaan Braams and Jerome Percus of the Courant Institute, and with Madhu Nayakkankuppam, who graduated from Courant in August 1999 with his Ph.D. The last chapter of Madhu's Ph.D. thesis was entirely about the quantum mechanics application. Bas Braams was very pleased with the results Madhu got with his code and feels there is a lot of potential here for future work. Our current student in physics, Zhengji Zhang, has continued investigations in this area, and a postdoctoral fellow, Mitsuhiro Fukuda, will be joining our group in November 2002 to continue these investigations.

III. Optimizing Matrix Stability

A problem that is closely related to semidefinite programming, but much harder, is what we call *semistable programming:* now the variable is a *nonsymmetric* matrix, and the constraint is that *its eigenvalues lie in the right (or left) half of the complex plane.* This constraint is called a *stability constraint.* The PI has had a long standing collaboration with Jim Burke of the University of Washington, investigating non-Lipschitz optimization problems of this kind. Problems like this arise in many kinds of applications, especially in control, but the problems are very difficult because the constraint is not differentiable, convex, or even Lipschitz. In Publications 4 and 5 below we were able to give quite a complete variational theory for the analysis of spectral objective and constraint functions of this kind. This led to optimality conditions for semistable programs, and other related optimization problems with stability constraints. We see this as a very substantial theoretical breakthrough. We investigated further theoretical questions with another colleague, Adrian Lewis, in Publications 6, 7 and 8. The next step was how to move forward with numerical methods for optimizing stability. We made a substantial step forward in this direction in Publication 9.

Publications Acknowledging the Grant

1. J.-P. Haeberly, M.V. Nayakkankuppam and M.L. Overton, *Extending Mehrotra and Gondzio Higher Order Methods to Mixed Semidefinite-Quadratic-Linear Programming*, Optimization Methods and Software 11 (1999), pp. 67-90.
2. J.-P. Haeberly, M.V. Nayakkankuppam and M.L. Overton, *Mixed Semidefinite-Quadratic-Linear Programs*, in: Recent Advances in LMI Methods for Control (L. El Ghaoui and S.I. Niculescu, eds), SIAM, 2000, pp. 41-54.
3. K.D. Andersen, E. Christiansen, A.R. Conn and M.L. Overton, *An Efficient Primal-Dual Interior-Point Method for Minimizing a Sum of Euclidean Norms*, SIAM J. Scient. Comp. 22 (2000), pp. 243-262.
4. J.V. Burke and M.L. Overton, *Variational Analysis of the Abscissa Mapping for Polynomials*, SIAM J. Control Optim. 39 (2001), pp. 1651-1676.
5. J.V. Burke and M.L. Overton, *Variational Analysis of Non-Lipschitz Spectral Functions*, Math. Programming 90 (2001), pp. 317-352.
6. J.V. Burke, A.S. Lewis and M.L. Overton, *Optimizing Matrix Stability*, Proceedings of the American Mathematical Society 129 (2001), pp. 1635-1642.
7. J.V. Burke, A.S. Lewis and M.L. Overton, *Optimal Stability and Eigenvalue Multiplicity*, Foundations of Computational Mathematics 1 (2001), pp. 205-225.
8. J.V. Burke, A.S. Lewis and M.L. Overton, *Approximating Subdifferentials by Random Sampling of Gradients*, to appear in Mathematics of Operations Research.
9. J.V. Burke, A.S. Lewis and M.L. Overton, *Two Numerical Methods for Optimizing Matrix Stability*, Linear Algebra and its Applications 351-352 (2002), pp. 117-145.

Students, Postdoctoral Associates, and other Collaborators

- Madhu V. Nayakkankuppam was supported by the grant in 1998 and 1999. He completed his Ph.D. in computer science at NYU in August 1999. He is now an Assistant Professor at the University of Maryland, Baltimore County, following a one-year postdoctoral position at the Pacific Institute of Mathematical Sciences in Vancouver, British Columbia.

- Zhengji Zhang is a fifth-year Ph.D. student in physics completing her thesis with the P.I. She is supported by the physics department.
- Emre Mengi is a second-year Ph.D. student in computer science, beginning research work with the P.I.
- François Oustry (INRIA, Grenoble, France) was a postdoctoral visitor at the Courant Institute, collaborating with the P.I., for the calendar year Jan-Dec, 1998. He received his Ph.D. in eigenvalue optimization in Paris, 1997. His Ph.D. thesis specifically built on eigenvalue optimization work done by the P.I. in the late 1980's. His visit was supported by the French Government, though NYU paid some of his expenses.
- During 1998 and 1999, Jean-Pierre Haeberly received some summer support from the P.I.'s NSF grant at NYU while he was an Associate Professor at Fordham University, collaborating closely with the P.I. on computational semidefinite programming.
- Adrian Lewis is a Professor at Simon Fraser University on leave from the University of Waterloo. He spent his sabbatical leave at NYU during the academic year 1999-2000 and is a frequent visitor to NYU. He has closely collaborated with the P.I. on matrix stability optimization. Some of his visiting expenses were covered by the grant.
- Other on-going collaborators: James Burke (Washington) and Bas Braams (Courant).

Other Activities of the P.I.

- Editor-in-Chief of *SIAM J. Optimization* through 1999.
- Invited plenary lectures at: an international workshop on Computational Semidefinite Programming in Berlin, November 1998; the West Coast Optimization Meeting, Seattle, February 1999; SIAM Conference on Optimization in Atlanta, May 1999; Householder Symposium XIV, Whistler, BC, June 1999; International Conference on Optimization, Aveiro, Portugal, July 2001; SIGOPT - International Conference on Optimization, Lambrecht, Germany, February 2002; International Linear Algebra Conference, Auburn, June 2002.
- Invited semiplenary lecture at Conference on Foundations of Computational Mathematics, Oxford, July 1999.
- Distinguished Speaker Series in High Performance Computation for Engineered Systems, MIT, May 9, 2001.

- Author of the book *Numerical Computing with IEEE Floating Point Arithmetic*, published by SIAM in 2001.
- Director of Undergraduate Studies in the Computer Science Department at NYU.
- Elected member of the SIAM Board of Trustees.

Information on the Web

For further information on the P.I.'s activities, including all publications, see

http://cs.nyu.edu/cs/faculty/overton/